International Journal of Mathematical Archive-4(6), 2013, 8-15
IJMA Available online through www.ijma.info ISSN 2229 – 5046

COMMON FIXED POINT THEOREM IN METRIC SPACE USING COMPATIBILITY TYPE (R)

[1]M. Koireng Meitei, [2]M. Popeshwar Singh* and [2]Sahidur Rahman

[1]*Department of Mathematics, Modern College, Imphal, Manipur, India*

[2]*C. M. J. University, Shillong, Meghalaya, India*

(Received on: 22-04-12; Revised & Accepted on: 16-05-13)

ABSTRACT

In this paper, we study the concept of compatible mappings of type (R) and discuss some common fixed point theorem for four and six compatible mappings of type (R) satisfying contractive conditions.

Key words: *Compatible mapping of type (R), contractive modulus function, fixed point, common fixed point theorem.*

2000 Mathematics Subject Classification: *47H10, 54H25.*

1. INTRODUCTION

Gerald Jungck [2] introduced the concept of compatible mappings by a generalization commuting mappings. Also he shows that weak commuting mappings are compatible mappings but converse need not hold. Pathak, Chang and Cho [5] introduced the concept of a new type of compatible mappings called compatible type of (P). Further, Rohen, Singh and Shambhu [6] introduced the concept of a new type of compatible mappings called compatible type of (R) by combining the concept of compatible and compatible mappings of type (P).

In this paper, we establish the existence of unique common fixed point of four and six self mappings through compatibility of type (R). Our results generalize, extend and modify some earlier result [7, 8].

2. PRELIMINARIES

Definition 2.1 Self mappings A and B of a metric space (X, d) are said to be weakly commuting pair if, for all $x \in X$ $d(ABx, BAx) \leq d(Ax, Bx)$.

Definition 2.2: Self mappings A and B of a metric space (X, d) are said to be compatible if, for all $x \in X$ $\lim_{n\to\infty} d(ABx_n, BAx_n) = 0$ whenever $\{x_n\}$ is a sequence in X and such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = t$ for some $t \in X$.

Definition 2.3: Self mappings A and B of a metric space (X, d) are said to be compatible of type (A) if, for all $x \in X$ $\lim_{n\to\infty} d(ABx_n, BBx_n) = 0$ and $\lim_{n\to\infty} d(BAx_n, AAx_n) = 0$

whenever $\{x_n\}$ is a sequence in X and such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = t$ for some $t \in X$.

Definition 2.4: Self mappings A and B of a metric space (X, d) are said to be compatible of type (P) if, for all $x \in X$ $\lim_{n\to\infty} d(AAx_n, BBx_n) = 0$ whenever $\{x_n\}$ is a sequence in X and such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = t$ for some $t \in X$.

Definition 2.5: Let S and T be mappings from a metric space X into itself. The mapping S and T are said to be compatible of type (R) if

$$\lim_{n\to\infty} d(STx_n, TSx_n) = 0 \text{ and } \lim_{n\to\infty} d(SSx_n, TTx_n) = 0$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$ for some t in X.

Corresponding author: [2]*M. Popeshwar Singh**
[2]*C. M. J. University, Shillong, Meghalaya, India*

Definition 2.6: Self mappings A and B of a metric space (X, d) are said to be weakly compatible if they commute at their coincidence points.

Definition 2.7: Two self mappings A and B of a set X are occasionally weakly compatible if and only if there is a point t in X which is a coincidence point of A and B at which A and B commute.

Definition 2.8: A function $\varphi: [0, \infty) \to [0, \infty)$ is said to be a contractive modulus if $\varphi(0) = 0$ and $\varphi(t) < t$ for $t > 0$.

Definition 2.9: A real valued function φ defined on $X \subset \boldsymbol{R}$ is said to be upper semi continuous if $\lim\limits_{n\to\infty} \varphi(t_n) \le \varphi(t)$, for every sequence $\{t_n\}$ in X with $t_n \to t$ as $n\to\infty$.

Rohen, Singh and Shambhu [6] prove the following propositions.

Proposition 2.9: Let S and T be mappings from a complete metric space (X, d) into itself. If a pair $\{S, T\}$ is compatible of type (R) on X and $Sz = Tz$ for $z \in X$, then

$$STz = TSz = SSz = TTz.$$

Proposition 2.10: Let S and T be mappings from a complete metric space (X, d) into itself. If a pair $\{S, T\}$ is compatible of type (R) on X and $\lim\limits_{n\to\infty} Sx_n = \lim\limits_{n\to\infty} Tx_n = z$ for some $z \in X$, then we have

(i) $d(TSx_n, Sz) \to 0$ as $n \to \infty$ if S is continuous,
(ii) $d(STx_n, Tz) \to 0$ as $n \to \infty$ if T is continuous and
(iii) $STz=TSz$ and $Sz=Tz$ if S and T are continuous at z.

