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Degree of Approximation of Function Belonging to the W (L^r, ξ(t)) class By (E, q) *A*-Product Means of its Fourier series

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ABSTRACT

In this paper, a new theorem on the degree of approximation of function belonging to the class W(L^r,ξ(t)) class by Euler and Matrix (E, q)A-product means of the Fourier series has been established.

Keywords: *Degree of approximation, W(L^r,ξ(t)) class of function,(E, q) mean, A-mean, (E, q) A-Product means, Fourier series, Lebesgue integral.*

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1. INTRODUCTION

The degree of approximation belonging to Lipα, Lip(α, p), Lip (ξ(t),),p) and W (L_p,ξ(t)()) classes by using Cesáro, Nörlund and generalized Nörlund summability methods has been discussed number of researchers like chandra [1], Holland, [3], Holland and Sahney [4], Khan [5], Qureshi [8]. Thereafter Lal and Kushwaha[6] discussed the degree of approximation of Lipschitz Function by (C,1)(E, q) product Summability methods. Recently Padhy *et al*. [7] discussed the degree of approximation of Fourier series by the product means (E, q)*A*. The Weighted W(L^r,ξ(t)) class is the generalization of Lip α, Lip(α,r) and Lip(ξ(t),r). Therefore, In the present paper, a new theorem on the degree of approximation of a functions belonging to the W(L^r, ξ(t)) class by (E, q)*A* -Product means of Fourier series .our theorem extands the result of Padhy *et al.* [7] on degree of approximation of Fourier by the product means (E, q)*A*.

2. DEFINITION AND NOTATION

Let *f* (x) be a 2π- periodic function and integrable in the Lebesgue sense. The Fourier series *f* (x) is given by

$$f(x) \sim \frac{1}{2}a_o + \sum_{n=1}^{\infty}\left(a_n \cos nx + b_n \sin nx\right) \equiv \sum_{n=0}^{\infty} A_n(x) \tag{2.1}$$

with n^{th} partial sum $s_n(f;\ x)$.

A function $f \in$ Lip α, if

$$f(x+t) - f(x) = O\left(|t|^{\alpha}\right), \text{ for } 0 < \alpha \le 1 \tag{2.2}$$

A function *f* (x) ∈ Lip (α, r), for $0 \le x \le 2\pi$ if

$$\left(\int_0^{2\pi} |\, f(x+t) - f(x)\,|^r\, dx\right)^{\frac{1}{r}} = O\left(|t|^{\alpha}\right), 0 < \alpha \le 1, r \ge 1 \tag{2.3}$$

Given a positive increasing function ξ(t) and integer r ≥ 1, *f* ∈ Lip (ξ(t), r), if

$$\left(\int_0^{2\pi} |\, f(x+t) - f(x)\,|^r\, dx\right)^{\frac{1}{r}} = O\left(\xi(t)\right) \tag{2.4}$$

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and $f \in W(L^r,\xi(t))$ if

$$\left(\int_0^{2\pi} \left| \left\{ f(x+t) - f(x) \right\} \sin^{\beta} x \right|^r dx \right)^{1/r} = O\big(\xi(t) \big),\ \beta \geq 0 \tag{2.5}$$

In case β=0, $W(L^r,\xi(t))$ class reduces to the Lip $(\xi(t),r)$ class and if $\xi(t)=t^{\alpha}$, then Lip $(\xi(t),r)$ class reduces to the Lip (α,r) class and if $r\to\infty$ then Lip (α,r) class reduces to the Lip α class. We observe that

$Lip\alpha \subseteq Lip(\alpha,r) \subseteq Lip(\xi(t),r) \subseteq W(L^r,\xi(t))$ *for* $0<\alpha\leq 1,\ r\geq 1.$ L_r- norm is defined by

$$\| f \|_r = \left(\int_0^{2\pi} | f(x) |^r dx \right)^{1/r},\quad r \geq 1 \tag{2.6}$$

L_∞-norm of a function $f: R\to R$ is defined by

$$\| f \|_\infty = \sup \left\{ | f(x) : x \in R | \right\} \tag{2.7}$$

The degree of approximation of a function *f*: R → R by trigonometric polynomials t_n of order n is defined by zygmund [11]

$$\left\| t_n - f \right\|_\infty = Sup \ \left\{ \left| t_n(x) - f(x) \right| : x \in R \right\} \tag{2.8}$$

Let $\sum_{n=0}^{\infty} u_n$ be a given infinite series with the sequence of its n^{th} partial sums $\{s_n\}$ and $A \equiv (a_{mn})$ be infinite matrix satisfying Töeplitz [10] conditions of regularity, i.e.

