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GABOR MULTIPLIERS FOR MODULATION SPACES

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ABSTRACT

Using the structure of the Heisenberg group we define madulation spaces $H_w^p(R^d)$, $1 \leq p \leq \infty$, and their antiduals $H_w^{p\sim}(R^d)$. We obtain atomic characterization of $H_w^p(R^d)$ in terms of Gabor atoms and study the boundedness properties of Gabor multipliers on these spaces of functions or distributions.

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1. INTRODUCTION

In Gabor analysis the basic problem is the expansion of an arbitrary function in terms of translations and modulations of an analyzing vector with coefficients as Gabor transforms of the function with respect to an analyzing vector. A number of problems in signal analysis in the time-frequency plane involve the pointwise multiplication of Gabor coefficient by some other functions satisfying suitable conditions. This technique is frequently used in signal processing and, in engineering terminology, it is known as masking operation (cf, [FZ 98] p.143). Recently, Feichtinger [Fei 02] has developed a fairly general theory of Gabor multipliers.

According to his definition, if Λ is a TF-lattice in $R^d \times \hat{R}^d, \{m(\lambda)\}_{\lambda\in\Lambda}$ a complex-valued sequence on Λ and g_1, g_2 are any two square - integrable functions, then the Gabor multipliers associated with the triple (g_1, g_2, Λ) with upper symbol m is given by

$$G_m(f) \equiv G_{g_1,g_2,\Lambda,m}(f) = \sum_{\lambda\in\Lambda} m(\lambda) < f, \pi(\lambda)g_1 > \pi(\lambda)g_2.$$

Feichtinger (loc.cit), in fact, has paved a new way to move from function space theory towards operator theory associated with Gabor expansions and laid the foundation of the theory of Gabor multipliers, which arise from pointwise multiplication of Gabor coefficients. He has discussed in details the boundedness properties of Gabor multipliers on the function spaces $L^2(R^d)$, Feichtinger algebra $S_0(R^d)$ and its dual space $S_0{}'(R^d)$.

More recently, Feichtinger and Nowak [FN 03, Chapter 5] have given the first systematic and extensive survey of Gabor multipliers. In this chapter our aim is to study the theory of Gabor multipliers on the Heisenberg group. In section 2, we present the basic notations and definitions for use in the sequel. In section 3, we define weighted Banach spaces on the Heisenberg group including some of their properties.

Using the structure of the Heisenberg group in section 4, we define weighted Banach spaces $H_w^p(R^d)$, $1 \leq p \leq \infty$, of test functions, which include the well known Feichtinger algebra $S_0(R^d)$ as a particular case for $w = 1$ and $p = 1$. Also, we define $H_w^{p\sim}(R^d)$, as the space of all continuous conjugate linear functionals on $H_w^p(R^d)$. Section 5 deals with the atomic characterization of the space $H_w^p(R^d)$ in terms of Gabor atoms. In Section 6 we prove three lemma for use in the proof of Theorem 5.1. In the last section of the chapter we demonstrate the boundedness of Gabor operators on the spaces $L_w^2(R^d)$, $H_w^p(R^d)$ and $H_w^{p\sim}(R^d)$.

2. NOTATIONS AND BASIC CONCEPTS

Let R^d be the d-dimensional Euclidean space and Z^d the set of all d-tuples of integers. Let $H^d = R^d \times \hat{R}^d \times \tau$ be the Heisenberg group with the group operations defined by

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$$h_1.h_2 = (x_1, \xi_1, t_1).(x_2, \xi_2, t_2)$$
$$= (x_1 + x_2, \xi_1 + \xi_2, t_1.t_2.e^{2\pi i x_2 \xi_1}); \qquad \forall\, h_1, h_2 \in H^d$$
$$\text{and } h^{-1} = (x, y, t)^{-1} = (-x, -y, t^{-1} e^{2\pi i x \xi}), \qquad \forall\, h \in H^d,$$

where $\hat{R}^d$ is the dual group of R^d and the elements (x_1, y_1, t_1) and (x_2, y_2, t_2) belong to the Heisenberg group H^d.