3. MAIN RESULTS

Theorem 3.1: Suppose F, G, A *and* B are four self mappings of a complete metric space (X, d) into itself satisfying the conditions

(i) $F(X) \subseteq A(X)$, $G(X) \subseteq B(X)$.

(ii) $d^2(Fx, Gy) \le \max\{\varphi(d^2(Bx, Ay)), \varphi(d(Bx, Fx))\varphi(d(Ay, Gy)),$
$\varphi(d(Bx, Gy))\varphi(d(Ay, Fx)), \varphi(d(Bx, Fx))\varphi(d(Ay, Fx))\}$ for all $x, y \in X$.

(iii) φ is contractive modulus as in definition (2.8).

(*iv*) one of F, G, A and B is continuous. *a*nd if

(v) the pairs (F, B) and (G, A) are compatible of type (R).

Then F, G, A and B have a unique common fixed point.

Proof: Let x_0 be an arbitrary point in X. Choose a point x_1 in X such that $Fx_0 = Ax_1$. Let x_2 be another point in X such that $Gx_1 = Ax_2$. In general we can choose $x_{2n}, x_{2n+1}, x_{2n+2},\ldots$ such that $Fx_{2n} = Ax_{2n+1}$ and $Gx_{2n+1} = Bx_{2n+2}$, so that we obtain a sequence

$$Fx_0, Gx_1, Fx_2, Gx_3, \ldots \tag{1}$$

By (ii), we have

$$d^2(Fx_{2n}, G\, x_{2n+1}) \le \max\{\varphi(d^2(Bx_{2n}, Fx_{2n})), \varphi(d(Bx_{2n}, Fx_{2n}))\varphi(d(Ax_{2n+1}, Gx_{2n+1})),$$
$$\varphi(d(Bx_{2n}, Gx_{2n+1}))\varphi(d(Ax_{2n+1}, Fx_{2n})), \varphi(d(Bx_{2n}, Fx_{2n}))\varphi(d(Ax_{2n+1}, Fx_{2n}))\},$$

$$\Rightarrow \quad d(Fx_{2n}, Gx_{2n+1}) \le \varphi(d(Fx_{2n}, Gx_{2n-1})) \le d(Fx_{2n}, Gx_{2n-1}). \tag{2}$$

By similar procedure we can prove that

$$d(Fx_{2n}, Gx_{2n-1}) < d(Fx_{2n-2}, Gx_{2n-1}).$$

Therefore the sequence $\{d(Fx_{2n}, Gx_{2n+1})\}$ is non decreasing and hence convergent, say converges to some real number c. Since φ is contractive modulus and letting $n\to\infty$, we get $c \le \varphi(c) \le c$ and so $c = 0$. To show that the sequence (1) is Cauchy, it is sufficient to show that $\{Fx_{2n}\}$ is Cauchy sequence. Suppose it is not so, hence there exist $\varepsilon > 0$ and a sequence of integers $\{m(k)\}$ and $\{n(k)\}$ with $m(k) \ge n(k) \ge k$ such that $d(Fx_{2m(k)}, Fx_{2n(k)}) > \varepsilon$. Let m be the smallest integer greater than $n(k)$ such that $d(Fx_{2m}, Fx_{2n}) > \varepsilon$ and $d(Fx_{2m-2}, Fx_{2n}) \le \varepsilon$. Therefore

$$\begin{aligned}\varepsilon &< d(Fx_{2m}, Fx_{2n})\\ &\le d(Fx_{2m}, Fx_{2m-1}) + d(Fx_{2m-1}, Fx_{2m-2}) + d(Fx_{2m-2}, Fx_{2n})\\ &\le d(Fx_{2m}, Fx_{2m-1}) + d(Fx_{2m}, Fx_{2m-2}) + \varepsilon\end{aligned}$$

Letting $k\to\infty$, we get

$$\lim_{k} d(Fx_{2m}, Fx_{2n}) = \varepsilon. \tag{3}$$

Also we have

$$d(Fx_{2m}, Fx_{2n}) \le d(Fx_{2m}, Tx_{2n+1}) + d(Fx_{2n}, Tx_{2n+1}) \tag{4}$$