$$\lim_{m\to\infty} a_{mn} = 0 \tag{2.9}$$

$$\lim_{m\to\infty} \sum_{n=0}^{\infty} a_{mn} = 1 \tag{2.10}$$

$$\sup_m \sum_{n=0}^{\infty} |a_{mn}| < M, \quad \text{where M is constant.} \tag{2.11}$$

Then the sequence-to-sequence transformation

$$t_n = \sum_{\upsilon=0}^{n} a_{m\upsilon}\, s_\upsilon \tag{2.12}$$

Defines the sequence $\{t_n\}$ of matrix *A*- mean of a sequence $\{s_n\}$ generated by the sequence of coefficients (a_{mn}). The series $\sum_{n=0}^{\infty} u_n$ is said to be *A*-summable to the sum s by matrix method if limit of t_n is exists and is equal to s (Zygmund [11]) and we write $t_n \to s$, as $n\to\infty$.

The sequence-to-sequence transformation (Hardy [2]),

$$T_n = \frac{1}{(1+q)^n} \sum_{\upsilon=0}^{n} \binom{n}{\upsilon} q^{n-\upsilon} s_\upsilon \tag{2.13}$$

Defines the sequence $\{T_n\}$ of the (E, q) means of the sequence $\{s_n\}$.

If $T_n \to s$ as $n \to \infty$, then the series $\sum_{n=0}^{\infty} u_n$ is said to be (E, q) summable to the sum s.

Cleary (E, q) method is regular.

The (E, q) transform of the *A*- transform of $\{s_n\}$ is defined by

$$\tau_n = \frac{1}{(1+q)^n}\sum_{k=0}^{n}\binom{n}{k} q^{n-k} t_k$$

$$\tau_n = \frac{1}{\left(1+q\right)^n}\sum_{k=0}^{n}\binom{n}{k} q^{n-k}\left\{\sum_{\upsilon=0}^{k} a_{k\upsilon}\, s_\upsilon\right\} \tag{2.14}$$

If $\tau_n \to s$ as $n \to \infty$, then the series $\sum_{n=0}^{\infty} u_n$ is said to be (E, q) *A*- summable to the sum s. we use the following notations:

$$(i)\quad \varphi(t) = f(x+t) + f(x-t) - 2f(x)$$

$$(ii)\quad K_n(t) = \frac{1}{2\pi(1+q)^n}\sum_{k=0}^{n}\binom{n}{k} q^{n-k}\left\{\sum_{\upsilon=0}^{k} a_{k\upsilon}\,\frac{\sin\left(\upsilon+\frac{1}{2}\right)t}{\sin t/2}\right\}$$

3. KNOWN THEOREM

Dealing with the degree of approximation by the product mean (E,q)*A* of Fourier series, Padhy *et al.* [7] proved the following theorem:

Theorem 3.1: Let $A = (a_{mn})_{\infty\times\infty}$ be a regular matrix. If f is a 2π– periodic function of class Lipα, then the degree of approximation by the product (E, q)*A*- summability means of its Fourier series is given by

$$\| \tau_n - f \|_\infty = O\left(\frac{1}{\left(n+1\right)^\alpha}\right), \quad 0<\alpha<1$$

$$where \quad \tau_n = \frac{1}{\left(1+q\right)^n}\sum_{k=0}^{n}\binom{n}{k} q^{n-k}\left\{\sum_{\upsilon=0}^{k} a_{k\upsilon}\, s_\upsilon\right\}.$$

4. MAIN THEOREM

In this paper our aim is to generalize the above result of Padhy *et al.* [7], In fact, we prove the following theorem:

Let $A=(a_{mn})$ be an infinite matrix. If f is a 2π- periodic, Lebesgue integrable function, belongs to W(L^r, ξ(t)) class, $r \geq 1$, then the degree of approximation by product (E, q)*A*-summability means of its Fourier series (2.1) is given by

$$\| \tau_n - f(x) \|_r = O\left[\left(n+1\right)^{\beta+\frac{1}{r}}\, \xi\left(\frac{1}{n+1}\right)\right] \tag{4.1}$$

Provided ξ(t) satisfies the following conditions:

$$\left\{\frac{\xi(t)}{t}\right\} \text{ be a decreasing sequence,} \tag{4.2}$$

$$\left\{\int_0^{\frac{1}{n+1}}\left(\frac{t\,|\varphi(t)|}{\xi(t)}\right)^r \sin^{\beta r} t\, dt\right\}^{1/r} = O\left(\frac{1}{n+1}\right) \tag{4.3}$$

and

$$\left\{\int_{\frac{1}{n+1}}^{\pi}\left(\frac{t^{-\delta}\,|\varphi(t)|}{\xi(t)}\right)^r dt\right\}^{1/r} = O\left((n+1)^\delta\right) \tag{4.4}$$

where δ is an arbitrary number such that s (1-δ) - 1 > 0, $\frac{1}{r}+\frac{1}{s}=1$ Such that $1\le r<\infty$, Conditions (4.3) and (4.4) holds uniformly in x .

