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AN UNSTEADY VISCOUS FLOW THROUGH A POROUS SLAB BOUNDED BETWEEN TWO IMPERMEABLE PARALLEL PLATES

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ABSTRACT

The present investigation deals with an unsteady flow of viscous incompressible Newtonian fluid through a porous medium bounded between two impermeable parallel plates. The momentum equation for the flow through a porous medium takes care of fluid inertia and the Newtonian stresses in addition to classical Darcy's friction. Initially, the flow is generated by a constant pressure gradient down the plates. When the steady state is reached, the pressure gradient is suddenly withdrawn and the subsequent fluid-flow is analyzed employing the Laplace transform technique to obtain the fluid-velocity field. Expressions for a further, flow-rate, skin friction on the boundary having been obtained.

Variations of these flow parameters with time and the porosity coefficient have been illustrated based on which some conclusions.

INTRODUCTION

Fluid flows through porous media have been attracting the attention of Engineers and Applied Mathematicians for the last one and half centuries because of their importance notably in the flows of oils through porous rocks, extraction of energy through geothermal regions, filtration of solids from liquids, drug permeation through human skin and so on. Knowledge of the flow through porous media is immensely useful in the efficient recovery of crude oil from the porous reservoir rocks by the displacement with immissible water (Rudraiah *et.al* [9]).

Flow through porous media occur in the ground-water hydrology, irrigation drainage problems and also in absorption and filtration processes in chemical engineering. The subject of fluid-flows through porous media has widespread applications to problems encountered in the civil engineering and agriculture engineering and several other areas of importance in industries. Diffusion and flow of fluids through ceramic materials as bricks and porous earthenware has long been problems of the ceramic industry. The construction of filter beds for municipal water system and water seepage through, around and beneath dams, earthen reservoirs, the scientific treatment of problems of irrigation, soil erosion and tile drainage are some of the present day developments of flows through porous media.

Some other areas of study having wide application include the run-up and spin-up flow of fluids between rigid boundaries. Kazakai and Rivlin [4] investigated run-up and spin-up flows of non-Newtonian liquids in parallel plate and circular geometry. The flow field arising from super- position of waves propagated into the fluid and reflected back and forth at the boundaries was investigated later by Pattabhi Ramacharyulu [5] and Appala Raju [6]. Run-up flows of a fluid with particle suspensions was considered by RamaKrishna [7]. Some such flows of second order fluids were examined by Ramana Murthy[8]. Recently Gnanaprasuna, Pattabhi Ramacharyulu and Ramanamurthi[2] and Gnanaprasuna[3] investigated a run-up flow of a visco elastic fluid through a porous medium between two parallel plates

In this paper, we investigate a special run-up retarding flow of a viscous incompressible homogeneous fluid through a porous slab of finite thickness bounded between two impermeable parallel plates.

Following Yamamoto and Yashida Z. [10], we adopt the following momentum equation for the flow through porous media which takes care of the fluid inertia and the viscous stress in addition to the classical Darcy's friction

$$\rho\left[\frac{\partial\bar{v}}{\partial t} + (\nabla.\bar{v})\nabla\right] = -\nabla p + \mu\nabla^2\bar{v} - \frac{\mu}{k}\bar{v} \tag{1}$$

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where

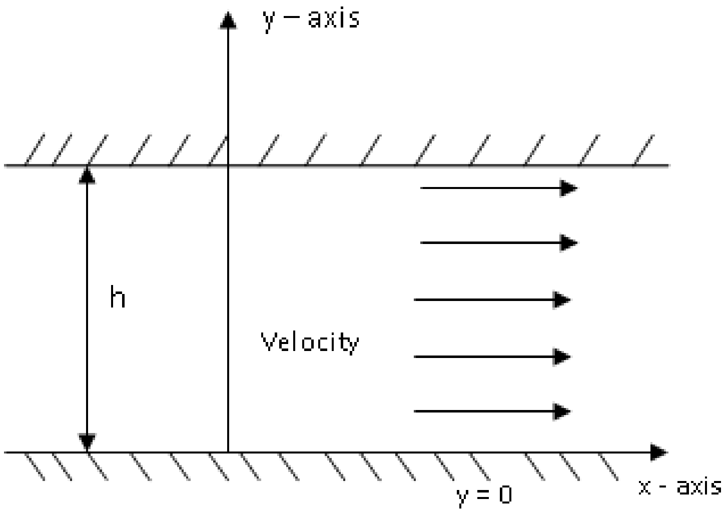
ρ is the fluid density
μ is the coefficient of viscosity and
k is the Darcy coefficient of porosity of the medium
p is the pressure and
$\bar{v}$ is the fluid velocity vector

Initially, the flow is generated by a constant pressure gradient down the plates. When the steady state is reached, the pressure gradient is suddenly withdrawn and the subsequent fluid-flow is examined.

Analytic expressions for the Velocity, Flow rate and Skin-Friction on the two plates are obtained, by employing Laplace Transformation technique and variation of all the flow parameters are shown with illustration and conclusions are drawn.

Formulation of the problem
Consider a Cartesian frame of reference O(x, y, z) with the origin fixed on one of the plates, x- axis along the flow direction, y-axis perpendicular to the plates, so that the plates can be represented by y=0 and y=h. where h is the gap between the plates. The fluid velocity vector can be taken as $\bar{v} = (u(y,t),0,0)$. This evidently satisfies the continuity equation:

$$\nabla . \bar{v} = 0 \tag{2}$$



Flow in porous medium:

The momentum equation is in x-direction characterizing the flow through porous media is

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \gamma \frac{\partial^2 u}{\partial y^2} - \frac{\gamma u}{k} \tag{3}$$

where $\gamma = \frac{\mu}{\rho}$

The problem under investigation is composed of two stages. The first stage is the steady flow between the parallel plates under a constant pressure gradient down the plates while the second stage concerns with the subsequent unsteady receding flow, when the pressure gradient (cited in stage – 1) is suddenly withdrawn.

Stage – 1:
This is given by the velocity $(u_1(y),0,0)$ which satisfies the momentum equation of steady state flow

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \gamma \frac{d^2 u_1}{dy^2} - \frac{\gamma}{k} u_1 \tag{4}$$

with the boundary conditions

$$u_1(y = 0) = 0 \ \text{ and } \ u_1(y = h) = 0 \tag{5}$$

Stage – 2:
When the steady state is reached, the pressure gradient is with drawn.

Let the subsequent flow (after the pressure gradient is with drawn) be characterized by the velocity $(u_2(y,t),0,0)$ this satisfies the momentum equation for the unsteady flow.

$$\frac{\partial u_2}{\partial t} = \gamma \frac{d^2 u_2}{dy^2} - \frac{\gamma}{k} u_2 \tag{6}$$

with the boundary conditions

$$u_2(y,t) = 0 \quad \text{on} \quad y = 0 \quad \text{and} \quad y = h \tag{7}$$

and with the initial condition

$$u_2(y,0) = u_1(y) \tag{8}$$

where $u_1(y)$ is the steady state solution obtained in stage - 1

The following non-dimensional quantities are introduced for simplicity of the analysis
$Y = hY$; $k = Kh^2$; $t = \frac{h^2}{\gamma} T$; $u_1(y) = u^* U_1(y)$;

$$u(y,t) = u^* U(Y,T); \; -\frac{h^2}{\rho\,\gamma\,u^*}\frac{\partial p}{\partial x} = C ; \tag{9}$$

where u^* is the mean velocity in the absence of the porous nature of the medium given by equation (6) (vide: Appendix A).

The solution of the problem in non-dimensional form:

Stage – 1:
The momentum equation is

$$\frac{d^2 U_1}{dY^2} - \frac{U_1}{K} = -c \tag{10}$$

with the boundary conditions

$$U_1(y=0) = 0 \quad \text{and} \quad U_1(y=1) = 0 \tag{11}$$

These equations yield the solution

$$U_1(Y) = 2CK \; \frac{\sin h\frac{Y}{2\sqrt{K}} \sinh\left(\frac{1-Y}{2\sqrt{K}}\right)}{\cos h\frac{1}{2\sqrt{K}}} \tag{12}$$

and this would be the initial conditions for the flow in the stage-2.