Using condition (ii), we have

$$\begin{aligned}d^2(Fx_{2n}, G\,x_{2n+1}) &\le \max\{\varphi(d^2(Bx_{2n}, Fx_{2n})), \varphi(d(Bx_{2n}, Fx_{2n}))\varphi(d(Ax_{2n+1}, Gx_{2n+1})),\\ &\qquad \varphi(d(Bx_{2n}, Gx_{2n+1}))\varphi(d(Ax_{2n+1}, Fx_{2n})), \varphi(d(Bx_{2n}, Fx_{2n}))\varphi(d(Ax_{2n+1}, Fx_{2n}))\}\\ &= \max\{\varphi(d^2(Gx_{2n-1}, Fx_{2n})), \varphi(d(Gx_{2n-1}, Fx_{2n}))\varphi(d(Fx_{2n}, Gx_{2n+1})),\\ &\qquad \varphi(d(Gx_{2n-1}, Gx_{2n+1}))\varphi(d(Fx_{2n}, Fx_{2n})), \varphi(d(Gx_{2n-1}, Fx_{2n}))\varphi(d(Fx_{2n}, Fx_{2n}))\}\\ &= \max\{\varphi(d^2(Gx_{2n-1}, Fx_{2n})), \varphi(d(Gx_{2n-1}, Fx_{2n}))\varphi(d(Fx_{2n}, Gx_{2n+1}))\}\end{aligned} \tag{5}$$

Also we have

$$d(Gx_{2n-1}, Gx_{2n+1}) \le d(Gx_{2n-1}, Fx_{2n-2}) + d(Fx_{2n-2}, Fx_{2n}) + d(Fx_{2n}, Gx_{2n+1}) \tag{6}$$

Using (2), (5) and letting $k\to\infty$, inequality (4) gives

$$\lim_{k} d(Fx_{2n}, Gx_{2n+1}) < \varepsilon.$$

Consequently (2) gives that

$$\varepsilon = \lim_{k} d(Fx_{2m}, Fx_{2n}) < \varepsilon$$

which is a contradiction. Hence $\{Fx_{2n}\}$ is a Cauchy sequence and consequently the sequence (1) is a Cauchy. Since X is complete, the sequence (1) converges to a limit z in X. Hence the sub-sequences $\{Fx_{2n}\} = \{Ax_{2n+1}\}$ and $\{Gx_{2n-1}\} = \{Bx_{2n}\}$ also converge to the limit point z.

Suppose that the mapping B is continuous, then

$$B^2x_{2n} \to Bz \text{ and } BFx_{2n} \to Bz \text{ as } n\to\infty.$$

Since the pair (F, B) is compatible of type (R) by proposition 2.10, we get

$$FBx_{2n} \to Bz \text{ as } n\to\infty.$$

Now by (ii)

$$\begin{aligned}d^2(FBx_{2n}, Gx_{2n+1}) &\le \max\{\varphi(d^2(BBx_{2n}, Ax_{2n+1})), \varphi(d(BBx_{2n}, FBx_{2n}))\varphi(d(Ax_{2n+1}, Gx_{2n+1})),\\ &\qquad \varphi(d(BBx_{2n}, Gx_{2n+1}))\varphi(d(Ax_{2n+1}, FBx_{2n})), \varphi(d(BBx_{2n}, FBx_{2n}))\varphi(d(Ax_{2n+1}, FBx_{2n}))\}\end{aligned}$$

Letting $n\to\infty$, we get,

$$\begin{aligned}d^2(Bz, z) &\le \max\{\varphi(d^2Bz, z)), \varphi(d(Bz, Bz))\, \varphi(d(z, z)), \varphi(d(Bz, z))\, \varphi(d(z, Bz)),\\ &\qquad \varphi(d(Bz, Bz)), \varphi(d(z, Bz))\}\\ &= \varphi(d(Bz, z))\varphi(d(Bz, z))\end{aligned}$$

i.e. $d(Bz, z) \le \varphi(d(Bz, z)) \le d(Bz, z)$. Hence $\varphi(d(Bz, z)) = 0$ i.e. $Bz = z$.

Further

$$d^2(Fz, Gx_{2n+1}) \le \max\{\varphi(d^2(Bz, Ax_{2n+1})), \varphi(d(Bz, Fz))\varphi(d(Ax_{2n+1}, Gx_{2n+1})), \varphi(d(Bz, Gx_{2n+1}))\varphi(d(Ax_{2n+1}, Fz)), \varphi(d(Bz, Fz))\varphi(d(Ax_{2n+1}, Fz))\}$$

$Ax_{2n+1} \to z$, $Gx_{2n+1} \to z$ as $n\to\infty$ and $Bz = z$, so letting $n\to\infty$ we get

$$d^2(Fz, z) \le \max\{\varphi(d^2(z, z)), \varphi(d(z, Fz))\varphi(d(z, z)), \varphi(d(z, z))\varphi(d(z, Fz)), \varphi(d(z, Fz))\varphi(d(z, Fz))\}$$

i.e. $d(Fz, z) \le \varphi(d(Fz, z)) \le d(Fz, z)$. Hence $\varphi(d(Fz, z)) = 0$ i.e. $Fz = z$. Thus $Fz = Bz = z$.

Since $F(X) \subseteq A(X)$, there is a point $u \in X$ such that $z = Fz = Au$. Now we prove that $Au = Gu$.