5. Lemmas: For the proof of our theorem, we required following lemmas:

Lemma 5.1: $\left|\,K_n(t)\,\right| = O(n+1); \quad for\ o \le t \le \frac{1}{n+1}.$

Proof: $For \quad 0 \le t \le \frac{1}{n+1},$ we have $\sin nt \le n \sin t$ then

$$|K_n(t)| = \frac{1}{2\pi(1+q)^n}\left|\left[\sum_{k=0}^{n}\binom{n}{k} q^{n-k}\left\{\sum_{\upsilon=0}^{k} a_{k\upsilon}\frac{\sin\left(\upsilon+\frac{1}{2}\right)t}{\sin t/2}\right\}\right]\right|$$

$$\le \frac{1}{2\pi(1+q)^n}\left|\sum_{k=0}^{n}\binom{n}{k} q^{n-k}\left\{\sum_{\upsilon=0}^{k} a_{k\upsilon}\frac{(2\upsilon+1)\sin {t/2}}{\sin {t/2}}\right\}\right|$$

$$\le \frac{1}{2\pi(1+q)^n}\left|\sum_{k=0}^{n}\binom{n}{k} q^{n-k}(2k+1)\left\{\sum_{\upsilon=0}^{k} a_{k\upsilon}\right\}\right|$$

$$\le \frac{M(2n+1)}{2\pi(1+q)^n}\left|\sum_{k=0}^{n}\binom{n}{k} q^{n-k}\right|$$

$$= O(n+1).$$

This complete proof of the lemma (5.1).

Lemma 5.5: $K_n(t)| = O\left(\frac{1}{t}\right),\ for\ \frac{1}{n+1} \le t \le \pi.$

Proof: $For\ \frac{1}{n+1} \le t \le \pi,\ we\ have \quad by \quad Jordans \quad Lemma, \quad \sin\left(\frac{t}{2}\right) \ge \frac{t}{\pi}, \quad \sin nt \le 1, then$,

$$|K_n(t)| = \frac{1}{2\pi(1+q)^n}\left|\left[\sum_{k=0}^{n}\binom{n}{k} q^{n-k}\left\{\sum_{\upsilon=0}^{k} a_{k\upsilon}\frac{\sin\left(\upsilon+\frac{1}{2}\right)t}{\sin t/2}\right\}\right]\right|$$

$$\le \frac{1}{2\pi(1+q)^n}\left|\left[\sum_{k=0}^{n}\binom{n}{k} q^{n-k}\left\{\sum_{\upsilon=0}^{k} a_{k\upsilon}\frac{1}{t/\pi}\right\}\right]\right|$$

$$= \frac{\pi}{2\pi t(1+q)^n}\left|\left[\sum_{k=0}^{n}\binom{n}{k} q^{n-k}\left\{\sum_{\upsilon=0}^{k} a_{k\upsilon}\right\}\right]\right|$$

$$\leq \frac{M}{2t(1+q)^n}\left|\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\right|$$

$$=O\left(\frac{1}{t}\right)..$$

This complete proof of the lemma (5.2).

6. PROOF OF THE MAIN THEOREM

Following Titchmarsh [9] and using Riemann-Lebesgue theorem, that the n^{th} partial sum s_n of Fourier series (2.1) at t=x may be written as :

$$s_n(f;x)-f(x)=\frac{1}{2\pi}\int_0^{\pi}\phi(t)\frac{\sin\left(n+\frac{1}{2}\right)t}{\sin\left(t/2\right)}\,dt$$

Therefore, the *A*-transform of $s_n(f;x)$ is given by

$$t_n-f(x)=\frac{1}{2\pi}\int_0^{\pi}\phi(t)\sum_{k=0}^{n}a_{nk}\frac{\sin\left(k+\frac{1}{2}\right)t}{\sin\left(t/2\right)}\,dt$$

Now denoting (E, q)*A*- transform of $s_n(f;x)$ by τ_n, we have

$$\tau_n-f(x)=\frac{1}{2\pi(1+q)^n}\int_0^{\pi}\phi(t)\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\left\{\sum_{\upsilon=0}^{k}a_{k\upsilon}\frac{\sin\left(\upsilon+\frac{1}{2}\right)t}{\sin\left(t/2\right)}\right\}dt$$

$$=\int_0^{\pi}\phi(t)K_n(t)\,dt$$

$$=\left[\int_0^{\frac{1}{n+1}}+\int_{\frac{1}{n+1}}^{\pi}\right]\phi(t)K_n(t)\,dt$$