The identity element in H^d is (0, 0, 1) and H^d is topologized by the product topology on $R^d \times \hat{R}^d \times \tau$. Since for the Euclidean space R^d, we have $R^d \cong \hat{R}^d$, hence it is convenient to use R^d in place of $\hat{R}^d$.

Let T_x be the translation (time- shift) operator such that

$$T_x f(y) = f(y - x) \text{ for all } x, y \in R^d.$$

We denote by M_ξ the modulation (frequency-shift) operator such that

$$M_\xi f(y) = e^{2\pi i \xi . y} f(y), \forall\, y \in R^d \text{ and } \xi \in \hat{R}^d.$$

By virtue of the above definitions, it is easy to verify that

$$\widehat{T_x f} = M_{-x} \hat{f}$$
$$\widehat{M_\xi f} = T_\xi \hat{f}$$
$$\text{and } \quad M_\xi T_x = e^{2\pi i x \xi} \; T_x \; M_\xi,$$

where $\hat{f}$ denotes the Fourier transform of f on R^d defined by

$$\hat{f}(\xi) = \int_{R^d} e^{-2\pi i x . \xi} f(x) dx.$$

It is well known that the Lebesgue measure $dh = dx\, dy\, d\tau$ is the Haar measure on H^d, which is unimodular.

We denote by $\pounds(X, Y)$ the space of all bounded linear operators $T: X \to Y$ with operator norm $|\; \| T|\pounds(X,Y)| \;\|$; X, Y being any two Banach spaces.

3. WEIGHTED BANACH SPACES ON H^d

A strictly positive and constant function $m: H^d \to R_+$ is called a submultiplicative weight on H^d provided

$$m(hh') \leq m(h) m(h'); \forall\, h, h' \in H^d.$$

A weight function u is called moderate with respect to m, if

$$u(hh') \leq m(h) u(h'); \forall\, h, h' \in H^d.$$

We denote by $L_u^p(H^d)$, $1 \leq p < \infty$, the Banach space of functions on H^d under the norm

$$\| \; F|L_u^p(H^d) \; \| = \left(\int_{H^d} |\, F(h)|^p u^p(h) dh \right)^{1/p} < \infty. \tag{3.1}$$

In case $p = \infty$, the space $L_u^\infty(H^d)$ denotes the space of all measurable functions F on H^d such that

$$\| \; F|L_u^\infty \; \| = ess \sup \{\, |\, F(h)| u(h) : h \in H^d\} < \infty. \tag{3.2}$$

We denote the conjugate space of $L_w^p(H^d)$ by $L_{w^{-1}}^{p'}(H^d)$, where $1/p + 1/p' = 1$.

The unimodularity of the Heisenberg group ensures that the left and right translation operators given by

$$L_h\, F(g) = F(h^{-1}g)$$

and

$$R_h\, F(g) = F(gh)$$

respectively act isometrically on $L_w^p(H)^d,\ 1 \le p \le \infty$.

We assume that

$$u(h) = |||\, L_h \mid L_u^p(H^d)\, |||$$

$$\text{and} \quad v(h) = \Delta\,(h^{-1}) = |||R_{h^{-1}}\, |L_v^p(H^d)\, |||,$$

where u, v are moderate weight functions and the operator norm $||| \cdot |||$ is the norm of left and right translations on $L_u^p(H^d)$ and $L_v^p(H^d)$.

In the sequel we assume that w is a moderate weight function on H^d such that

$$w(h) \ge C \max\{u(h), u(h^{-1}), v(h), v(h^{-1})\ \Delta\,(h^{-1})\},$$

C being a positive constant not necessarily the same at each occurrence.

In particular, we suppose that $w(h) \ge 1$ and

$$\| F \mid L_w^p\ ||| = \| F^{\nabla} L_w^p\ \|,$$

where the involution F^{∇} is given by

$$F^{\nabla}(h) = \overline{F(h^{-1})}.$$

If $C_c(H^d)$ denotes the space of all continuous complex-valued functions on H^d with compact support, then the convolution of any two functions $F,\, G \in C_c(H^d)$ is given by

$$(F * G)(g) = \int_{H^d} G\,(h^{-1}g)\, F(h) dh = \left(\int_{H^d} F(h) L_h G\, dh\right)(g).$$

It is well known that convolution of functions on H^d is associative, but not commutative. The following basic properties of convolution of functions on H^d can be easily verified as in [FG 92c, pp.370-371]:

(i) $L_h(F * G) = L_h\, F * G$ and $R_h(F * G) = F * R_h G$.

