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A CLASS OF ITERATIVE METHODS WITH CUBIC CONVERGENCE TO SOLVE NONLINEAR EQUATIONS

[1]Hamidreza Mofidi*, [2]Behzad Ghanbari and [3]Mehdi Gholami Porshokouhi

[1] Department of Applied Mathematics, Tarbiat Modares University, Tehran, Iran

[2, 3] Department of Mathematics, Takestan Branch, Islamic Azad University, Takestan, Iran

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ABSTRACT

A new family of methods of third order is developed. Starting with a suitably chosen x_0*, the methods generate a sequence of iterates converging to the root. The efficacy of the methods is investigated with a number of numerical examples. It is observed that our method takes less number of iterations than Newton's method and some known third-order methods.*

Keywords and Phrases: *Iterative methods; Cubic convergence; Newton's method.*

Mathematics Subject Classification: *47J30, 49S05, 35G25.*

1. INTRODUCTION

As the nonlinear equations frequently arise in applied mathematics and engineering, so it is significant to find robust and efficient methods in the case of finding their roots.

Newton's method which is a well-known iterative method for finding simple root, say α, as follows

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}. \tag{1}$$

We here remark that this method shows a quadratic convergence behavior in a suitable neighborhood containing α.

In literature, several variants of the third-order methods have been introduced. For instance, Kou's third-order method [1] as follows

$$x_{n+1} = x_n - \frac{f(x_n)}{3f(x_n) - f(x_n + f(x_n)/f'(x_n))} \frac{f(x_n)}{f'(x_n)},$$

and so-called Newton Steffensen method [2] as

$$x_{n+1} = x_n - \frac{f(x_n)}{f(x_n) - f(x_n + f(x_n)/f'(x_n))} \frac{f(x_n)}{f'(x_n)}, \tag{2}$$

where

$$x_{n+1} = x_n + \frac{f(x_n)}{f'(x_n)}.$$

For further methods, the interested reader is referred to [3-8] and the references therein. This contribution is structured as follows. General iterative schemes of root-finders have been proposed and analyzed in Section 2. Numerical examples are given in Section 3. Finally, Section 4 is devoted to the concluding comments.

Corresponding author: [1]*Hamidreza Mofidi**
[1] Department of Applied Mathematics, Tarbiat Modares University, Tehran, Iran

2. DERIVATION OF METHOD AND CONVERGENCE ANALYSIS

To illustrate the idea of the present paper, we take into account the following general iteration scheme:

$$x_{n+1} = x_n - H(t_n) f(x_n), \tag{3}$$

where

$$y_n = x_n - m\frac{f(x_n)}{f'(x_n)}, m \neq 1 \tag{4}$$

and $t_n = \dfrac{f'(y_n)}{f'(x_n)}$, is a (real valued) weight functions to be determined such that iterative method defined by (3) and (4) have a three order of convergence.

In fact, it is proven below that how we can obtain third-order family of methods from (2).

Theorem 1: *Let $\alpha \in I$ be a simple zero of a sufficiently differentiable function $f : I \to \mathbb{R}$ for an open interval, which contains x_0 as an initial approximation of α. If $H(t_n)$ satisfies the conditions:*

$$H(1) = 1, \quad H'(1) = \frac{1}{2m},$$

then the class of methods defined by (3) and (4) can be of third-order convergence.