The flow variables in the dimensional form at this stage are

Velocity:

$$u(y) = u^* \frac{2ck}{h^2} \; \frac{\sin h\frac{y}{2\sqrt{k}} \sinh\left(\frac{h-y}{2\sqrt{k}}\right)}{\cos h\frac{h}{2\sqrt{k}}} \tag{13}$$

Flow Rate:

$$Q(y) = \frac{2u^*ck}{h^2}\left[\, h - \frac{2}{\sqrt{k}} \tanh\frac{h}{2\sqrt{k}} \,\right] \tag{14}$$

and

$$\text{Mean Velocity} = \frac{2u^*ck}{h^3}\left[\, h - \frac{2}{\sqrt{k}} \tanh\frac{h}{2\sqrt{k}} \,\right] = u_{mean} \tag{15}$$

Stage – 2:
The unsteady flow-velocity $U_2(Y,T)$ satisfies the equation

$$\frac{\partial U_2}{\partial T} = \frac{\partial^2 U_2}{\partial y^2} - \frac{U_2}{K} \tag{16}$$

together with the boundary conditions

$$U_2(Y,T) = 0 \quad \text{on} \quad Y = 0 \quad \text{and} \quad Y = 1 \tag{17}$$

and the initial condition

$$U_2(Y,0) = U_1(Y) \tag{18}$$

where $U_1(Y)$ is the steady state velocity obtained in stage – 1 given by equ.(13).

Laplace Transform Technique is employed to realize the solution

Let $\overline{U_2}$ (Y,s) = Laplace Transform of $U_2 = \int_0^\infty U_2(Y,T)\, e^{-sT} dT$ (19)

Taking Laplace Transform of the equation (16), we get

$$\frac{\partial^2 \overline{U_2}}{\partial y^2} - \frac{\overline{U_2}}{K} = s\overline{U_2}` - U_1(Y) \tag{20}$$

i.e with the boundary conditions

$\overline{U_2}$ (Y,s) = 0 on Y = 0 and Y= 1 (21)

The solution of (20) that satisfies conditions (21) is

$$\overline{U_2}\,(Y,s) = \frac{C\cosh m\left(\frac{2Y-1}{2}\right)}{s\left(s+\frac{1}{K}\right)\cosh\left(\frac{m}{2}\right)} - \frac{CK\cosh\left(\frac{2Y-1}{2\sqrt{K}}\right)}{s\cosh\left(\frac{1}{2\sqrt{K}}\right)} + \frac{CK}{s+\frac{1}{K}} \tag{22}$$

where

$$m = \sqrt{s + \frac{1}{K}} \tag{23}$$

Taking Inverse Laplace Transform of $\overline{U_2}(Y, s)$ (vide:Appendix B), we get the velocity

$$U_2(Y,T) = L^{-1}\left\{\frac{C\cosh m\left(\frac{2Y-1}{2}\right)}{s\left(s+\frac{1}{K}\right)\cosh\left(\frac{m}{2}\right)} + \frac{CK}{s+\frac{1}{K}} - \frac{CK\cosh\left(\frac{2Y-1}{2\sqrt{K}}\right)}{s\cosh\left(\frac{1}{2\sqrt{K}}\right)}\right\} \tag{24}$$

$$\Rightarrow \; U_2(Y,T) = \frac{4C}{\pi}\sum_{n=1,3,5\ldots}^{\infty} \frac{e^{-\left(n^2\pi^2+\frac{1}{K}\right)T}\sin(n\pi Y)}{n\left(n^2\pi^2+\frac{1}{K}\right)} \tag{25}$$

From this, we get the flow rate

$$Q = \int_0^1 U(Y,T)\,dY$$

$$= \frac{8C}{\pi^2}\sum_{n=1,3,5\ldots}^{\infty} \frac{e^{-\left(n^2\pi^2+\frac{1}{K}\right)T}}{n^2\left(n^2\pi^2+\frac{1}{K}\right)} \tag{26}$$

and the Shear stress

$$\tau = \mu\frac{du_2}{dy} = \frac{\mu u^*}{h}\frac{dU_2}{dY}$$

$$= \frac{4\mu u^* C}{h}\sum_{n=1,3,5\ldots}^{\infty} \frac{e^{-\left(n^2\pi^2+\frac{1}{K}\right)T}\cos(n\pi Y)}{\left(n^2\pi^2+\frac{1}{K}\right)}$$

This on the plate Y = 0 is

$$\tau_{Y=0} = \frac{4\mu u^{*}C}{h} \sum_{n=1,3,5\ldots}^{\infty} \frac{e^{-(n^2\pi^2+\frac{1}{K})T}}{(n^2\pi^2+\frac{1}{K})}$$

and on the plate Y= 1 is

$$\tau_{Y=1} = -\frac{4\mu u^{*}C}{h} \sum_{n=1,3,5\ldots}^{\infty} \frac{e^{-(n^2\pi^2+\frac{1}{K})T}}{(n^2\pi^2+\frac{1}{K})}$$

Also the rate of work done = $\frac{1}{2}\int_0^1 U^2 dY$

$$= \frac{4C^2}{\pi^2} \sum_{n=1,3,5\ldots}^{\infty} \frac{e^{-2(n^2\pi^2+\frac{1}{K})T}}{n^2(n^2\pi^2+\frac{1}{K})^2}$$

and dissipation of Energy = $\mu(\frac{dU}{dY})^2$

$$= \mu 16c^2 \left[\sum_{n=1,3,5,\ldots}^{\infty} \frac{e^{-2\left(n^2\pi^2+\frac{1}{K}\right)T} \cos^2(n\pi Y)}{\left(n^2\pi^2+\frac{1}{K}\right)^2} + 2\sum_{\substack{m,n=1,3,5,\ldots \\ m\neq n}}^{\infty} \frac{e^{-\left\{(n^2+m^2)\pi^2+\frac{2}{K}\right\}T} \cos(n\pi Y)\cos(m\pi Y)}{(n^2\pi^2+\frac{1}{K})(m^2\pi^2+\frac{1}{K})}\right]$$

The velocity in stage – 2 (final stage) in the dimensional form

$$u = u_2\,(y, t) = \frac{4C}{\pi} \sum_{n=1,3,5\ldots}^{\infty} \frac{e^{-(n^2\pi^2+\frac{h^2}{k})\frac{\gamma}{h^2}t} \sin(\frac{n\pi y}{h})}{n(n^2\pi^2+\frac{h^2}{k})}$$

CONCLUSION

The flow parameters like velocity, flow-rate etc. have been estimated numerically for different values of the porosity coefficient at different time instants. The velocity profiles are parabolic type with the velocity attaining maximum at the middle of the plates and due to friction, it decrease towards the plates.

Fig. 1(a) – 1(h) shows the velocity profiles when k=0.01, 0.02, 0.05, 0.08, 0.1, 0.2, 0.5, 1 at different time instants.

Fig. 2(a) – 2(e) shows the velocity profiles when t = 0.01, 0.05, 0.15, 0.1, 1 at different values of porosity coefficient.

Shear stress increases with k and T and it becomes linear in the middle from both the plates.

Fig. 3(a) shows the shear stress at upper plate (i.e. at $\tau_{Y=1}$) and

Fig. 3(b) shows the shear stress at lower plate (i.e. at $\tau_{Y=0}$)

Flow rate is decreasing slowly approaching ultimately to zero as the time increases. Fig. 4 shows the flow-rate for different values of porosity coefficient.

Fig. 5 shows the variation of the dissipation of energy at different time- instant. And Fig. 6 shows the variation of the workdone for different values of porosity coefficient.

Appendix A:

Flow in non-Porous median

When the flow region is clear i.e. not porous the momentum equation in the steady state is

$$0 = -\frac{\partial p}{\partial x} + \mu\frac{d^2u}{dy^2} \qquad (2)$$

with the boundary conditions

$$u\,(0) = 0, \quad u\,(h) = 0 \qquad (3)$$

These equations yield the solution

$$u = \frac{1}{2\mu}\frac{\partial p}{\partial x}(y^2 - hy) \tag{4}$$

The flow rate

$$Q = \int_0^h u\, dy = -\frac{h^3}{12\mu}\frac{\partial p}{\partial x} \tag{5}$$

Mean Velocity: $u^* = \frac{Q}{h}$

$$\Rightarrow u^* = -\frac{h^2}{12\mu}\frac{\partial p}{\partial x} \tag{6}$$

This mean velocity has been employed as the standard velocity in the non-dimensional scheme defined in equ. (9).