(ii) $L_w^1(H^d)$ is a Banach algebra with respect to convolution as multiplication.

(iii) $L_w^p(H^d)$ is a convolution module over $L_w^1(H^d)$, i.e., the following properties are satisfied :

$L_w^p * L_w^1 \subseteq L_w^p, \quad 1 \le p \le \infty$

and $\| (F * G)|L_w^p \| \le \| F|L_w^p \|\ \| G|L_w^1 \|$, for all $F \in L_w^p(H^d)$ and $G \in L_w^1(H^d)$.

4. GABOR TRANSFORM FOR WEIGHTED BANACH

The Gabor Heisenberg transform of a function $f \in L^2(R^d)$ with respect to a window function $g \in L^2(R^d)$ is given by (cf. [FG 92c], p.371):

$$V_g f(h) = \langle \pi(h)g, f \rangle, \qquad \forall\, h \in H^d.$$

Feichtinger and Gröchenig [FG 92c, pp.372-373] have shown that the transform $f \to V_g f$ is a linear mapping from the Hilbert space $L^2(R^d)$ into the space of bounded and continuous functions on H^d, intertwining property

$$V_g(\pi(h)f) = L_h(V_g f), \qquad \forall\, h \in H^d, \tag{4.1}$$

holds true, L_h being the left translation operator on H^d and

$$V_g f * V_g g = V_g f, \qquad \forall\, f \in L^2(R^d) \text{ and } \| g \|_2 = 1. \tag{4.2}$$

We define a class of analyzing vectors

$$A_w^1(R^d) = \{g \in L^2(R^d): V_g g \in L_w^1(H^d)\}, \quad 1 \le p \le 2. \tag{4.3}$$

On the lines of Feichtinger and Grӧchenig [FG 89], for a fixed non-zero $g \in A_w^1(R^d)$, we define

$$H_w^p(R^d) = \{f: f \in L^2(R^d), V_g f \in L_w^p(H^d)\} \tag{4.4}$$

and endow it with the norm

$$\| f | H_w^p(R^d) \| = \| V_g f | L_w^p(M^d) \| , 1 \le p \le 2. \tag{4.5}$$

It can be easily verified that $H_w^p(R^d)$ is independent of the choicely of g and complete under the norm defined by (4.5). These spaces were, originally, introduced by Feichtinger in (cf. [Fei 80]), who named them as modulation spaces.

By virtue of the above definitions, it is clear that the embeddings

$$H_w^p(R^d) \hookrightarrow L^2(R^d) \hookrightarrow M_w^{p\sim}(R^d),\ 1 \le p \le 2.$$

are continuous, where $H_w^{p\sim}(R^d)$ is the space of all continuous conjugate linear functionals on the Banach Space $H_w^p(R^d)$.

Since the inner product on $L^2(R^d) \times L^2(R^d)$ extends to a sesquilinear form on $H_w^p(R^d) \times H_w^{p\sim}(R^d)$, the extended Gabor transform takes the form

$V_g f(h) =< f, \pi(h) g >$, for all $g \in H_w^p(R^d)$, $f \in H_w^{p\sim}(R^d)$, $h \in H^d$ and $1 \le p \le 2$.

In case $p = 1$ and $w = 1$, the space $H_w^p(R^d)$ reduces to the well known Feichtinger algebra $S_0(R^d)$ of test functions on R^d.

5. ATOMIC CHARACTERIZATION

We denote by $W(C_0,\ L_w^1)(H^d)$ the Wiener amalgam space with the local and global components $C_0(H^d)$ and $L_w^1(H^d)$, where $C_0(H^d)$ is the space of all continuous functions on H^d vanishing at infinitely.