Now by (ii)

$$d^2(Fz, Gu) \le \max\{\varphi(d^2(Bz, Au)), \varphi(d(Bz, Fz))\varphi(d(Au, Gu)), \varphi(d(Bz, Gu))\varphi(d(Au, Fz)), \varphi(d(Bz, Fz))\varphi(d(Au, Fz))\}$$

$$= \max\{\varphi(d^2(z, z)), \varphi(d(z, z))\varphi(d(z, Gu)), \varphi(d(z, Gu))\varphi(d(z, z)), \varphi(d(z, z))\varphi(d(z, z))\}$$

So $d^2(Fz, Gu) \le 0$ implies $d(z, Gu) = 0$ and $Gu = z$, hence $z = Au = Gu$. Take $yn = u$ for $n \ge 1$, then $Gyn \to Gu = z$ and $Ayn \to Au = z$ as $n\to\infty$. Since the pair (G, A) is compatible of type (R), by proposition 2.9, we have $AAu = GGu = AGu = GAu$ hence $Gu = Au$.

Now

$$d^2(z, Gz) = d^2(Fz, Gz)$$

$$\le \max\{\varphi(d^2(Bz, Az)), \varphi(d(Bz, Fz))\varphi(d(Az, Gz)), \varphi(d(Bz, Gz))\varphi(d(Az, Fz)), \varphi(d(Bz, Fz))\varphi(d(Az, Fz))\}$$

$$= \max\{\varphi(d^2(z, Gz)), \varphi(d(z, z))\varphi(d(Gz, Gz)), \varphi(d(z, Gz))\varphi(d(Gz, z)), \varphi(d(z, z))\varphi(d(Gz, z))\}$$

$$= \varphi(d(z, Gz))\varphi(d(Gz, z))$$

$\Rightarrow$ $d(Gz, z) \le \varphi(d(Gz, z)) \le d(Gz, z)$.

Hence $\varphi(d(Gz, z)) = 0$ i.e. $Gz = z$ and $z = Gz = Az$. So z is a common fixed point of F, A, B and G, when continuity of B is assumed. The proof that z is a common fixed point of F, B, A and G is similar, when any one from F, A and G is continuous.

Now suppose that F is continuous, then F^2x_{2n}, $FBx_{2n} \to Fz$ as $n \to \infty$.

Since the pair (F, B) is compatible of type (R) we get $BFx_{2n} \to Bz$ and $BBx_{2n} \to Fz$ as $n\to\infty$.

Now by (ii)

$$d^2(F^2x_{2n}, Gx_{2n+1}) \le \max\{\varphi(d^2(BFx_{2n}, Ax_{2n+1})), \varphi(d(BFx_{2n}, F^2x_{2n}))\varphi(d(Ax_{2n+1}, Gx_{2n+1})), \varphi(d(BFx_{2n}, Gx_{2n+1}))\varphi(d(Ax_{2n+1}, F^2x_{2n})), \varphi(d(BFx_{2n}, F^2x_{2n}))\varphi(d(Ax_{2n+1}, F^2x_{2n}))\}$$

letting $n\to\infty$, using the compatible of type (R) of the pair (F, B) we get

$$d^2(Fz, z) \le \max\{\varphi(d^2(Fz, z)), \varphi(d(Fz, Fz))\varphi(d(z, z)), \varphi(d(Fz, z))\varphi(d(z, Fz)), \varphi(d(Fz, Fz))\varphi(d(z, Fz))\}$$

$$= \varphi(d(Fz, z))\varphi(d(Fz, z))$$

i.e. $d(Fz, z) \le \varphi(d(Fz, z)) \le d(Fz, z)$.

Hence $\varphi(d(Fz, z)) = 0$ i.e. $Fz = z$. $G(X) \subseteq A(X)$, there is a point $u \in X$ such that $z = Gz = Au$.

Now by (ii)

$$d^2(F^2x_{2n}, Gu) \le \max\{\varphi(d^2(BFx_{2n}, Au)), \varphi(d(BFx_{2n}, F^2x_{2n}))\varphi(d(Au, Gu)), \varphi(d(BFx_{2n}, Gu))\varphi(d(Au, F^2x_{2n})), \varphi(d(BFx_{2n}, F^2x_{2n}))\varphi(d(Au, F^2x_{2n}))\}$$

letting $n\to\infty$, we get

$$d^2(z, Gu) = d^2(Fz, Gu)$$

$$\le \max\{\varphi(d^2(z, z)),\ \varphi(d(z, z))\varphi(d(z, Gu)),\ \varphi(d(z, Gu))\varphi(d(Gu, z)),\ \varphi(d(z, z))\varphi(d(Gu, z)\}$$

i.e. $d(z, Gu) \le \varphi(d(z, Gu)) \le d(z, Gu)$.

Hence $\varphi(d(z, Gu)) = 0$ i.e. $Gu = z$.

Let $yn = u$, then $Gyn \to Gu = z$ and $Ayn \to Au = z$. Since (G, A) is compatible mappings of type (R), $\lim_{n\to\infty} d(GAyn, AGyn) = 0$ and $\lim_{n\to\infty} d(GGyn, AAyn) = 0$. This gives $Gz = Az$.