Proof: If $\alpha \in I$ is the root and e_n is the error at $n\, th$ iteration, then $e_n = x_n - \alpha$ using Taylors expansion, we have

$$f(x_n) = f'(\alpha)[e_n + C_2 e_n^2 + C_3 e_n^3 + C_4 e_n^4 + O(e_n^5)], \tag{5}$$

$$f'(x_n) = f'(\alpha)[1 + 2C_2 e_n + 3C_3 e_n^2 + 4C_4 e_n^3 + O(e_n^4)], \tag{6}$$

where
$c_k = f^{(k)}(\alpha)/k!f'(\alpha), k = 2,3,...$

Furthermore, using (5), (6) and (4), we obtain

$$y_n = \alpha + (1-m)e_n + mC_2 e_n^2 + 2m(C_3 - 2C_1^2)e_n^3 + O(e_n^4). \tag{7}$$

Again by Taylor's series expanding around the simple root of (7), we have

$$\begin{aligned} f'(y_n) = f'(\alpha)[&-2mC_2 e_n + \frac{1}{2}m\left[12C_2^2 m^2 - 12C_3 + 6C_3 m\right)\right]e_n^2 \\ &- \frac{1}{6}m\left[-168C_2 C_3 + 72mC_2 C_3 + 96C_2^3 - 72C_4 + 24C_4 m^2\right)\right]e_n^3 + O(e_n^4). \end{aligned} \tag{8}$$

Now from (5)-(8) and (3) and by the aid of symbolic computation in Maple, we have the following general error equation

$$x_{n+1} = K_1 e_n + K_2 e_n^2 + O(e_n^3). \tag{9}$$

where

$$K_1 = 1 - H(1).$$

and

$$K_2 = \left[2mH'(1) - H(1)\right]C_2.$$

This shows that to make the third-order of convergence for Iterative schemes (3), (4), weight functions should be chosen as follows

$$H(1)=1, H'(1)=\frac{1}{2m}, \tag{10}$$

Thus using (9), we attain the following error equation of scheme (3), (4):

$$e_{n+1}=O(e_n^3).$$

This reveals the third-order convergence of our suggested class, and concludes the proof.

As can be observed from the iteration (10), iterative schemes (3) and (4) introduce a general class of third-order methods. By way of illustration let us look at two following cases that can be derived immediately.

Case 1: By choosing $m=1$ and $H(t)=\frac{1}{2}+\frac{1}{2}t,$ which satisfy (10), a new third-order method is given by

$$x_{n+1}=x_n-\frac{1}{2}\left[1+\frac{f'(x_n-f(x_n)/f'(x_n))}{f'(x_n)}\right]f(x_n), \tag{11}$$

Case 2: By choosing $m=-1$ and $H(t)=\frac{1}{2}-\frac{1}{2}t,$ which satisfy (10), a new third-order method is given by

$$x_{n+1}=x_n-\frac{1}{2}\left[1-\frac{f'(x_n+f(x_n)/f'(x_n))}{f'(x_n)}\right]f(x_n), \tag{12}$$

Per iteration of above methods requires two evaluations of the function and one of its derivatives. In terms of computational point of view, if we consider the definition of efficiency index in [4] as $p^{1/d}$, where p is the order of the method and d is the number of functional evaluations, the efficiency index of our class (3) and (4) is $3^{1/3}\approx 1.4422,$ which is better than that of Newton's method $2^{1/2}\approx 1.4142$.

3. NUMERICAL EXAMPLES

The main goal of this section is a comparison of the new methods (11) (NM1) and (12) (NM2). But before that, let us mention some known root-solvers as follows.

Chun's method [5] (CM):

$$x_{n+1}=x_n+\frac{f^{2}(x_n)}{f'^{2}(x_n)}-\frac{f(x_n)\left[f(x_n)+f'(x_n)\right]f'(y_n)}{f'^{3}(x_n)},$$

where

$$y_n=x_n-\frac{1}{2}\frac{f(x_n)}{f(x_n)-f'(x_n)}.$$

The iteration scheme called Chebyshev's method [6] (CH):