We suppose that T is the convolution operator on $L_w^p(H^d)$ such that

$$T\,F = F * G; \qquad \forall\, F, G \in L_w^p(H^d),$$

where $F = V_g f$ and $G = V_g g$.

We suppose that $X = (h_i)_{i \in I}$ is a relatively separated and U-dense family in H^d. Also, let $\Psi = (\psi_i(h))_{i \in I}$ be a bounded uniform partition of unity of size U (U-BUPU). Now, on the lines of Feichtinger and Grӧchenig [FG 89a, p.329], we define the approximation operator

$$T_\Psi\colon F \longrightarrow \sum_{i \in I} < \psi_i,\ F > L_{h_i} G, \tag{5.1}$$

which is composed of a coefficient mapping

$$F \longrightarrow < \psi_i,\ F >_{i \in I}$$

and a convolution operator

$$(\lambda_i)_{i \in I} \longrightarrow \sum_{i \in I} \lambda_i\, L_{h_i} G\ = (\sum_{i \in I} \lambda_i\, \delta_{h_i}) * G,$$

where δ_{h_i} is the point measure at h_i.

Using the above approximation operator, we obtain an atomic characterization of the space $H_w^p(R^d)$.

Precisely, we prove the following:

Theorem 5.1: *If $g \in A_w^1(R^d)$ and $G \in W(C_0,\ L_w^1)(H^d)$ and $\| G_U^\# \| < 1$, then there exists a neighborhood U of the identity in H^d and a constant $c > 0$, both depending on H^d, such that for every U-dense and relatively separated family $X = (h_i)_{i \in I}$ in H^d any $f \in H_w^p(R^d)$, $1 \le p \le 2$, can be expressed in the form*

$$f = \sum_{i \in I} \alpha_i(f)\, \pi(h_i)\, g, \tag{5.2}$$

$$with \;\; \| \alpha_i | l_w^p(I) \| \leq c \, \| f | H_w^p \|, \tag{5.3}$$

where $G_U^{\#}$ is the modules of the continuity of G with respect to the norm $\|.\|_{1,w}$, $\alpha_i(f) = < \psi_i, T_\psi^{-1} V_g f >$ and the series in (5.2) is absolutely convergent in the norm topology of $H_w^p(R^d)$.

6. NECESSARY LEMMAS

We shall use the following lemmas in the proof of Theorem 5.1:

Lemma 6.1: *If $G \in W(C_0\, L_w^1)(H^d)$, $X = (h_i)_{i\in I}$ is a U-dense and relatively separated family in H^d and $\Lambda = (\lambda_i)_{i \in I}$ is defined by*

$$\lambda_i = < \psi_i, F >_{i \in I}, \; F \in L_w^p(H^d), \qquad \text{then} \quad F = \sum_{i \in I} \lambda_i\, L_{x_i} G \in L_w^p(H^d)$$

If and only if $\Lambda \in l_w^p(I)$. Also, there exists a positive constant C such that

$$\| F \|_{p,w} \leq C \, \| \Lambda \|_{p,w}.$$

Proof: Since w is a submultiplicative weight function on H^d, we have

$$w^p(h_i) \leq C_0\, w^p\,(g),$$

where $C_0 = \sup_{k \in U_0} w^p\,(k)$, $g \in h_i U_0$, C_0 is a positive constant and $U_0 \subseteq U$ is a neighborhood of the identity in H^d.

Hence we see that

$$| < \psi_i, F > |^p w^p(h_i) \leq C_0 \max\, | \psi_i |^p \;\; \| F \|_{p,w}.$$

$$\Rightarrow \| \Lambda \|_{p,w} \leq C \;\; \| F \|_{p,w},$$

C being a positive constant not necessarily the same at each occurrence.

Conversely, we have

$$\begin{aligned} \| \sum_{i \in I} \lambda_i\, L_{x_i} G \, \|_{p,w} &\leq \sum_{i \in I} | \lambda_i | \;\; \| L_{h_i} G \, \|_{p,w} \\ &\leq C \sum_{i \in I} | \lambda_i | w^p(h_i) \, \| G | W(C_0, L_w^1) \, \|. \end{aligned}$$

Hence the lemma holds true.