Further

$$d^2(Fx_{2n}, Gz) \le \max\{\varphi(d^2(Bx_{2n}, Az)),\ \varphi(d(Bx_{2n}, Fx_{2n}))\varphi(d(Az, Gz)),\ \varphi(d(Bx_{2n}, Gz))\varphi(d(Az, Fx_{2n})),\ \varphi(d(Bx_{2n}, Fx_{2n}))\varphi(d(Az, Fx_{2n}))\}$$

letting $n\to\infty$, we get

$$d^2(z, Gz) \le \max\{\varphi(d^2(z, Gz)),\ \varphi(d(z, z))\varphi(d(Gz, Gz)),\ \varphi(d(z, Gz))\varphi(d(Gz, z)),\ \varphi(d(z, z)\varphi(d(Gz, z))\}$$

$$= \varphi(d(z, Gz))\varphi(d(z, Gz))$$

i.e. $d(Gz, z) \le \varphi(d(Gz, z)) \le d(Gz, z)$. Hence $\varphi(d(Gz, z)) = d(Gz, z) = 0$ i.e. $Gz = z$. Hence $z = Gz$ and $z = Az = Gz$. Since $G(X) \subseteq B(X)$, There is a point $v \in X$ such that $z = Gz = Bv$. Now

$$d^2(Fv, z) = d^2(Fv, Gz) \le \max\{\varphi(d^2(Bv, Az)),\ \varphi(d(Bv, Fv))\varphi(d(Az, Gz)),\ \varphi(d(Bv, Gz))\varphi(d(Az, Fv)),\ \varphi(d(Bv, Fv))\varphi(d(Az, Fv))\}$$

$$= \max\{\varphi(d^2(z, z)),\ \varphi(d(z, Fv))\varphi(d(z, z)),\ \varphi(d(z, z))\varphi(d(z, Fv)),\ \varphi(d(z, Fv))\varphi(d(z, Fv))\}$$

i.e. $d(Fv, z) \le \varphi(d(Fv, z)) \le d(Fv, z)$.

Hence $\varphi(d(Fv, z)) = 0$ i.e. $Fv = z$.

Thus we have $d(Fv, z) = 0$ and $Fv = z$. Take $yn = v$ then $Fyn \to Fv = z$, $Byn \to Bv = z$. Since (F, B) is compatible of type (R), by the proposition 2.9. This implies that $Fz = Bz$. Hence z is a common fixed point of F, B, A and G when F is a continuous. The proof is similar that z is common fixed point of F, B, A and G when G is continuous.

UNIQUENESS:
Let z and w be two common fixed point of F, B, A and G, i.e. $z = Fz = Bz = Gz = Az$ and $w = Fw = Bw = Gw = Aw$.

From condition (ii) we have

$$d^2(z, w) = d^2(Fz, Gw) \le \max\{\varphi(d^2(Bz, Aw)),\ \varphi(d(Bz, Fz))\varphi(d(Aw, Gw)),\ \varphi(d(Bz, Gw))\varphi(d(Aw, Fz)),\ \varphi(d(Bz, Fz))\varphi(d(Aw, Fz))\}$$

$$= \max\{\varphi(d^2(z, w)),\ \varphi(d(z, z))\varphi(d(w, w)),\ \varphi(d(z, w))\varphi(d(w, z)),\ \varphi(d(z, z))\varphi(d(w, z))\}$$

Therefore $d(z, w) \le \varphi(d(z, w)) \le d(z, w)$ i.e. $\varphi(d(z, w)) = d(z, w)$. Thus $d(z, w) = 0$ i.e. $w = z$.

Hence the common fixed point is unique.

Theorem 3.2: Suppose E, F, H, G, A and B are six self mappings of a complete metric space (X, d) into itself satisfying the conditions

(i) $EF(X) \subseteq A(X)$, $HG(X) \subseteq B(X)$.
(ii) $d^2(EFx, HGy) \le \max\{\varphi(d^2(Bx, Ay)),\ \varphi(d(Bx, EFx))\varphi(d(Ay, HGy)),\ \varphi(d(Bx, HGy))\varphi(d(Ay, EFx)),\ \varphi(d(Bx, EFx))\varphi(d(Ay, EFx))\}$ for all $x, y \in X$.
(iii) φ is contractive modulus as in definition (2.4).
(iv) one of F, G, A and B is continuous.*a* nd if
(v) the pairs (EF, B) and (HG, A) are compatible of type (R).

Then EF, HG, A and B have a unique common fixed point.