Appendix B:

Let $L\{f(T)\} = F(s)$, be the Laplace Transform of (t).

Then the inverse transform is given by

$$f(T) = L^{-1}\{F(s)\} = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} f(s)\, e^{st} ds$$

where γ is a vertical contour in the complex plane chosen so that all the poles of the integrand are to the left of it [10]. Eric W. Weirstein, CRC, Concise Encyclopedia of Mathematics, 1999, page 176.

Therefore $L^{-1}\{f(s)\}$ = sum of the residues of all the poles of $f(s)\, e^{st}$

i.e. of $f(s)$ inside the Bromwich Contour.from (24)

$$U_2(Y,T) = L^{-1}\left\{\frac{C\cosh m(\frac{2Y-1}{2})}{s\left(s+\frac{1}{K}\right)\cosh(\frac{m}{2})} + \frac{CK}{s+\frac{1}{K}} - \frac{CK\cosh(\frac{2Y-1}{2\sqrt{K}})}{s\cosh(\frac{1}{2\sqrt{K}})}\right\}$$

The poles of the first term are at

$s = 0,\ -\frac{1}{K}$ and $\frac{m_n}{2} = \frac{n\pi i}{2}$, $n = 1, 3, 5, \ldots$

$\Rightarrow s_n = -\left(n\pi^2 + \frac{1}{K}\right)$ (where $m_n = \sqrt{s_n + \frac{1}{K}}$)

Residue at s = 0: $\frac{CK\cosh(\frac{2Y-1}{2\sqrt{K}})}{\cosh(\frac{1}{2\sqrt{K}})}$

Residue at $s = -\frac{1}{K}$: $-CK\, e^{-\frac{T}{K}}$

Residue at $s = s_n$: $\sum_{n=1,3,5\ldots}^{\infty} \frac{e^{-(n^2\pi^2+\frac{1}{k})T}\sin(n\pi Y)}{n\pi(n^2\pi^2+\frac{1}{k})}$

Thus,

$$L^{-1}\left\{\frac{C\cosh m(\frac{2Y-1}{2})}{s\left(s+\frac{1}{K}\right)\cosh(\frac{m}{2})}\right\} = \sum_{n=1,3,5\ldots}^{\infty} \frac{e^{-(n^2\pi^2+\frac{1}{k})T}\sin(n\pi Y)}{n\pi(n^2\pi^2+\frac{1}{k})}$$

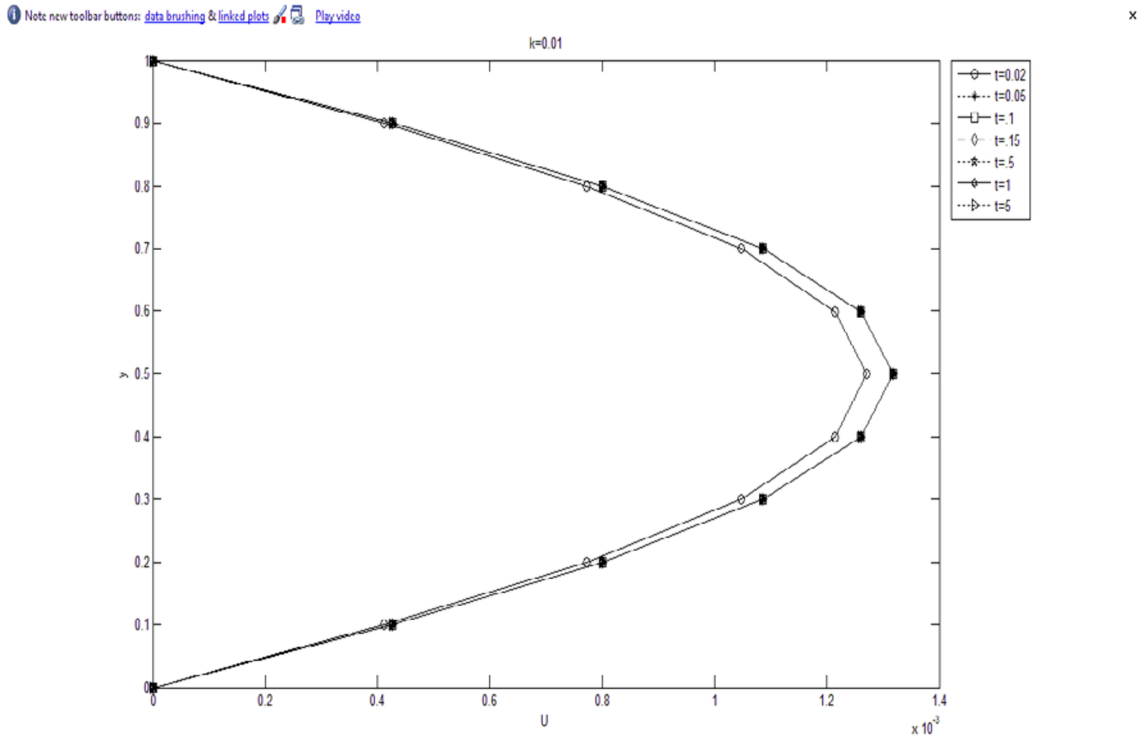


Fig. 1(a): Velocity profiles for k=0.01 at different time instants (t)

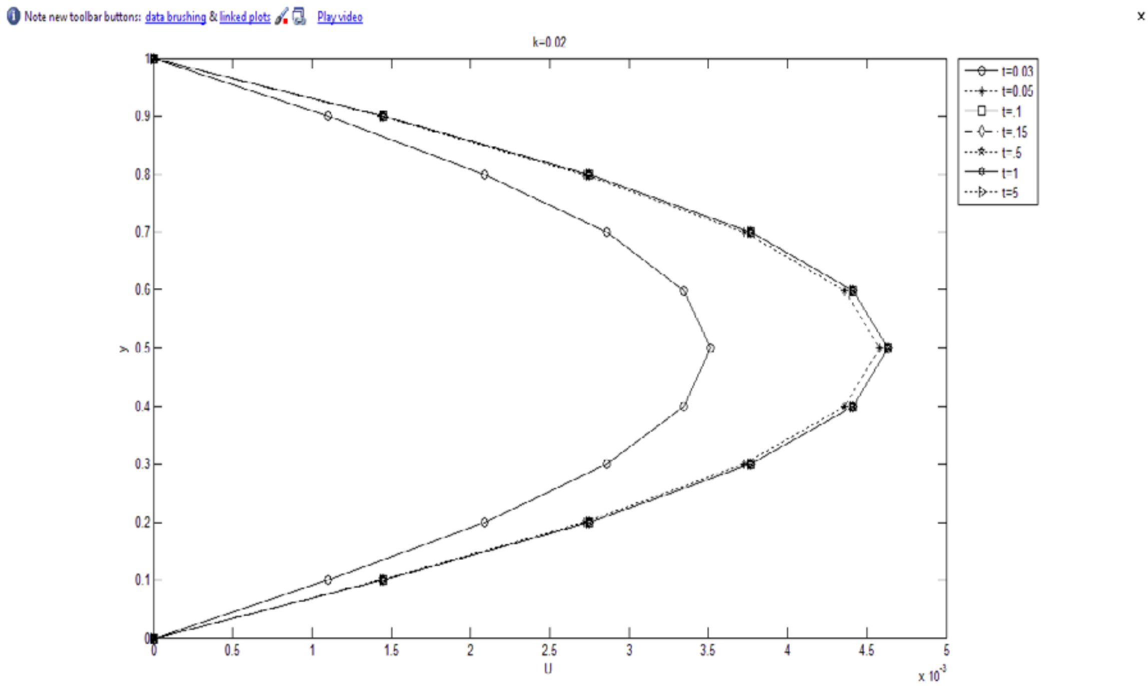


Fig. 1(b): Velocity profiles for k=0.02 at different time instants (t)