Lemma 6.2: *The set of operators $\{T_\psi\}$, when ψ runs through the family of U_0 -BUPUS, acts uniformly bounded on $L_w^p(H^d)$.*

Proof Let $F \in L_w^p(H^d)$. Then we have

$$\begin{aligned} \| T_\psi F \, \|_{p,w} &= \| \sum_{i \in I} < \psi_i, F > L_{h_i} G \, \|_{p,w} \\ &= \| \sum_{i \in I} < \psi_i, F > \delta_{h_i} * G \, \|_{p,w} \\ &\leq \| \sum_{i \in I} < \psi_i, F > \delta_{h_i} \|_{p,w} \, \| G | L_w^1 \, \| \\ &\leq C \;\; \| F \|_{p,w} \;\; \| G | W(C_0, L_w^1) \, \| \leq C \;\; \| F \|_{p,w} \\ &\leq C \, \| \Lambda \|_{p,w} \text{ by lemma 6.1.} \end{aligned}$$

Lemma 6.3: *If* $\{T_\psi\}$ *is a net of U-BAPU's, then*

$$\lim_{\psi\to\infty} |||\, T_\psi - T|||_{p,w} = 0.$$

Proof: Let $F \in L_w^p(G)$. Then, as in [FG 89, p.331], we have

$$\begin{aligned}
\| T_\psi F - TF \|_{p,w} &= \| \sum_{i\in I} (<\psi_i, F> \delta_{h_i} - F\psi_i) * G \|_{p,w} \\
&\le \sum_{i\in I} \| (<\psi_i, F>)\delta_{h_i} - F\psi_i^* G \|_{p,w} \\
&\le \sum_{i\in I} \| \int_{h_i U} (L_{h_i} G - L_g G)^* F(g)\psi_i(g)dg \|_{p,w} \\
&\le \sum_{i\in I} \| \int_{h_i U} (L_{h_i} G - L_g G) \|_{1,w} \| F(g)\psi_i(g) \|_{p,w} \\
&\le \sum_{i\in I} \sup_{u\in U} \| L_{h_i} G - L_u G \|_{1,w} \| F(g)\psi_i(g) \|_{p,w} \\
&\le C \sup_{u\in U} \| G - L_u G \|_{1,w} \| F \|_{p,w}. \\
&\le C\omega_U(G) \cdot \| F \|_{p,w},
\end{aligned}$$

where $\omega_U(G) = \sup_{u\in U} \| G - L_u G \|_{1,w}$ is the modulus of continuity of G with respect to the norm $\|\cdot\|_{1,w}$. Finally, choosing U sufficiently small, we obtain

$$\| |T_\psi - T \| |_{p,w} \le C \;\; \omega_U(G) \;\; \longrightarrow 0 \quad \text{as} \quad U \to \{e\}.$$

7. PROOF OF THEOREM 5.1

We write $F = V_g f$ and $G = V_g g$, which imply that

$$G * G = G$$

$$\| F * G \|_{p,w} \le \| F \|_{p,w} \cdot \| G \|_{1,w}$$

$$\le C \| F \|_{p,w} \cdot \| G \|_{1,w},$$

C being a positive constant not necessarily the same at each occurrence

Also, on account of the relation (4.2), $F \in L_w^p(H^d) * G(H^d)$ if and only if $F = V_g f$ for a function $f \in H_w^p(R^d)$.

Thus the convolution $L_w^p(H)^d * G(H^d)$ is a bounded projection from $L_w^p(H^d)$ onto the closed subspace $L_w^p(H^d) * G(H^d)$.