Proof: Let x_0 in X be arbitrary. Choose a point x_1 in X such that $EFx_0 = Ax_1$. This can be done since $EF(X) \subseteq A(X)$. Let x_2 be a point in X such that $HGx_1 = Ax_2$. This can be done since $HG(X) \subseteq A(X)$. In general we can choose x_{2n}, x_{2n+1}, x_{2n+2},… such that $EFx_{2n} = Ax_{2n+1}$ and $HGx_{2n+1} = Bx_{2n+2}$, so that we obtain a sequence

$$EFx_0, HGx_1, EFx_2, HGx_3, \ldots \tag{7}$$

Similarly, to prove of theorem 3.1, we have $\{EFx_{2n}\}$ is a Cauchy sequence. Since X is complete, the sequence (7) converges to a limit z in X. Hence the sub-sequences $\{EFx_{2n}\} = \{Ax_{2n+1}\}$ and $\{HGx_{2n-1}\} = \{Bx_{2n}\}$ also converge to the limit point z.

Suppose that the mapping B is continuous, then $B^2x_{2n} \to Bz$ and $B(EF)x_{2n} \to Bz$ as $n\to\infty$. Since the pair (EF, B) is compatible of type (R) by proposition 2.10, we get $EFBx_{2n} \to Bz$ as $n\to\infty$.

Now by (vii)

$$d^2(EFBx_{2n}, Gx_{2n+1}) \le \max\{\varphi(d^2(BBx_{2n}, Ax_{2n+1})), \varphi(d(BBx_{2n}, (EF)Bx_{2n}))\varphi(d(Ax_{2n+1}, HGx_{2n+1})), \varphi(d(BBx_{2n}, HGx_{2n+1}))\varphi(d(Ax_{2n+1}, (EF)Bx_{2n})), \varphi(d(BBx_{2n}, (EF)Bx_{2n}))\varphi(d(Ax_{2n+1}, (EF)Bx_{2n}))\},$$

Letting $n\to\infty$, we get,

$$d^2(Bz, z) \le \max\{\varphi(d^2Bz, z)), \varphi(d(Bz, Bz))\varphi(d(z, z)), \varphi(d(Bz, z))\varphi(d(z, Bz)), \varphi(d(Bz, Bz))\varphi(d(z, Bz))\}$$

$$= \varphi(d(Bz, z))\varphi(d(Bz, z))$$

i.e. $d(Bz, z) \le \varphi(d(Bz, z)) \le d(Bz, z)$.

Hence $\varphi(d(Bz, z)) = 0$ i.e. $Bz = z$.

Further

$$d^2(EFz, HGx_{2n+1}) \le \max\{\varphi(d^2(Bz, Ax_{2n+1})), \varphi(d(Bz, EFz))\varphi(d(Ax_{2n+1}, HGx_{2n+1})), \varphi(d(Bz, HGx_{2n+1}))\varphi(d(Ax_{2n+1}, EFz)), \varphi(d(Bz, EFz))\varphi(d(Ax_{2n+1}, EFz))\}$$

$Ax_{2n+1}\to z$, $HGx_{2n+1}\to z$ as $n\to\infty$ and $Bz = z$, so letting $n\to\infty$ we get

$$d^2(EFz, z) \le \max\{\varphi(d^2(z, z)), \varphi(d(z, EFz))\varphi(d(z, z)), \varphi(d(z, z))\varphi(d(z, EFz)), \varphi(d(z, EFz))\varphi(d(z, EFz))\}$$

i.e. $d(EFz, z) \le \varphi(d(EFz, z)) \le d(EFz, z)$. Hence $\varphi(d(EFz, z)) = 0$ i.e. $EFz = z$.

Thus $EFz = Bz = z$. Since $EF(X) \subseteq A(X)$, there is a point $u \in X$ such that $z = EFz = Au$. Now we prove that $Au = HGu$.

Now by (vii)

$$d^2(HFz, Gu) \le \max\{\varphi(d^2(Bz, Au)), \varphi(d(Bz, EFz))\varphi(d(Au, Gu)), \varphi(d(Bz, HGu))\varphi(d(Au, EFz)), \varphi(d(Bz, EFz))\varphi(d(Au, EFz))\}$$

$$= \max\{\varphi(d^2(z, z)), \varphi(d(z, z))\varphi(d(z, HGu)), \varphi(d(z, HGu))\varphi(d(z, z)), \varphi(d(z, z))\varphi(d(z, z))\}$$

so $d^2(EFz, HGu) \le 0$ implies $d(z, HGu) = 0$ and $HGu = z$, hence $z = Au = HGu$. Take $yn = u$ for $n \ge 1$, then $HGyn \to HGu = z$ and $Ayn \to Au = z$ as $n\to\infty$. Since the pair (HG, A) is compatible of type (R), by proposition 2.10, we have $AAu = (HG)(HG)u = A(HG)u = (HG)Au$, hence $HGu = Au$.