$$x_{n+1}=x_n-\left(1+\frac{L(x_n)}{2}\right)\frac{f(x_n)}{f'(x_n))},$$

Table 1: Comparison of various third-order convergent iterative methods

	IT	NFE	$f(x_{IT})$
$f_1, x_0 = 1.27$			
NM	5	10	2.1e-41
CM	4	12	20e-10
CH	4	12	3.0e-35
HM	3	9	1.2e-19
NM1	3	9	1.7e-33
NM2	3	9	2.4e-26
$f_2, x_0 = 2.0$			
NM	6	12	2.7e-16
CM	5	15	6.9e-42
CH	5	15	4.1e-31
HM	4	12	2.0e-34
NM1	4	12	5.4e-46
NM2	4	12	6.7e-36
$f_3, x_0 = 2.5$			
NM	6	12	9.1e-28
CM	5	15	3.6e-34
CH	4	12	7.1e-24
HM	5	15	9.3e-43
NM1	4	12	3.3e-39
NM2	3	9	2.2e-19
$f_4, x_0 = 1.7$			
NM	5	10	1.8e-16
CM	4	12	5.3e-28
CH	4	12	6.3e-24
HM	4	12	1.9e-30
NM1	4	12	2.3e-16
NM2	3	9	3.9e-42
$f_5, x_0 = 1.27$			
NM	7	14	4.3e-29
CM	5	15	2.7e-29
CH	5	15	8.4e-32
HM	4	12	6.7e-26
NM1	4	12	3.5e-27
NM2	4	12	3.6e-23

And the scheme called Halley's method [7] (HM):

$$x_{n+1} = x_n - \left(\frac{2}{2 - L(x_n)}\right)\frac{f(x_n)}{f'(x_n))},$$

where

$$L(x_n) = \frac{f(x_n) f''(x_n)}{f'^2(x_n)},$$

Also Newton method (NM) (1) is included to the compared methods.

The test instances in this case are given as follows:

$f_1(x) = x^3 + 4x^2 - 10,$
$f_2(x) = \sin^2(x) - x^2 + 1,$
$f_3(x) = x^2 - e^x - 3x + 2,$
$f_4(x) = e^{x^2+7x-30} - 1,$
$f_5(x) = x^{10} - 1.$

Moreover, all numerical computations in Table 1, have been carried out via Maple13 with 64 digit floating point arithmetic (Digits: = 64) by using the stopping criteria of $|f\ (x_n)| < 10^{-15}$.

In Table 1, (IT) stands for the number of iterations the number of functional evaluations; (NFE) stands for needed number of functional evaluations to reach the desire accuracy. Table 1 shows that our method takes less number of iterations in comparison with other methods.

4. CONCLUSION

In this paper, we have constructed an iterative family of methods for solving nonlinear equations. The class is also free from second order derivative. It has been shown by illustration that the proposed third order family of methods can be effectively used for solving nonlinear equations.

REFERENCES

[1] Kou. J., Li. Y., Modified Chebyshevs method free from second derivative for nonlinear equations, Appl. Math. Comp., 187, (2007), 1027-1032.

[2] J.R. Sharma, A composite third order Newton Steffensen method for solving nonlinear equations, Appl. Math. Comput. 169 (2005), 242-246.

[3] J. Kou, Y. Li, X. Wang, Modified Halley s method free from second derivative, Appl. Math. Comput.183 (2006) 704-708.

[4] C. Chun, Iterative methods improving Newton’s method by the decomposition method, Comput. Math. Appl. 50 (2005) 1559–1568..

[5] C. Chun, Some third-order families of iterative methods for solving nonlinear equations. Math. Comput. 140 (2003) 419-426.

[6] J.F. Traub, Iterative Methods for the Solution of Equations, Chelsea Publishing Company, New York, 1977.

[7] E. Halley, A new exact and easy method of finding the roots of equations generally, and that without any previous reduction, Phil. Roy. Soc. London 18 (1964) 136–145.

[8] E. Hansen, M. Patrick, A family of root finding methods, Numer. Math. 27 (1977) 257-269.

[9] C. Chun, Iterative Methods Improving Newton’s Method by the Decomposition Method, Computers Math. Applic. 50 (2005) 1559-1568.

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