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COMMON FIXED POINT THEOREM IN CONE METRIC SPACES

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ABSTRACT

In this paper, we prove a unique common fixed point theorem in cone metric spaces without appealing to commutativity condition. These results generalize some recent results.

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Keywords: *Common fixed Point, Cone metric space, Coincidence points.*

1. INTRODUCTION AND PRELIMINARIES

In 2007 Huang and Zhang [3] have generalized the concept of a metric space, replacing the set of real numbers by an ordered Banach space and obtained some fixed point theorems for mapping satisfying different contractive conditions. Subsequently, Abbas and Jungck [1] and Abbas and Rhoades [2] have studied common fixed point theorems in cone metric spaces (see also [3], [5] and the references mentioned therein). Recently, Abbas and Jungck [1] have obtained coincidence point results for two mappings in cone metric spaces. In this paper, we prove a fixed point theorem in cone metric spaces without appealing to commutativity, which generalizes the Theorem of [1]

.

In all that follows B is a real Banach Space, and θ denotes the zero element of B. For the mapping f, g: X $\to$ X, let C (f, g) denote the set of coincidence points of f and g, that is C(f, g) = $\{z \in X : fz = gz\}$.

The following definitions are due to Huang and Zhang [3].

Definition 1.1: Let B be a real Banach Space and P a subset of B .The set P is called a cone if and only if:

(a). P is closed , non –empty and P $\neq \{\theta\}$;

(b). a, b $\in R$, a, b ≥ 0, x, y $\in P$ implies ax+by $\in P$;

(c). x $\in$ P and -x $\in P$ implies x = θ.

Definition 1.2: Let P be a cone in a Banach Space B, define partial ordering '$\leq$' with respect to P by x $\leq$ y if and only if y-x $\in P$. We shall write x<y to indicate x $\leq y$ but x $\neq y$ while x<<y will stand for y-x $\in$ Int P, where Int P denotes the interior of the set P. This Cone P is called an order cone.

Definition 1.3: Let B be a Banach Space and P $\subset$ B be an order cone .The order cone P is called normal if there exists K>0 such that for all x, y $\in$ B.

$\theta \leq x \leq y$ implies $\|x\| \leq K \|y\|$.

The least positive number K satisfying the above inequality is called the normal constant of P.

Definition 1.4: Let X be a nonempty set of B. Suppose that the map d: X $\times$X $\to$ B satisfies:

(d1). $\theta \leq d(x, y)$ for all x, y $\in X$ and d(x, y) = θ if and only if x = y;

(d2). d(x, y) = d(y, x) for all x, y $\in X$;

(d3). d(x, y) $\leq$ d(x, z) +d(y, z) for all x, y, z $\in X$.

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Then d is called a cone metric on X and (X, d) is called a cone metric space.

The concept of a cone metric space is more general than that of a metric space.

Definition 1.5: Let (X, d) be a cone metric space. We say that $\{x_n\}$ is

(i) a Cauchy sequence if for every c in B with $c >> \theta$, there is N such that for all n, m > N, $d(x_n, x_m) << c$;

(ii) a convergent sequence if for any $c >> \theta$, there is an N such that for all n>N, $d(x_n, x) << c$, for some fixed x in X.

We denote this $x_n \to x$ (as $n \to \infty$) .

Lemma 1.6: Let (X, d) be a cone metric space, and let P be a normal cone with normal constant K. Let $\{x_n\}$ be a sequence in X. Then
(i). $\{x_n\}$ converges to x if and only if $d(x_n, x) \to 0$ ($n \to \infty$).
(ii). $\{x_n\}$ is a Cauchy sequence if and only if $d(x_n, x_m) \to 0$ ($n, m \to \infty$).

2. MAIN RESULTS

In this section we obtain a common fixed point theorem for self-mappings without appealing to commutativity condition, defined on a cone metric space.

The following Theorem generalizes the Theorem of [1].

Theorem 2.1: Let (X, d) be a complete cone metric space and P a normal cone with normal constant K. Suppose that the mappings f, g: $X \to X$ are such that for some constant $\lambda \in (0,1)$ and for every x, $y \in X$ are two self-maps of X satisfying

$$d(fx, fy) \leq \lambda\, d(gx, gy) \tag{1}$$

If the range of g contains the range of f and g(X) is a complete subspace of X, then f and g have coincidence point. Then, f and g have a unique common fixed point in X.