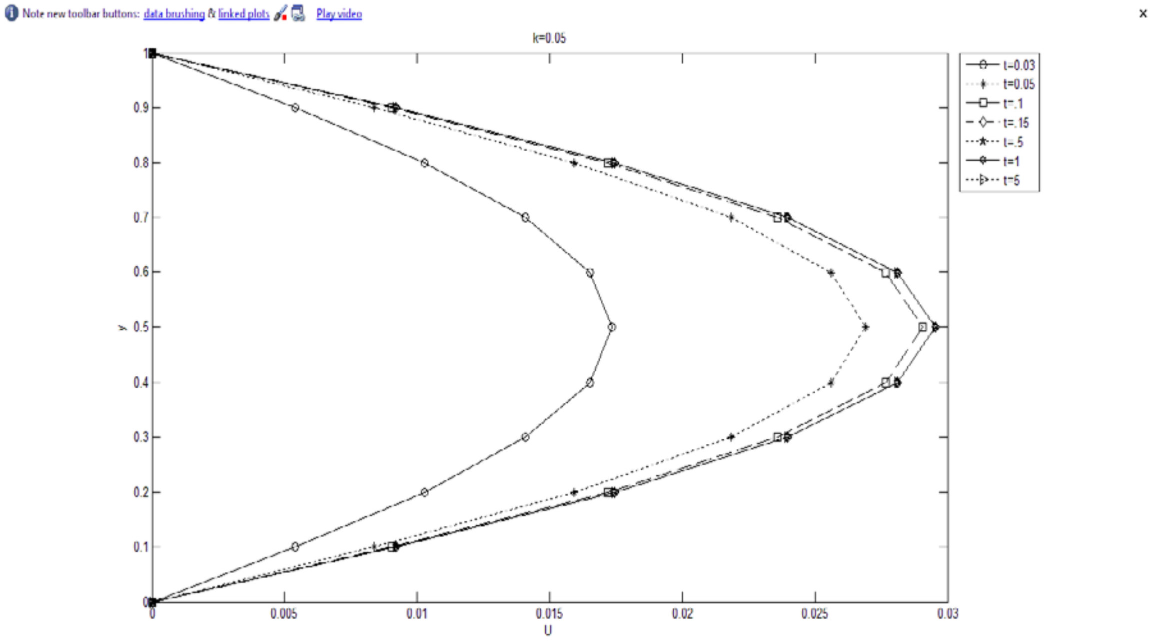


Fig. 1(c): Velocity profiles for k=0.05 at different time instants (t)

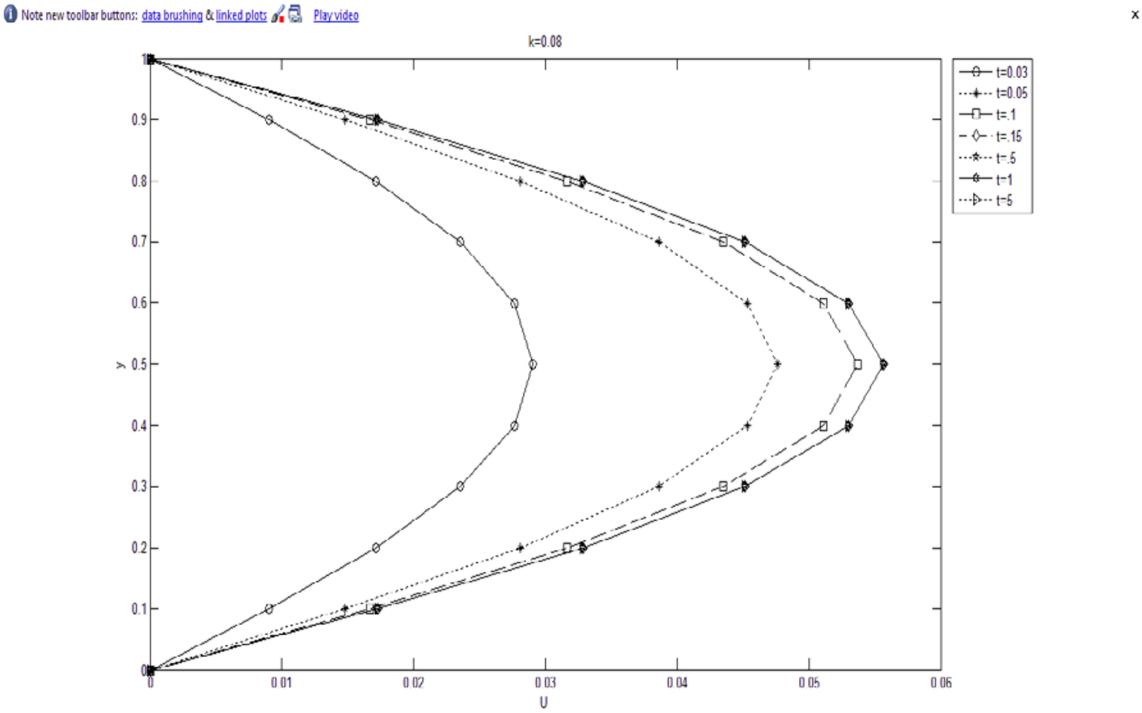


Fig. 1(d): Velocity profiles for k=0.08 at different time instants (t)

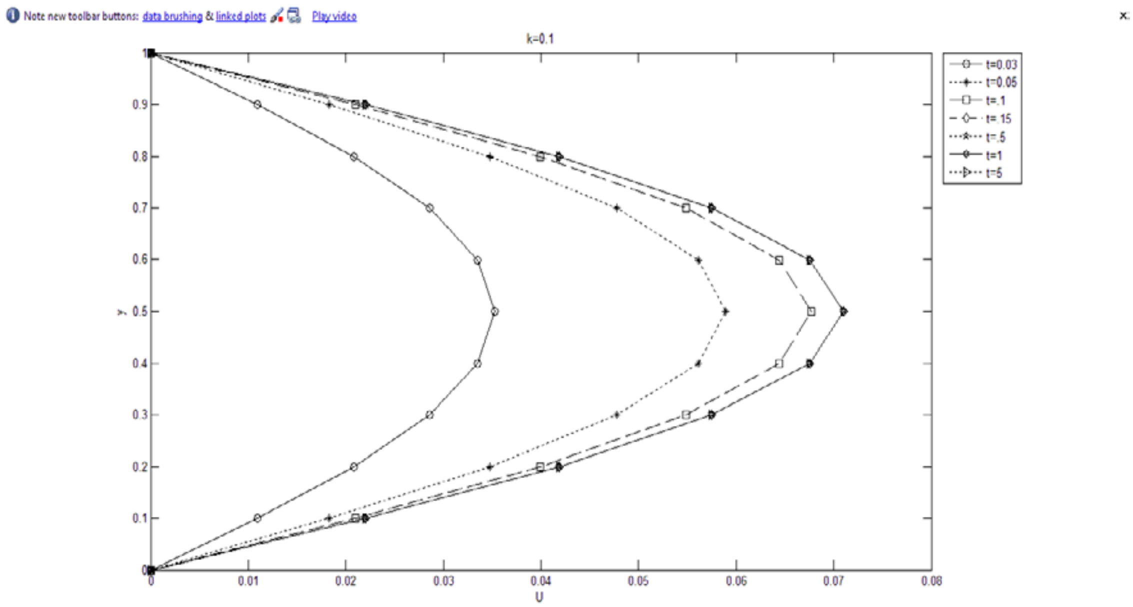


Fig. 1(e): Velocity profiles for k=0.1 at different time instants (t)

