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COMMUTATIVITY OF ALTERNATIVE LEFT s-UNITAL RINGS WITH $x[x^n, y] = y^r[x, y^m]y$

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ABSTRACT

Let R be an alternative left s-unital ring. In this paper we show that if $n > 1, m, r$ are fixed non-negative integers and an alternative ring R with unity 1 satisfies the polynomial identity (i) $x[x^n, y] = y^r[x, y^m]y$ for all x, y in R, then $C(R)$ is nil and if R is n-torsion free, then $N(R) \subseteq Z(R)$. Also we show that an alternative left s-unital ring R satisfying the polynomial identity (i) is commutative.

AMS Mathematics Subject Classification: *17.*

Key words: *Alternative ring, s-unital ring, center.*

INTRODUCTION

Abujabal and M.S. Khan [2] studied the commutativity of a left s-unital ring R satisfying the polynomial identity $x^t[x^n, y] = y^r[x, y^m]y^s$, for all x, y in R. In this section, we prove that if n>1, m, r are fixed nonnegative integers and an alternative ring R with unity 1 satisfies the polynomial identity (i) $x[x^n, y] = y^r[x, y^m]y$ for all x, y in R, then $C(R)$ is nil and if R is n-torsion free, then $N(R) \subseteq Z(R)$. Also we show that an alternative left s-unital ring R satisfying the polynomial identity (i) is commutative.

PRELIMINARIES

Throughout this section R denotes an alternative left s-unital ring, The center $Z(R)$ of R is defined as $Z(R) = \{z \in R \,/\, [z, R] = 0\}$ and a ring R is called a left (respectively right) s-unital ring if $x \in Rx$(respectively $x \in xR$) for each $x \in R$. Further R is called s-unital if it is both left as well as right s-unital. i.e., if $x \in xR \cap Rx$, for each $x \in R$. Here $C(R)$ the commutator ideal of R, $N(R)$ the set of all nilpotent elements of R, $N'(R)$ the set of all zero divisors in R, $GF(p)$ the Galois field with p elements and $(GF(p))_2$ the ring of all 2x2 matrices over $GF(p)$.

In order to prove our results, we shall require the following well-known results.

Lemma 1: Let R be a ring such that $[x, [x, y]] = 0$ for all x and y in R, then $[x^k, y] = kx^{k-1}[x, y]$ for any positive integer k.

Proof: We prove this by induction on k.

The identity $[x^k, y] = kx^{k-1}[x, y]$ is true for integer $k = 1$.

Suppose we assume that$[x^k, y] = kx^{k-1}[x, y]$.

Consider $[x^{k+1}, y] = [x^k x, y]$
$= x^k[x, y] + [x^k, y]x$
$= x^k[x, y] + kx^{k-1}[x, y]x$
$= x^k[x, y] + kx^k[x, y]$, since $[x, [x, y]] = 0$.
$= (k + 1)x^k[x, y]$, for all $k > 1$.

Therefore by induction for all positive integers k, $[x^k, y] = kx^{k-1}[x, y]$.

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Lemma 2[2, Lemma 2]: Let R be a ring with unity 1, and let x and y be elements in R. If $kx^m[x,y] = 0$ and $k(x+1)m[x,y] = 0$, for some integers $m \geq 1$ and $k \geq 1$, then necessarily $k[x,y] = 0$.

Lemma 3[6, Lemma 3]: Let R be a ring with unity 1, and let x and y be elements in R. If $(1-y^k)x = 0$, then $(1-y^{km})x = 0$, for some integers $k > 0$ and $m > 0$.

Lemma 4[1]: Let x and y be elements in a ring R. Suppose that there exists relatively prime positive integers m and n such that $m[x,y] = 0$ and $n[x,y] = 0$ then $[x,y] = 0$.

Lemma 5[3, Theorem 4(c)]: Let R be a ring with unity 1. Suppose that for each x in R there exists a pair n and m of relatively prime positive integers for which $x^n \in Z(R)$ and $x^m \in Z(R)$, then R is commutative.

Lemma 6[4, Theorem 18]: Let R be a ring and let $n>1$ be an integer. Suppose that $(x^n - x) \in Z(R)$, for all x in R, then R is commutative.

Lemma 7[5] If for every x and y in a ring R we can find a polynomial $p_{x,y}(t)$ with integral coefficients which depends on x and y such that $[x^2 p_{x,y}(x) - x, y] = 0$, then R is commutative.