Proof: Let x_0 be an arbitrary point in X, and let $x_1 \in X$ be chosen such that $y_0 = f(x_0) = g(x_1)$. Since $f(X) \subseteq g(X)$. Let $x_2 \in X$ be chosen such that $y_1 = f(x_1) = g(x_2)$. Continuing this process, having chosen $x_n \in X$, we chose $x_{n+1} \in X$ such that $y_n = f(x_n) = g(x_{n+1})$.

We first show that

$$d(y_n, y_{n-1}) \leq \lambda\, d(y_{n-1}, y_{n-2}) \quad \text{for } n = 2,3,\ldots. \tag{2}$$

Indeed,
$d(y_n, y_{n-1}) = d(fx_n, fx_{n-1}) \leq \lambda\, d(gx_n, gx_{n-1})$.

(2) Implies that

$$d(y_n, y_{n-1}) \leq \lambda d(y_{n-1}, y_{n-2}) \leq \ldots\ldots. \leq \lambda^{n-1} d(y_1, y_0). \tag{3}$$

Now we shall show that $\{y_n\}$ is a Cauchy sequence. By the triangle inequality,

for n > m we have

$d(y_n, y_m) \leq d(y_n, y_{n-1}) + d(y_{n-1}, y_{n-2}) + \ldots\ldots.+ d(y_{m+1}, y_m)$.

Hence, as p is a normal cone,

$d(y_n, y_m) \leq d(y_n, y_{n-1}) + d(y_{n-1}, y_{n-2}) + \ldots\ldots.+ d(y_{m+1}, y_m)$.

Now by (3),

$d(y_n, y_m) \leq (\lambda^{n-1} + \lambda^{n-2} + \ldots\ldots\ldots.+ \lambda^m) d(y_1, y_0)$.

From (1.3)

$\|d(y_n , y_m)\| \leq \frac{\lambda^m}{1-\lambda} K \|d(y_1, y_0)\| \to 0$ as $m\to\infty$. $\lambda \in (0,1)$

From ([3], Lemma 4) it follows that $\{y_n\}$ is a Cauchy sequence. Since g(X) is complete, there exists a q in g(X) such that $y_n \to q$ as $n\to\infty$. Consequently, we can find p in X such that g(p)= q. We shall show that f(p) = q. From (1)
$d(gx_n , fp) = d(fx_{n-1} , fp) \leq \lambda d(gx_{n-1}, gp)$,

$\Rightarrow$ $d(gp, fp) \leq \lambda d(gp, gp) = 0$.

That is, d(gp, fp) = 0.

Hence, gp = q= fp, p is a coincidence point of f and g. (4)

Now using (1),

$d(p, gp) \leq d(p, y_n) + d(y_n, gp)$ (by the triangle inequality)

$= d(p, y_n)+ d(fx_n, fp)$ (Since, fp = gp)

$\leq d(p, y_n) + \lambda d(gx_n, gp)$

From (1.3),

$\|d(p, gp)\| \leq K(\|(d (p, fx_n) + \lambda d(gx_n, gp))\|)$

$\leq K (\|d(p, fx_n)\| + \lambda \|d(gx_n, gp)\|)$ as $n \to \infty$

$\leq K(\|(d(p, q)\| + \lambda\|d(q, gp)\|)$

$\leq K (\|(d(p, gp)\| + \lambda\|d(gp, gp)\|)$

$\leq K\|d(p, gp)\|$.

$\Rightarrow \|d(p, gp)\|= 0$

Hence, p = gp.

Now,
d(fp,p) = d(fp, gp)

= d(fp, fp) (since fp=gp)

$\leq \lambda d(gp, gp_1) \leq \lambda d(p, p_1) = 0$ (by (1))

$\Rightarrow$ d(fp, p) = 0

That is, fp = p.

Since, fp = gp.

Therefore, fp = gp = p, f and g have a common fixed point.

Uniqueness, let p_1 be another common fixed point of f and g, then

$d(p, p_1) = d(fp, g p_1)$

$= d(fp, f p_1)$

$\leq \lambda d(gp, gp_1)$ (by (1))

$\leq \lambda d (p, p_1) < d (p,p_1)$ (since $\lambda<1$), a contradiction.

Therefore, $d(p, p_1) = 0$,

$\Rightarrow$ $p = p_1$.

Therefore, f and g have a unique common fixed point.

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