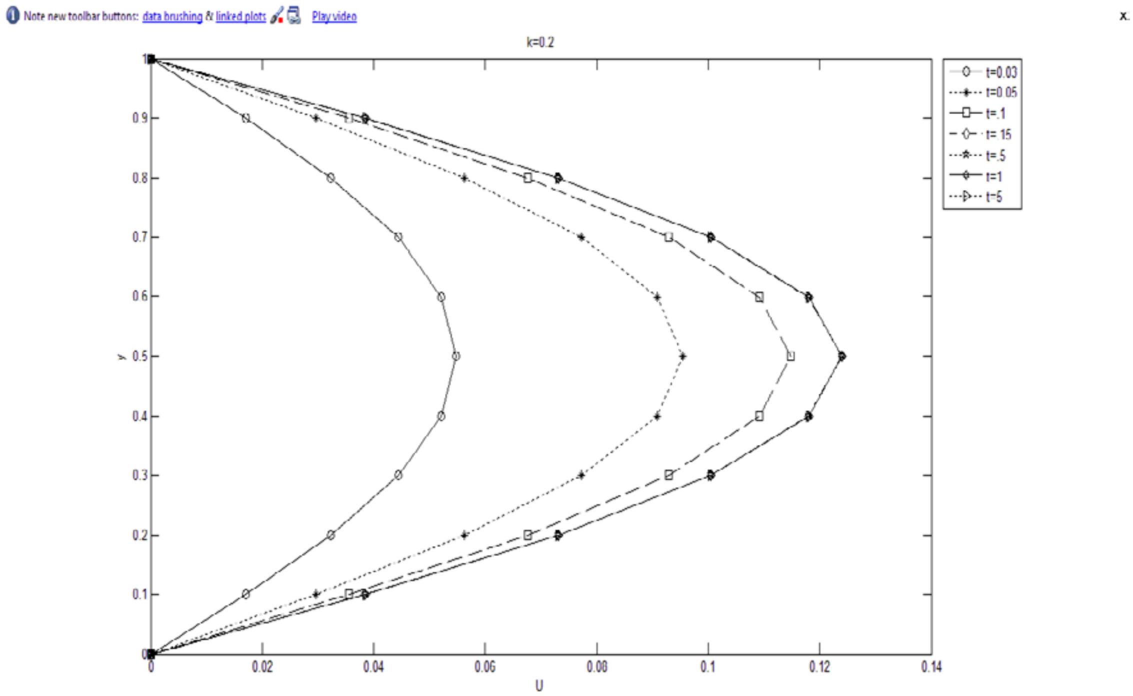


Fig. 1(f): Velocity profiles for k=0.2 at different time instants (t)

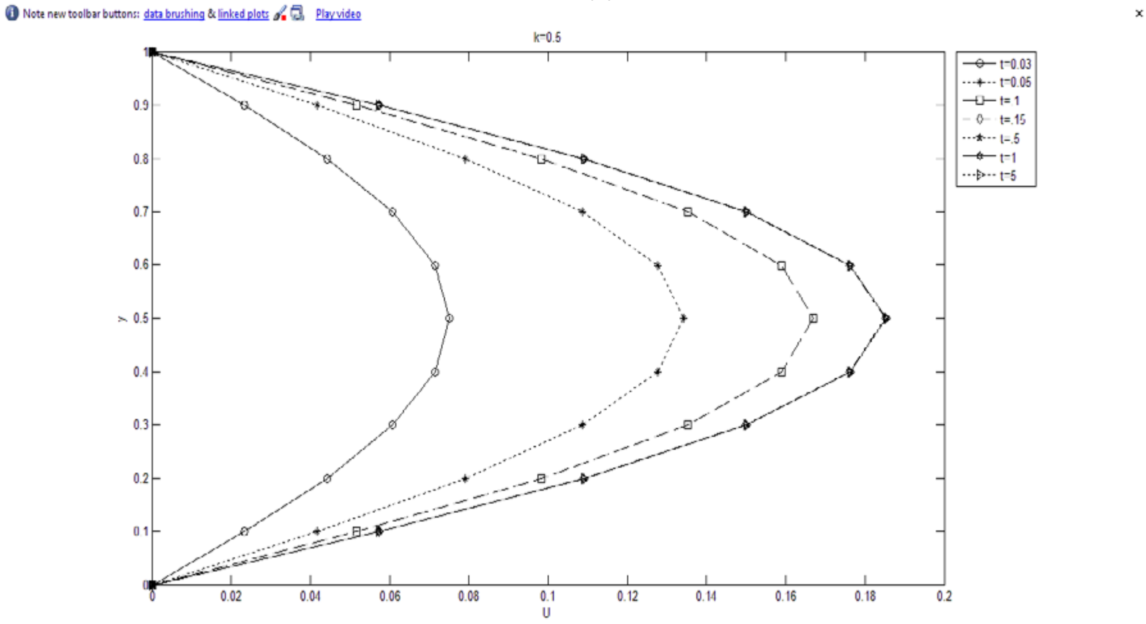


Fig. 1(g): Velocity profiles for k=0.5 at different time instants (t)

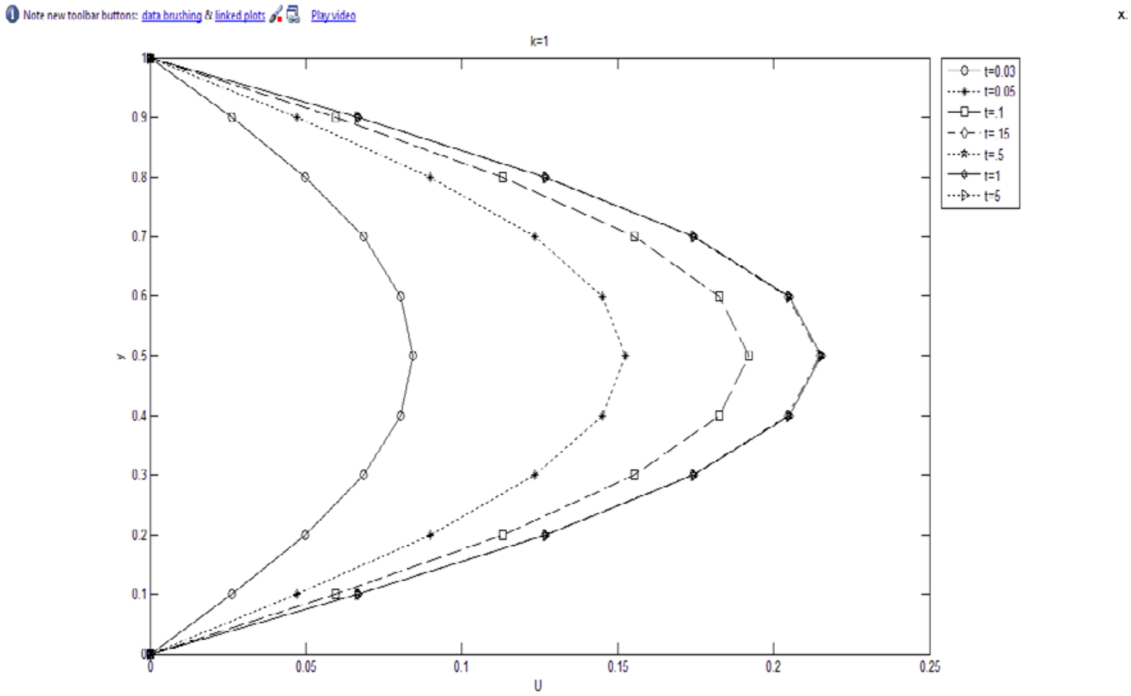


Fig. 1(h): Velocity profiles for k=1 at different time instants (t)