Now

$$d^2(z, HGz) = d^2(EFz, HGz)$$

$$\le \max\{\varphi(d^2(\mathrm{B}z, Az)), \varphi(d(Bz, EFz))\varphi(d(Az, HGz)), \varphi(d(Bz, HGz))\varphi(d(Az, EFz)), \varphi(d(Bz, EFz))\varphi(d(Az, EFz))\}$$

$$= \max\{\varphi(d^2(z, HGz)), \varphi(d(z, z))\varphi(d(HGz, HGz)), \varphi(d(z, HGz))\varphi(d(HGz, z)), \varphi(d(z, z))\varphi(d(HGz, z))\}$$

$$= \varphi(d(z, HGz))\varphi(d(HGz, z))$$

$\Rightarrow \quad d(HGz, z) \le \varphi(d(HGz, z)) \le d(HGz, z)$.

Hence $\varphi(d(HGz, z)) = 0$ i.e. $HGz = z$ and $z = HGz = Az$. So z is a common fixed point of EF, A, B and HG when continuity of B is assumed. The proof that z is a common fixed point of EF, B, A and HG is similar, when any one from EF or A or HG is continuous.

Now suppose that EF is continuous, then EF^2x_{2n}, $EFBx_{2n} \to EFz$ as $n\to\infty$. Since the pair (EF, B) is compatible of type (R) we get $BEFx_{2n} \to EFz$ and $BBx_{2n} \to EFz$ as $n\to\infty$.

Now by (vii)

$$d^2(EF^2x_{2n}, HGx_{2n+1}) \le \max\{\varphi(d^2(BEFx_{2n}, Ax_{2n+1})), \varphi(d(BEFx_{2n}, EF^2x_{2n}))\varphi(d(Ax_{2n+1}, HGx_{2n+1})),$$
$$\varphi(d(BEFx_{2n}, HGx_{2n+1}))\varphi(d(Ax_{2n+1}, EF^2x_{2n})), \ \varphi(d(BEFx_{2n}, EF^2x_{2n}))\varphi(d(Ax_{2n+1}, EF^2x_{2n}))\}$$

letting $n\to\infty$, using the compatible of type (R) of the pair (F, B) we get

$$d^2(EFz, z) \le \max\{\varphi(d^2(EFz, z)), \varphi(d(EFz, EFz))\varphi(d(z, z)), \varphi(d(EFz, z))\varphi(d(z, EFz)), \varphi(d(EFz, EFz))\varphi(d(z, EFz))\}$$

$$= \varphi(d(EFz, z))\varphi(d(EFz, z))$$

i.e. $d(EFz, z) \le \varphi(d(EFz, z)) \le d(EFz, z)$.

Hence $\varphi(d(EFz, z)) = 0$ i.e. $EFz = z$. $HG(X) \subseteq A(X)$, there is a point $u \in X$ such that

$$z = HGz = Au.$$

Now by (vii)

$$d^2(EF^2x_{2n}, HGu) \le \max\{\varphi(d^2(BEFx_{2n}, Au)), \varphi(d(BEFx_{2n}, EF^2x_{2n}))\varphi(d(Au, Gu)),$$
$$\varphi(d(BEFx_{2n}, Gu))\varphi(d(Au, EF^2x_{2n})), \varphi(d(BEFx_{2n}, EF^2x_{2n}))\varphi(d(Au, EF^2x_{2n}))\}$$

letting $n\to\infty$, we get

$$d^2(z, HGu) = d^2(EFz, HGu)$$

$$\le \max\{\varphi(d^2(z, z)), \varphi(d(z, z))\varphi(d(z, HGu)), \varphi(d(z, HGu))\varphi(d(HGu, z)), \varphi(d(z, z))\varphi(d(HGu, z)\}$$

i.e. $\quad d(z, HGu) \le \varphi(d(z, HGu)) \le d(z, HGu)$.

Hence $\varphi(d(z, HGu)) = 0$ i.e. $HGu = z$.

Let $yn = u$, then $HGyn \to HGu = z$ and $Ayn \to Au = z$. Since (HG, A) is compatible mappings of type (R) by proposition 2.10, we get

$$(HG)(HG)u = AAu.$$ This gives $HGz = Az$.

Further

$$d^2(EFx_{2n}, HGz) \le \max\{\varphi(d^2(Bx_{2n}, Az)), \varphi(d(Bx_{2n}, EFx_{2n}))\varphi(d(Az, HGz)),$$
$$\varphi(d(Bx_{2n}, HGz))\varphi(d(Az, EFx_{2n})), \varphi(d(Bx_{2n}, EFx_{2n}))\varphi(d(Az, EFx_{2n}))\}$$

letting $n\to\infty$, we get

$$d^2(z, HGz) \le \max\{\varphi(d^2(z, HGz)), \varphi(d(z, z))\varphi(d(HGz, HGz))\ \varphi(d(z, HGz))\varphi(d(HGz, z)), \varphi(d(z, z)\varphi(d(HGz, z))\}$$

$$= \varphi(d(z, HGz))\varphi(d(z, HGz))$$

i.e. $d(HGz, z) \le \varphi(d(HGz, z)) \le d(HGz, z)$. Hence $\varphi(d(HGz, z)) = d(HGz, z) = 0$ i.e. $HGz = z$.