Since $TF = F * G$, the operator T acts as identity operator on $L_w^p * G$. Hence, by Lemma 6.3, there exists a net $\{T_\psi\}$ of U-BUPU's, which is norm convergent to T and we have

$$\lim_{\psi\to\infty} |||\, (T_\psi - T)|L_w^p(H^d) * G \,||| = 0.$$

Thus as in [FG 88, p.58], we may choose $a > 0$ such that

$$|||(T - T_\psi)|L_w^p * G \;\; ||| < a < 1,.$$

in a sufficiently small neighborhood U of e in H^d. Hence, on account of Neuman's series, we have

$$T_\psi^{-1} \,||| \le (1-a)^{-1}.$$

Thus, finally, we see that

$$\begin{aligned}
V_g f &= F \\
&= T_\psi\left(T_\psi^{-1} F\right) \\
&= \sum_{i\in I} < \psi_i T_\psi^{-1} F > L_{h_i} G.
\end{aligned}$$

$$\Rightarrow f = \sum_{i\in I} < \psi_i, T_\psi^{-1} F > \ V_g^{-1}\big(L_{h_i} G\big)$$

$$= \textstyle\sum_{i\in I} < \psi_i, T_\psi^{-1} V_g f > \pi(h_i) g \qquad \text{(by 4.1)}$$

$$= \textstyle\sum_{i\in I} \alpha_i\, \pi(h_i) g, \text{ where } \quad \alpha_i = < \psi_i, T_\psi^{-1} V_g f >.$$

Next, Since $T_\psi^{-1} F \in L_w^p * G$, we have

$$\begin{aligned}
\| \alpha_i | l_w^p(I) \| &= \Big(\sum_{i\in I} |\alpha_i|^p w^p(h_i)\Big)^{1/p} \\
&= \Big(\sum_{i\in I} |< \psi_i, T_\psi^{-1} F >|^p w^p(h_i)\Big)^{1/p} \\
&\le C \sum_{i\in I} |< \psi_i, T_\Psi^{-1} F\, w(h_i) >|^p \\
&= C \| T_\psi^{-1} F \|_{p,w} \\
&\le C \| f \| H_w^p(R^d) \|.
\end{aligned}$$

This completes the proof of the theorem.

8. GABOR MULTIPLIERS ON $H_w^p(R^d)$

Feichtinger, in a very recent paper [Fei 02], has initiated the study of Gabor multipliers on the spaces $L^2(R^d), S_0(R^d)$ and $S'_0(R^d)$. In this section, using the concept of Heisenberg group, we study Gabor multiplier for the spaces $L^2(R^d), H_w^p(R^d)$ and $H_w^{p\sim}(R^d)$.

Definition 8.1: If g, h are any two function in $L^2(R^d)$and Λ is a lattice in H^d, then a complex-valued sequence $(m_\lambda)_{\lambda\in\Lambda}$ is called a Gabor multipliers associated with the triple (g, h, Λ) provided

$$G_m(f) \equiv G_{g,h,\Lambda,m} = \sum_{\lambda\in\Lambda} m(\lambda)\alpha_\lambda(f, g)\pi(\lambda) h. \tag{8.1}$$

In case $h = g$, we simply write

$$G_m(f) \equiv G_{g,\Lambda,m}(f).$$

We prove the following:

Theorem 8.1: *Let Λ be a lattice in $H^d, m = \{m(\lambda)\}_{\lambda\in\Lambda}$ and the linear operator G_m is defined by (8.1). Then the following results holds true:*

(i) *If $f, g \in L^2(R^d)$, $h \in H_w^p(R^d)$ and $m \in l^\infty(\Lambda)$, then $G_m \in £(L^2(R^d))$ with*

$$|||G_m | £(L^2(R^d))||| \le C_\Lambda \| h | H_w^p \| \| m | l^\infty \| \| g \|_2, \tag{8.2}$$

C_Λ being a positive constant depending on Λ and $1 \le p \le 2$.