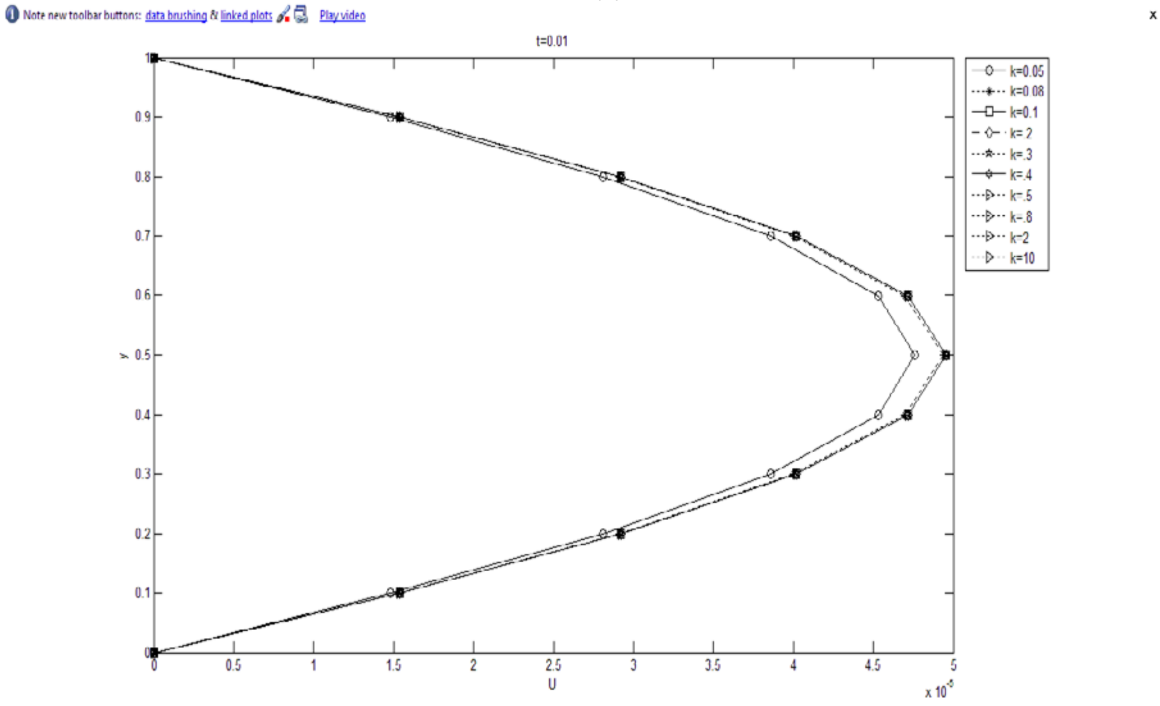


Fig. 2(a): Velocity profiles for t=0.01for different values of *k*

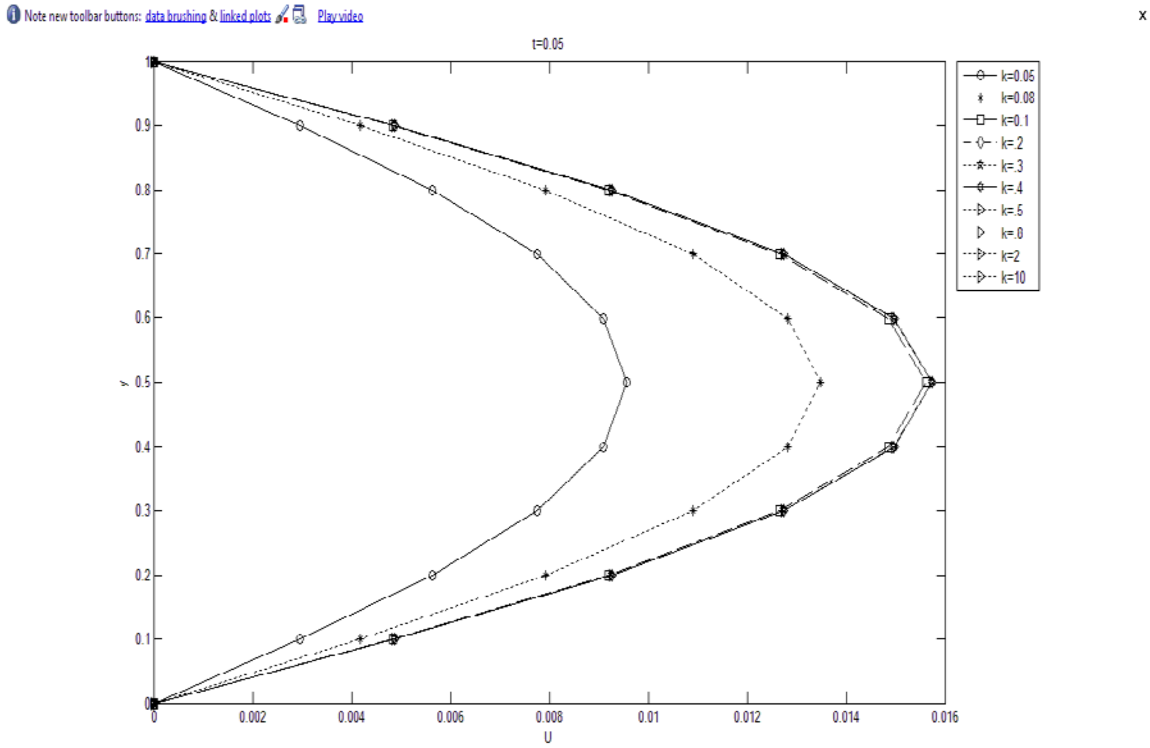


Fig. 2(b): Velocity profiles for t=0.05for different values of k

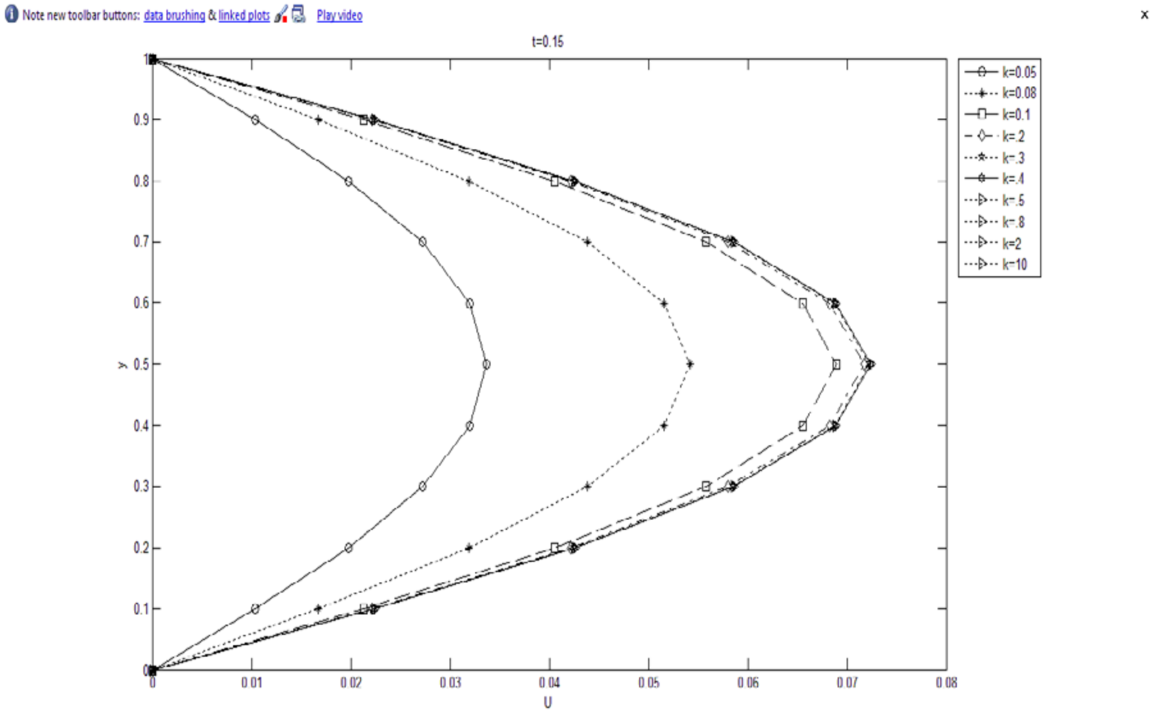


Fig. 2(c): Velocity profiles for t=0.15for different values of k

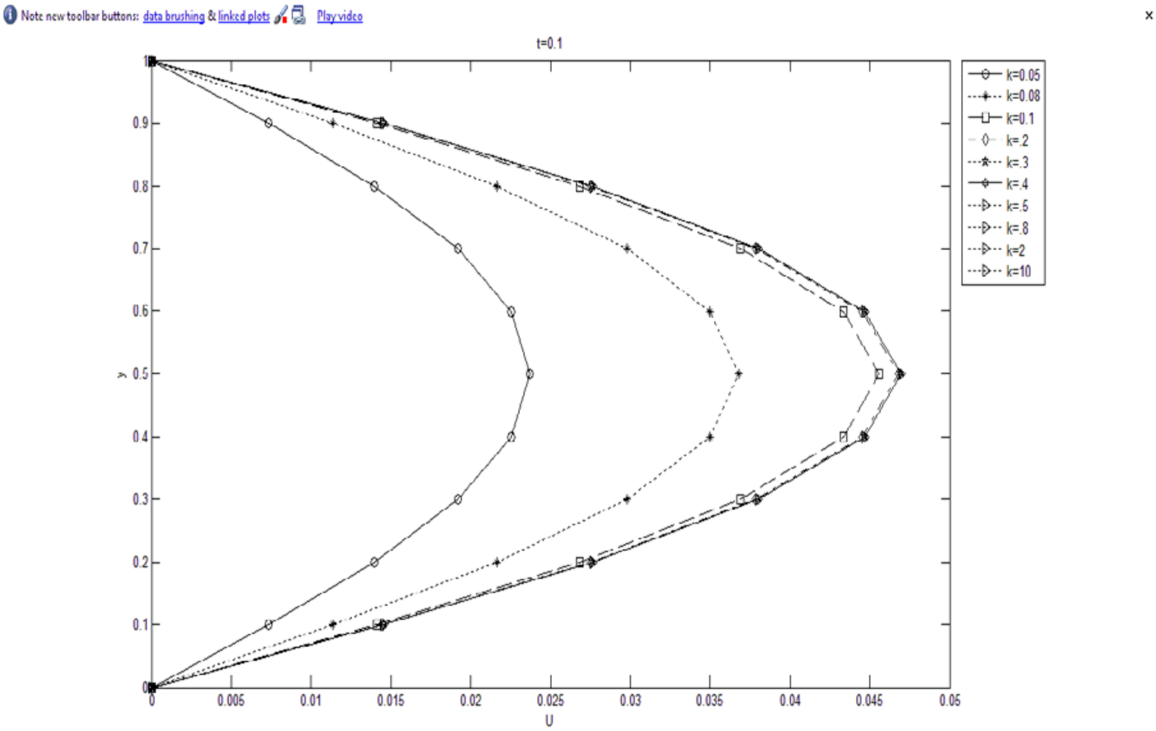