MAIN RESULTS

Lemma 8: Let $n>0$, m and r be fixed non negative integers such that $(r, n, m) \neq (0, 1, 1)$ and let R be an alternative left s-unital ring satisfying the polynomial identity

$$x[x^n, y] = y^r[x, y^m]y, \text{ for all } x, y \text{ in } R, \tag{1}$$

then R is an s-unital ring.

Proof: Let x and y be arbitrary elements in R. Suppose that R is an alternative s-unital ring. Then there exists an element $e \in R$ such that $ex = x$ and $ey = y$. By replacing x by e in (1), we get

$$e[e^n, y] = y^r[e, y^m]y$$
$$e(e^n y - ye^n) = y^r(ey^m - y^m e)y$$
$$e(y - ye^n) = y^r(y^m - y^m e)y$$
$$ey - eye^n = (y^{r+m} - y^{r+m}e)y$$
$$y - ye^n = y^{r+m+1} - y^{r+m}ey$$
$$y - ye^n = y^{r+m+1} - y^{r+m+1}$$
$$y - ye^n = 0.$$

So, $y = ye^n \in yR$, for all y in R.

Thus R is an s-unital ring.

Lemma 9: Let $n>0$, r, m be fixed non-negative integers and let R be an alternative ring satisfying the polynomial identity $x[x^n, y] = y^r[x, y^m]y$, for all x, y in R, then $C(R)$ is nil.

Proof: Let $x = e_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $y = e_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Then x and y fail to satisfy the polynomial identity whenever $n>0$ except for $r = 0$, $m = 1$.

In this later case we can choose $x = e_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $y = e_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. Hence Lemma 7 ensures that

$$C(R) \subseteq N(R). \tag{2}$$

Lemma 10: Let $n>1$, m and R be fixed non-negative integers and let R be an alternative ring with unity 1. Suppose that R satisfies the polynomial identity $x[x^n, y] = y^r[x, y^m]y$, for all x, y in R. Further, if R is n-torsion free then $N(R) \subseteq Z(R)$.

Proof: Let $a \in N(R)$ then there exists a positive integer p such that $a^k \in Z(R)$ for all $k \geq p$ and p minimal. (3)

If $p = 1$ then $a \in Z(R)$.

Now suppose that $p>1$ and $b = a^{p-1}$.

By replacing x by b in the polynomial identity, we get

$b[b^n, y] = y^r[b, y^m]y$, for all x, y in R.

By using (3) and the fact that $(p-1)n \geq p$ for $n > 1$,

$$\text{we get } a^{p-1}\left[a^{(p-1)n}, y\right] = y^r[a^{p-1}, y^m]y$$
$$= y^r[b, y^m]y = 0, \text{ for all } y \text{ in } R. \quad (4)$$

By replacing x by $1+b$ in the polynomial identity, we get

$(1 + b)[(1 + b)^n, y] = y^r[1 + b, y^m]y$, for all y in R.

As $(1+b)$ is invertible and using (4), we get

$[(1 + b)^n, y] = 0$, for all y in R. (5)

By using (3) and (5), we get$[(1 + b)^n, y] = 0$.

That is, $[1 + nb), y] = 0$.

So, $n[b, y] = 0$, for all y in R.

Since R is n-torsion free, we get $[b, y] = 0$, for all y in R.

So, $b \in Z(R)$.

That is, $a^{\,p-1} \in Z(R)$.

This contradicts the minimality of p.

So we conclude that $p = 1$ and hence $a \in Z(R)$.

Therefore, $N(R) \subseteq Z(R)$. (6)

Combining (2) and (6), we get

$C(R) \subseteq N(R) \subseteq Z(R)$. (7)

Theorem 1: Let $n > 1$, m, r be fixed non-negative integers and let R be an alternative left s-unital ring satisfying the polynomial identity $x[x^n, y] = y^r[x, y^m]y$, for all x, y in R. Further, if R is n-torsion free, then R is commutative.

Proof: According to Lemma 8, R is an s-unital ring.

Therefore, in view of proposition 1 of [7], it is sufficient to prove the theorem for R with unity.

If $m = 0$, then (1) gives $x[x^n, y] = 0$, for all x, y in R.

Hence $nx^n[x, y] = 0$, for all x, y in R.

By replacing x by $x+1$ and applying Lemma 2, we obtain $n[x,y] = 0$, for all x, y in R.