(ii) *For $g, h \in H_w^p(R^d)$,* and $m \in l^1(\Lambda)$, *we have*

$$G_m \in £(H_w^p(R^d)) \text{ with}$$

$$|||G_m | H_w^p(R^d)||| \le C_\Lambda \| h | H_w^p \| \| g | H_w^p \| \| m | l^1(\Lambda) \|. \tag{8.3}$$

(iii) *If $g \in H_w^p(R^d); f, h \in H_w^{p\sim}(R^d)$ and $m \in l^1(\Lambda)$, then $G_m \in £(H_w^{p\sim}(R^d))$ with*

$$|||G_m | £(H_w^{p\sim}(R^d))||| \le C_\Lambda \| g | H_w^p \| \| h | H_w^{p\sim} \| \| m | l^1(\Lambda) \|. \tag{8.4}$$

Proof: (i) By virtue of the relation (5.2), we have

$$|G_m f| = \textstyle\sum_{\lambda\in\Lambda} \alpha_\lambda(f) m(\lambda)\pi(\lambda) h$$

$$= \textstyle\sum_{\lambda\in\Lambda} < \psi_\lambda, T_\psi^{-1} V_g f(\lambda) > m(\lambda)\pi(\lambda) h$$

$$\leq \sup_{\lambda\in\Lambda} < \psi_\lambda, T_\psi^{-1} V_g f(\lambda) > \sum_{\lambda\in\Lambda} |\,\pi(\lambda)h| \;\; |m(\lambda)|.$$
$$\leq C \parallel f \parallel_2 \parallel g \parallel_2 . \parallel h|H_w^p(R^d) \parallel . \parallel m|l^\infty(\Lambda) \parallel,$$

$\Rightarrow G_m \in \pounds(L^2(R^d))$ and the inequality in (8.2) follows.

(ii) We have

$$\begin{aligned} |G_m f| &= \sum_{\lambda\in\Lambda} < \psi_\lambda, T_\psi^{-1} V_g f(\lambda) > m(\lambda)\pi(\lambda)h \\ &\leq \sup_{\lambda\in\Lambda} < \psi_\lambda, T_\psi^{-1} V_g f(\lambda) > \sum_{\lambda\in\Lambda} |\,m(\lambda)| \;\; \parallel \pi(\lambda)h|H_w^p \parallel \\ &\leq C \parallel g|H_w^p \parallel \parallel f|H_w^{p\sim} \parallel \parallel m|l^1(\Lambda) \parallel \parallel h \parallel_{H_w^p}. \end{aligned}$$

$\Rightarrow G_m \in \pounds(H_w^p(R^d))$ and the inequality in (8.3) holds true, for $H_w^p \hookrightarrow H_w^{p\sim}$.

(iii) As above, we have

$$\begin{aligned} |G_m f| &\leq \sup_{\lambda\in\Lambda} < \psi_\lambda, T_\psi^{-1} V_g f(\lambda) > \sum_{\lambda\in\Lambda} |\,m(\lambda). |\pi(\lambda)h| \\ &\leq C \parallel g|H_w^p \parallel \parallel f|H_w^{p\sim} \parallel \parallel m|l^1(\Lambda) \parallel \parallel h|H_w^{p\sim} \parallel. \end{aligned}$$

$\Rightarrow G_m \in \pounds(H_w^{p\sim}(R^d))$ and the inequality in (8.4) follows.

This completes the proof of the theorem.

9. COMPACTNESS OF GABOR MULTIPLIERS

It is well known that any operator in $L(X)$ is compact provided it maps norm bounded subsets of X to compact subsets. In this section we prove the following theorem on the compactness of Gabor multipliers:

Theorem 9.1: *If $g, h \in H_w^p(R^d)$, then G_m is a compact operator on $L^2(R^d)$ and $H_w^p(R^d)$ for $m(\gamma) \to 0$ as $\gamma \to \infty$.*

This theorem provides an extension to the corresponding results of Feichtinger [Fei 02, Theorem 4.7 (iv)] for $g = h$ and $g \in S_0(R^d)$.

We shall use the following lemma in the proof of our theorem, which is an extension of an analogous result by Dörfler, Feichtinger and Gröchenig [DFG 03, Theorem 2] on a d-dimensional Euclidean space:

Lemma 9.2: *A closed and bounded set $S \subseteq L^2(R^d)$ is compact if and only the set $\{V_g f : f \in S\}$ is tight in $L^2(H^d)$.*

Proof of the lemma: We suppose that $S \subseteq L^2(R^d)$ is compact. This ensures that there exists a finite number of functions $f_1, f_2, \ldots\ldots, f_j$, say, such that

$$\min_{j=1,2,3\ldots,n} \parallel f - f_j \parallel < \epsilon/2, \quad \forall\, f \in S.$$