Fig. 2(d): Velocity profiles for t=0.1for different values of k

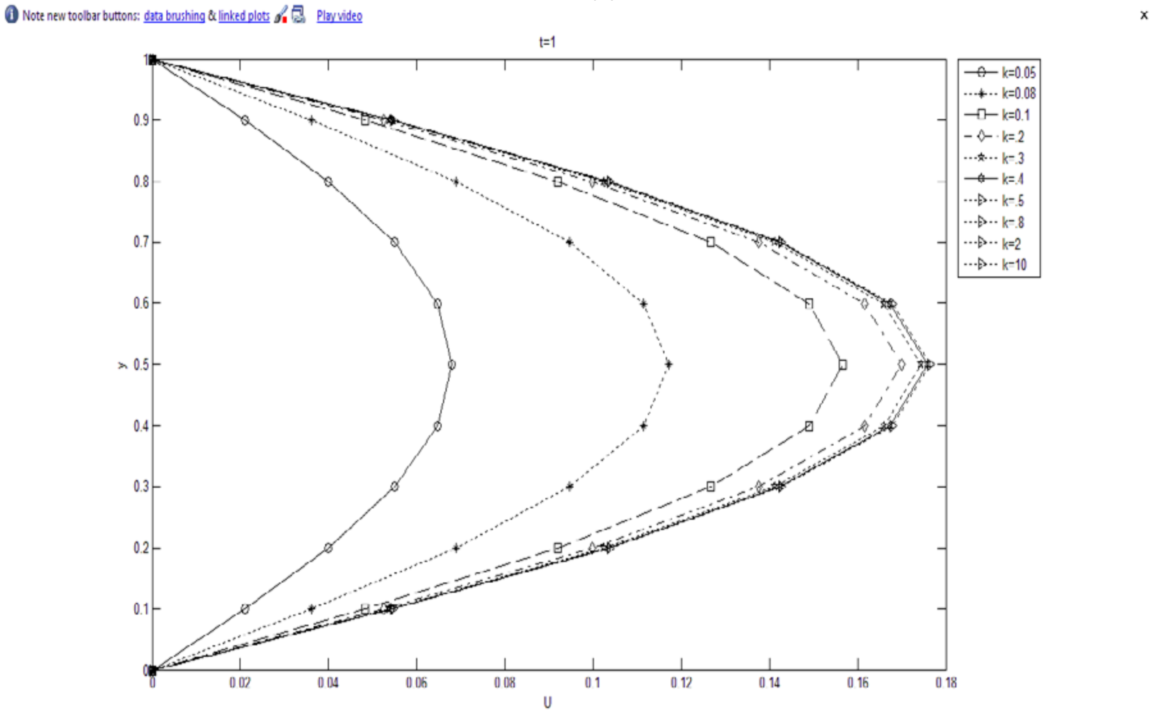


Fig. 2(e): Velocity profiles for t=1for different values of k

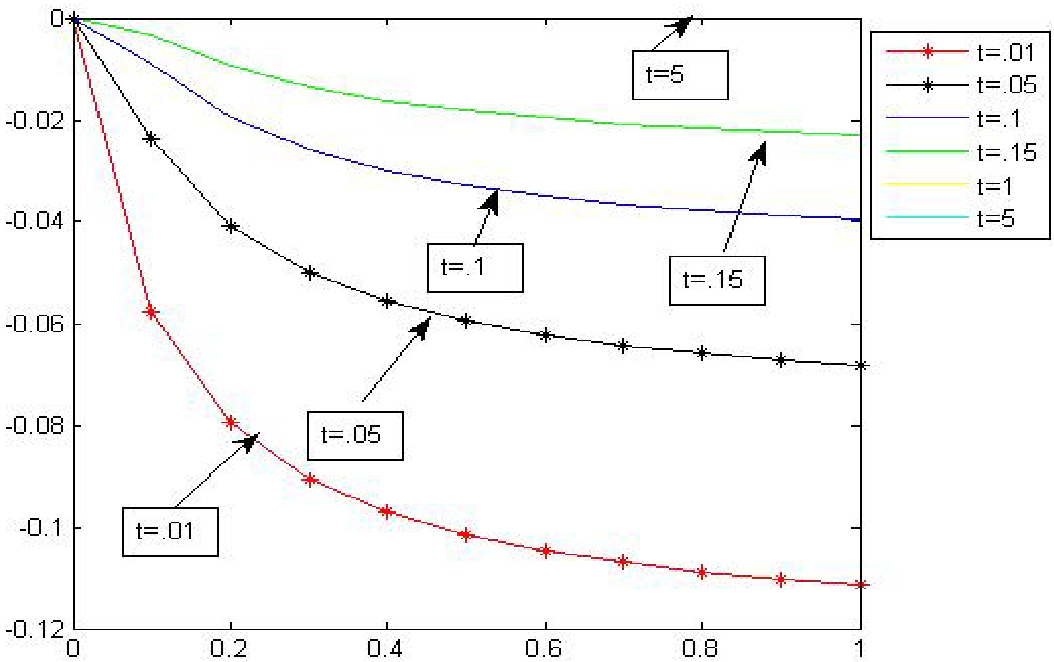


Fig. 3(a): Shear-Stress at at $\tau_{Y=1}$

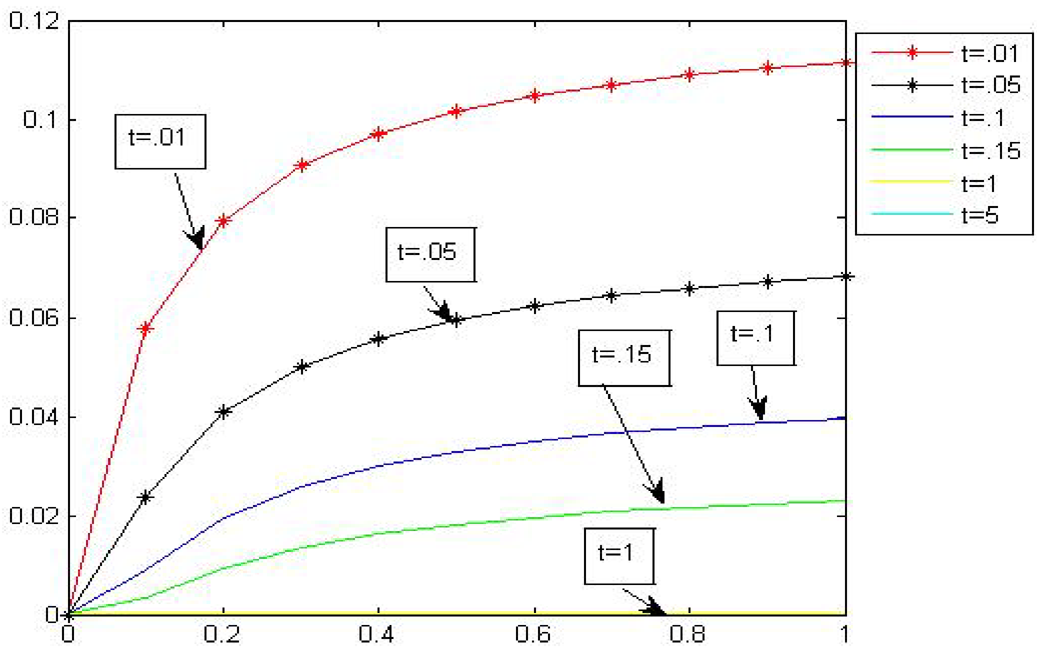


Fig. 3(b): Shear-Stress at at $\tau_{Y=0}$

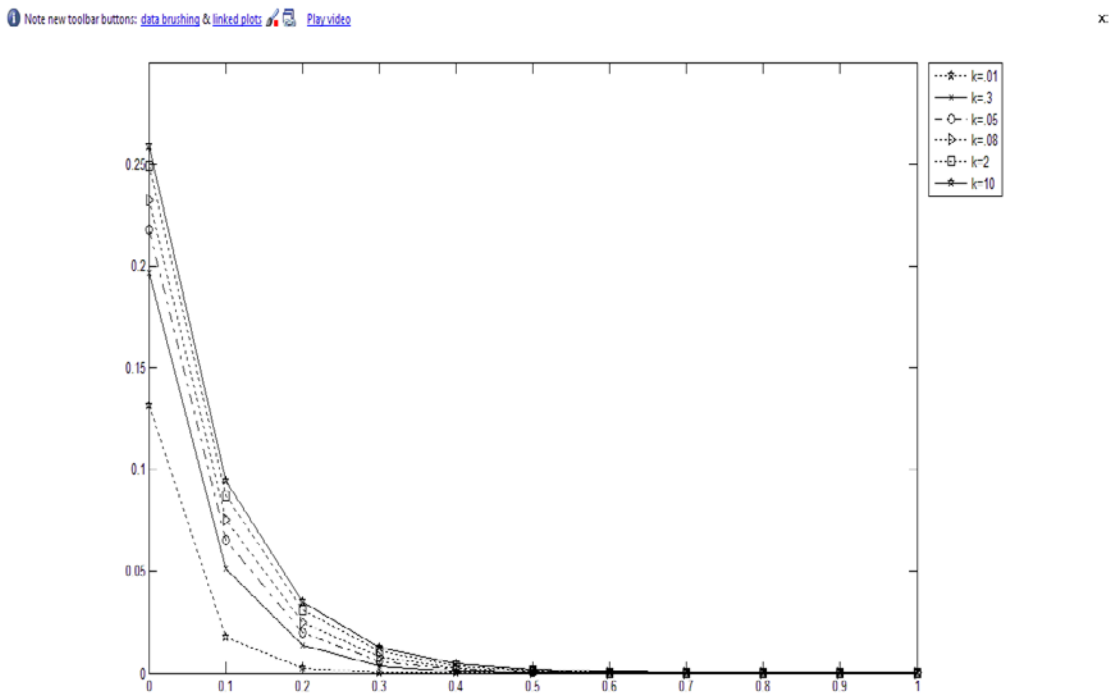