Since R is n-torsion free, we get $[x, y] = 0$, for all x, y in R.

Therefore, R is commutative.

Now, we consider $m \geq 1$. Let $q = (2^{n+1} - 2)$. Then from (1) we have

$$\begin{aligned} qx[x^n, y] &= (2^{n+1} - 2)\, x[x^n, y] \\ &= 2^{n+1} x[x^n, y] - 2x[x^n, y] \\ &= (2x)\,[(2x)^n, y] - 2y^r[x, y^m]y \\ &= (2x)\,[(2x)^n, y] - y^r[(2x), y^m]y \\ &= 0. \end{aligned}$$

Therefore, $qx[x^n, y] = 0$.

So, $qnx^n[x, y] = 0$, for all *x, y* in *R*.

By replacing *qn* by *k* and using Lemma 2, we obtain $k[x, y] = 0$, for all *x, y* in *R*.

Thus $[x^k, y] = kx^{k-1}\,[x, y] = 0$, for all *x, y* in *R*.

So $x^k \in Z(R)$, for all *x, y* in *R*. (8)

Here we distinguish between the two cases.

Case (a): Let $m > 1$. Then from (1) and (7) we have,

$x[x^n, y] = m[x, y]y^{r+m}$, for all *x, y* in *R*.

By replacing *y* by y^m, we get $x[x^n, y^m] = m[x, y^m]y^{m(r+m)}$.

So, $mx[x^n, y]y^{m-1} = m[x, y^m]y^{m(r+m)}$, for all *x, y* in *R*.

By using (1), we get $my^r[x, y^m]y^m = m[x, y^m]y^{m(r+m)}$.

$m[x, y^m]y^{m+r} - m[x, y^m]y^{m(r+m)} = 0$.

$m[x, y^m]y^{r+m}\,(1 - y^{(m-1)(r+m)}) = 0$, for all *x, y* in *R*.

By using Lemma 3, we get

$m[x, y^m]y^{r+m}\,(1 - y^{k(m-1)(r+m)}) = 0$, for all *x, y* in *R*. (9)

Now by using (6) the polynomial identity (1) becomes

$nx^n[x, y] = my^{r+m}[x, y] = m[x, y]y^{r+m}$. (10)

It is well known that *R* is isomorphic to a subdirect sum of subdirectly irreducible rings R_i, $i \in I$, the Index set. Each R_i satisfies (1), (7), (8), (9) and (10) but not necessarily n-torsion free.

We consider the ring R_i, $i \in I$. Let *S* be the intersection of all nonzero ideals of R_i, then $S \neq (0)$ and $Sd = 0$, for any central zero-divisor *d*.

Let $a \in N'(R_i)$, the set of all zero-divisors of *R* then by using (9), we have

$m[x, a^m]a^{r+m}\,(1 - a^{k(m-1)(r+m)}) = 0$, for all *x* in R_i.

Suppose $m[x, a^m]a^{r+m} \neq 0$, for *x* in R_i.

So, $a^{k(m-1)(r+m)}$ and $1 - a^{k(m-1)(r+m)}$ are central zerodvisors.

That is, $(0) = S(1 - a^{k(m-1)(r+m)}) = S \neq (0)$, which is a contradiction.

Therefore $m[x, a^m]a^{r+m} = 0$, for all *x* in R_i. (11)

From (10) and (11), we have $nx^n[x, a^m] = m[x, a^m]a^{m(r+m)} = 0$.

Therefore by Lemma 2, we get $n[x, a^m] = 0$, for all *x* in R_i.

Hence $nm[x,a]a^{m-1} = 0$, for all for x in R_i

Now by Lemma 1, we have $n^2x^n[x,a] = n(nx^n[x,a])$

$$= nm[x,a]a^{r+m}, \text{for all } x \text{ in } R_i.$$

By replacing x by x+1 and applying Lemma 2, we get $n^2[x,a] = 0$, for all x in R_i. But $\left[x^{n^2}, a\right] = n^2x^{n^2-1}[x,a]$.

Therefore $\left[x^{n^2}, a\right] = 0$, for all x in R_i. and a in $N'(R_i)$. (12)

Let $c \in Z(R_i)$. Then by (1), we have

$$\begin{aligned}(c^{n+1} - c)x[x^n, y] &= c^{n+1}\, x[x^n, y] - cx[x^n, y].\\ &= (cx)\,[(cx)^n, y] - cy^r[x, y^n]y.\\ &= (cx)[(cx)^n, y] - y^r[(cx), y^m]y.\\ &= 0, \text{ for all } x, y \text{ in } R_i.\end{aligned}$$

By applying Lemma 1, we obtain $n(c^{n+1} - c)x^n[x^n, y] = 0$, for all x, y in R_i.