Hence $z = HGz$ and $z = Az = HGz$. Since $HG(X) \subseteq B(X)$, There is a point $v \in X$ such that

$$z = HGz = Bv.$$

Now

$$d^2(EFv, z) = d^2(EFv, HGz) \le \max\{\varphi(d^2(Bv, Az)), \varphi(d(Bv, EFv))\varphi(d(Az, Gz)),$$
$$\varphi(d(Bv, HGz))\varphi(d(Az, EFv)), \varphi(d(Bv, EFv))\varphi(d(Az, EFv))\}$$
$$= \max\{\varphi(d^2(z, z)), \varphi(d(z, EFv))\varphi(d(z, z)), \varphi(d(z, z))\varphi(d(z, EFv)), \varphi(d(z, EFv))\varphi(d(z, EFv))\}$$

i.e. $d(EFv, z) \le \varphi(d(EFv, z)) \le d(EFv, z)$.

Hence $\varphi(d(EFv, z)) = 0$ i.e. $EFv = z$.

Thus we have $d(EFv, z) = 0$ and $EFv = z$. Take $yn = v$ then $EFyn \to EFv = z$, $Byn \to Bv = z$. Since (EF, B) is compatible of type (R), by the proposition 2.10. This implies that $Fz = Bz$. Hence z is a common fixed point of EF, B, A and HG when EF is a continuous. The proof is similar that z is common fixed point of EF, B, A and HG when HG is continuous.

UNIQUENESS:
Let z and w be two common fixed point of EF, B, A and HG, i.e. $z = EFz = Bz = HGz = Az$ and $w = EFw = Bw = HGw = Aw$.

From condition (vii) we have

$$d^2(z, w) = d^2(EFz, HGw) \le \max\{\varphi(d^2(Bz, Aw)), \varphi(d(Bz, EFz))\varphi(d(Aw, HGw)), \varphi(d(Bz, HGw))\varphi(d(Aw, EFz)), \varphi(d(Bz, EFz))\varphi(d(Aw, EFz))\}$$

$$= \max\{\varphi(d^2(z, w)), \varphi(d(z, z))\varphi(d(w, w)), \varphi(d(z, w))\varphi(d(w, z)), \varphi(d(z, z))\varphi(d(w, z))\}$$

Therefore $d(z, w) \le \varphi(d(z, w)) \le d(z, w)$ i.e. $\varphi(d(z, w)) = d(z, w)$. Thus $d(z, w) = 0$ i.e. $w = z$.

Hence the common fixed point is unique.

REFERENCES

[1] Sessa S., On weakly commutativity condition in a fixed point consideration, *Publ. Int. Math.* **32** (46) (1986), pp. 149-153.

[2] G. Jungck, Compatible mappings and common fixed points, *Internat. J. Math. Math. Sci.* **9,** (1986), 771-779.

[3] S. T. Patil, Results on fixed points and its applications, *Ph.D. thesis*, Marathwada University, Aurangabad, India, 1990.

[4] G. Jungck, P. P. Murthy and Y. J. Cho, Compatible mappings of type (A) and common fixed points, *Math. Japonica*, 38, (1993), 381-386.

[5] Pathak, Chang and Cho, Fixed point theorems for compatible mappings of type (P), *Indian Journal of Math*., 36(2), (1994), 151-166.

[6] Y. Rohen, M.R. Singh and L. Shambhu, Common fixed points of compatible mappings of type (C) in Banach spaces, *Proc. of Math. Soc.,* BHU, 20(2004), 77-78.

[7] Aage, C. T., Salunke, J. N., A note on common fixed point theorems, *Int. J. Math. Anal.* 2, 28 (2008), 1369–1380.

[8] D. Turkoglu, Ishak Altun, Brian Fisher, Common fixed point theorems for four mappings with some weak conditions of commutativity, *Novi Sad J. Math*., Vol. 36 No. 1, (2006), 75-80.

[9] V. Sriniwas and R. Uma Maheswar Rao, A Common Fixed Point Theorem for Four Selfmaps, *The Mathematics Education*, Vol. XL, No. 2, June. (2006), 99-104.

[10] M. Rangamma, V. Sriniwas and R. Uma Maheswar Rao, A Common Fixed Point Theorem for Three self maps, *The Mathematics Education*, Vol. XL, No. 3, Sept. (2006), 180-185.

[11] B. D. Pant and Sunny Chauhan, Fixed point theorems in Menger space using semi-compatibility, *Int. J. Contemp. Math. Sciences*, Vol. 5, 2010, no. 19, 943-951.

Source of support: Nil, Conflict of interest: None Declared