Fig.4: Flow-rate for different values of porosity coeff.(t)

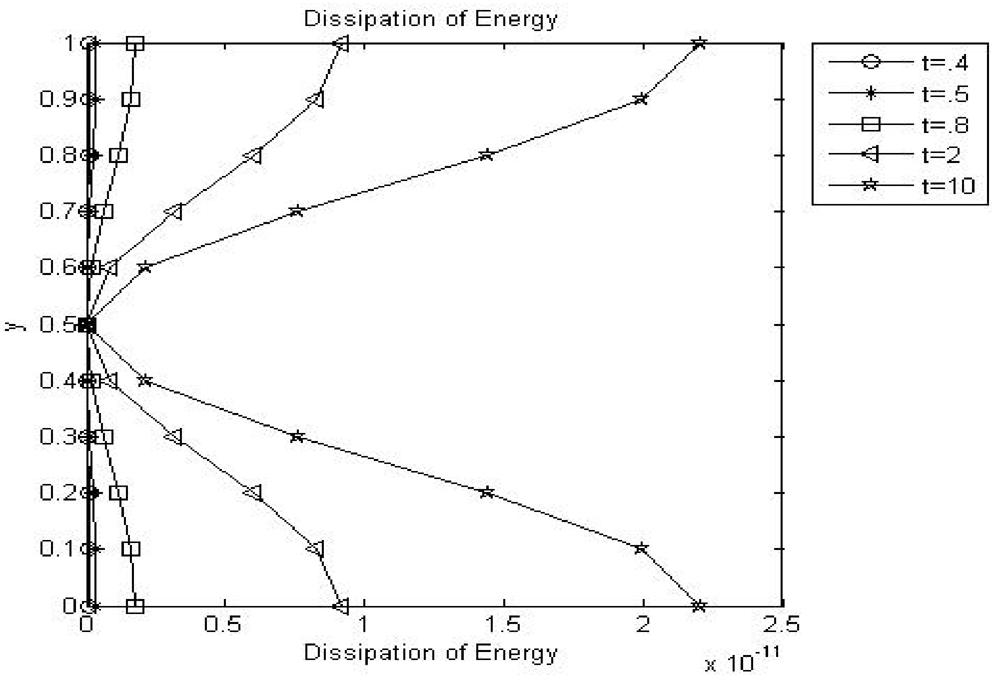


Fig. 5: Variation of dissipation of energy at different instant of time (t)

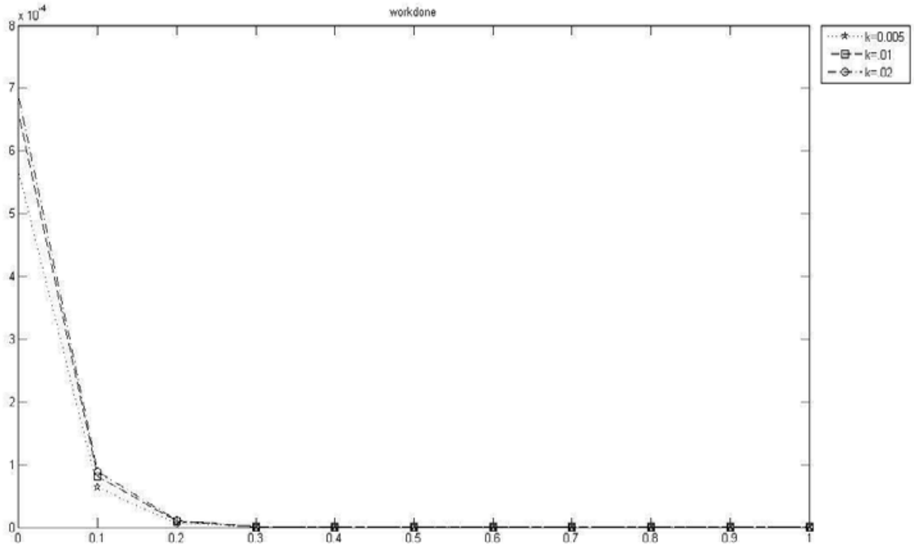


Fig. 6: Variation of Work done for different Values of coeff. Of porosity.

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