Next, since $V_g f_j \in L^2(H^d); j = 1,2,\ldots, n$ there exists a compact set $U \subseteq H^d$ such that

$$\int_{U^c} |\,V_g f_j|^2 d\gamma < \epsilon^2/4; \quad \forall\, \gamma \in H^d,$$

U^c being the complementary set of U in H^d. Hence, for each $f \in S$, we have

$$\begin{aligned} \left(\int_{U^c} |\,V_g f|^2 d\gamma\right)^{1/2} &\leq \min_{j=1,2,3\ldots,n} [\left(\int_{U^c} |\,V_g (f - f_j)(\gamma)|^2 d\gamma\right)^{1/2} \left(\int_{U^c} |\,V_g f_j(\gamma)|^2 d\gamma\right)^{1/2}] \\ &\leq \min_{j=1,2,3\ldots,n} \parallel f - f_j \parallel_2 + \epsilon/2 \\ &< \epsilon. \end{aligned}$$

Conversely, we suppose that the set $\{V_g f : f \in S\}$ is tight in $L^2(H^d)$. Hence, for any $\epsilon > 0$, we can find a compact set $U \subseteq H^d$ such that

$$\int_{U^c} (\,|V_g f(\gamma)|^2 d\gamma) < \epsilon^2, \quad \forall\, f \in S. \tag{9.1}$$

Let $\{f_n\}$ be a sequence of functions in S. Since S is bounded in $L^2(R^d)$, it is weakly compact in $L^2(R^d)$.

Now, putting $k = \pi(\gamma)g$, we obtain

$$V_g f_j(\gamma) \to V_g f(\gamma), \quad \forall\, \gamma \in H^d. \tag{9.2}$$

Now, applying Cauchy - Schwarz inequality, we get

$$\begin{aligned} |V_g(f - f_j)(\gamma)| &\leq \| f - f_j \|_2 \\ &\leq \sup_j \| f_j \|_2 + \| f \|_2 \\ &\leq C, \end{aligned}$$

where C is a positive constant not necessarily the same at each occurrence.

Thus the restriction of $|V_g(f - f_j)|^2$ to U is dominated by $C^2 \chi_U \in L^2(R^d)$, where χ_U is the characteristic function of U.

Hence, using (9.2) and the dominated convergence theorem, we obtain

$$\int_{U^c} | V_g(f - f_j)(\gamma)|^2 d\gamma \to 0 \text{ as } j \to \infty. \tag{9.3}$$

Finally, combining (9.1) and (9.3), we have

$$\begin{aligned} \lim_{j\to\infty} \| f - f_j \|_2 &= \overline{\lim_{j\to\infty}} \| V_g\big(f - f_j\big) \|_2 \\ &\leq \overline{\lim_{j\to\infty}} \left(\int_U | V_g\big(f - f_j\big)(\gamma)|^2 d\gamma \right)^{1/2} \\ &+ \overline{\lim_{j\to\infty}} \left(\int_{U^c} | V_g\big(f - f_j\big)(\gamma)|^2 d\gamma \right)^{1/2} \\ &\leq 2\epsilon. \end{aligned}$$

Choosing $\epsilon \to 0$, we see that

$$\lim_{j\to\infty} \| f - f_j \|_2 = 0.$$

$\Longrightarrow$ Every sequence $\{f_n\}$ in S has a convergent subsequence.

$\Longrightarrow S$ is compact.

Proof of theorem 9.1: On account of the above lemma $H_w^p(R^d)$ is a compact subset of $L^2(R^d)$. Hence on account of Theorem 8.1 (i), any operator $G_m \in £^2(R^d)$ has norm bounded from $L^2(R^d)$ onto $H_w^p(R^d)$ for all $m(\gamma) \to 0$ as $\gamma \to \infty$.

$\Longrightarrow$ G_m is compact on $L^2(R^d)$.

Also, from Theorem 8.1(ii), we see that any $G_m \in £(H_w^p(R^d))$ is compact.

This compete the proof of the theorem.

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