By using Lemma 2, we obtain $n(c^{n+1} - c)[x, y] = 0$ which implies

$(c^{n+1} - c)[x^n, y] = 0$, for all x, y in R_i and $c \in Z(R_i)$. (13)

In particular, by (8), we have

$(y^{k(n+1)} - y^k)[x^n, y] = 0$, for all x, y in R_i (14)

Consider $y \in R_i$. If $[x^n, y] = 0$ then clearly $\left[x^{n^2}, y^j - y\right] = 0$, for all positive integers j and x in R_i.

If $\left[x^{n^2}, y\right] \neq 0$ then $[x^n, y] \neq 0$. For $[x^n, y] = 0$ implies that $\left[x^{n^2}, y\right] = 0$, which is a contradiction.

Since $[x^n, y] \neq 0$, then by (14), $(y^{k(n+1)} - y^k)$ is a zerodivisor.

Therefore $(y^{kn+1} - y)$ is also a zerodivisor.

Hence by (12), $\left[x^{n^2}, y^{kn+1} - y\right] = 0$, for all x, y in R_i. (15)

As each R_i satisfies (15), the original ring R also satisfies (15). But R is n-torsion free. Therefore combining (15) with Lemma 1, we finally obtain $\left[x, y^{kn+1} - y\right] = 0$, for all x, y in R.

Thus R is commutative by Lemma 6.

Case (b): Let m = 1, Then we get $x[x^n, y] = y^r[x, y]y$, for all x, y in R.

Thus $nx^n[x, y] = [x, y]y^{r+1}$, for all x, y in R. (16)

By replacing x by x^n in (16), we get

$$\begin{aligned}nx^{n^2}[x^n, y] &= [x^n, y]y^{r+1}\\ &= nx^{n-1}[x, y]y^{r+1}\\ &= nx^n[x^n, y], \text{ for all } x, y \text{ in } R.\end{aligned}$$

Therefore, $n\left(1 - x^{(n-1)n}\right)x^n[x^n, y] = 0$, for all x, y in R.

By using Lemma 3, we get

$n(1 - x^{k(n-1)n})x^n[x^n, y] = 0$, for all x, y in R. (17)

As in case (a), if $a \in N'(R_i)$ then by (17), we obtain

$n(1 - a^{k(n-1)n})a^n[a^n, y] = 0$, for all $y \in R_i$.

By similar argument as in case (a), we can prove that

$na^n[a^n, y] = 0$, for all $y \in R_i$. (18)

Now we have $[a^n, y]y^{r+1} = na^{n^2}[a^n, y] = 0$.

By using Lemma 2, we get $[a^n, y] = 0$, for all y in R_i.

Therefore, $[a, y]y^{r+1} = a[a^n, y] = 0$.

So $[a, y] = 0$, for all y in R_i and $a \in N'(R_i)$. (19)

If $c \in Z(R_i)$, then as in case (a), we obtain $(c^{n+1} - c)\,[x, y] = 0$, for all x, y in R_i.

In particular by (8), we have $(x^{k(n+1)} - x^k)\,[x, y] = 0$, for all x, y in R_i.

If $[x, y] = 0$ for all x, y in R_i, then R satisfies $[x, y] = 0$, for all x, y in R. Therefore, R is commutative.

Now if for each x, y in R_i, $[x, y] \neq 0$ then $(x^{kn+1} - x) \in N'(R_i)$ and hence $(x^{kn+1} - x) \in N'(R)$.

But the identity (19) is satisfied by the original ring R.

Therefore, $(x^{kn+1} - x, y) = 0$, for all x, y in R.

Hence R is commutative by Lemma 6.

In Theorem 1, n-torsion free property is essential. Consider the following example :

Example: Let $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $C = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ be the elements of the ring of all 3x3 matrices over Z_2, the ring of integers mod 2. If R is the ring generated by the matrices A, B, C, then using Dooroh construction with Z_2, we obtain with unity 1. Then R is not commutative and satisfies $[x^2, y] = [x, y^2]$, for all x, y in R.

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