$$=I_1+I_2. \quad \text{(Say)} \tag{6.1}$$

We consider

$$|I_1|\leq\int_0^{\frac{1}{n+1}}|\phi(t)||K_n(t)|\,dt$$

Now using Hölder's inequality and the fact that $\phi(t)\in W(L^r,\xi(t))$, condition(4.3), Lemma(5.1) and second mean value theorem for integrals,we have

$$|I_1|\leq\left[\int_0^{\frac{1}{n+1}}\left\{\frac{t\,|\phi(t)|\sin^{\beta}t}{\xi(t)}\right\}^r dt\right]^{1/r}\left[\int_0^{\frac{1}{n+1}}\left\{\frac{\xi(t)\,|K_n(t)|}{t\sin^{\beta}t}\right\}^s dt\right]^{1/s}$$

$$= O\left(\frac{1}{n+1}\right)\left[\int_0^{\frac{1}{n+1}}\left\{\frac{(n+1)\,\xi(t)}{t^{1+\beta}}\right\}^s dt\right]^{1/s}$$

$$= O\left\{\xi\left(\frac{1}{n+1}\right)\right\}\left[\int_{\in}^{\frac{1}{n+1}}\frac{dt}{t^{(1+\beta)s}}\right]^{1/s} \quad \text{for some } 0 < \in < \frac{1}{n+1}$$

$$= O\left[(n+1)^{\beta+\frac{1}{r}}\,\xi\left(\frac{1}{n+1}\right)\right]; \quad \text{Since } r^{-1}+s^{-1}=1 \qquad (6.2)$$

Now using Hölder's inequality, sin ≥ (2t/π) , condition (4.2), (4.4), lemma (5.2) and second mean value theorem for integrals, we have

$$|I_2| \le \int_{\frac{1}{n+1}}^{\pi}|\phi(t)||K_n(t)|\,dt$$

$$|I_2| \le \left[\int_{\frac{1}{n+1}}^{\pi}\left(\frac{t^{-\delta}\,|\phi(t)|\sin^\beta t}{\xi(t)}\right)^r dt\right]^{\frac{1}{r}}\left[\int_{\frac{1}{n+1}}^{\pi}\left(\frac{\xi(t)|K_n(t)|}{t^{-\delta}\sin^\beta t}\right)^s dt\right]^{\frac{1}{s}}$$

$$= O\left\{(n+1)^\delta\right\}\left[\int_{\frac{1}{n+1}}^{\pi}\left(\frac{\xi(t)}{t^{\beta+1-\delta}}\right)^s dt\right]^{\frac{1}{s}}$$

$$= O\left\{(n+1)^\delta\right\}\left\{\int_{\frac{1}{\pi}}^{n+1}\left[\frac{\xi\left(\frac{1}{y}\right)}{t^{\delta-1-\beta}}\right]^s\frac{dy}{y^2}\right\}^{\frac{1}{s}} \quad ;\left(put\ t=\frac{1}{y}\right)$$

$$= O\left\{(n+1)^\delta\ \xi\left(\frac{1}{n+1}\right)\right\}\left[\int_{\frac{1}{\pi}}^{n+1}\frac{dy}{y^{s(\delta-1-\beta)+2}}\right]^{\frac{1}{s}}$$

$$= O\left\{(n+1)^\delta\,\xi\left(\frac{1}{n+1}\right)\right\}\left[\frac{(n+1)^{s(1+\beta-\delta)-1}-\pi^{s(\delta-1-\beta)+1}}{s(1+\beta-\delta)-1}\right]^{\frac{1}{s}}$$

$$= O\left\{(n+1)^\delta\,\xi\left(\frac{1}{n+1}\right)\right\}\left[(n+1)^{(1+\beta-\delta)-\frac{1}{s}}\right]$$

$$= O\left\{(n+1)^{\beta+\frac{1}{r}}\,\xi\left(\frac{1}{n+1}\right)\right\}, \quad \text{since } r^{-1}+s^{-1}=1 \qquad (6.3)$$

At last, combining (6.1),(6.2) and (6.3), we have

$$|\tau_n - f(x)| = O\left[(n+1)^{\beta+\frac{1}{r}}\,\xi\left(\frac{1}{n+1}\right)\right]$$

Using the L_r-norm, we get

$$\| \tau_n - f(x) \|_r = \left(\int_0^{2\pi} | \tau_n - f(x) |^r dx \right)^{\frac{1}{r}} (1 \le r < \infty)$$

$$= O\left[\int_0^{2\pi} \left\{ (n+1)^{\beta+\frac{1}{r}} \xi\left(\frac{1}{n+1}\right) \right\}^r dx \right]^{\frac{1}{r}}$$

$$= O\left[(n+1)^{\beta+\frac{1}{r}} \xi\left(\frac{1}{n+1}\right) \right] \left\{ \int_0^{2\pi} dx \right\}^{\frac{1}{r}}$$

$$\| \tau_n - f(x) \|_r = O\left[(n+1)^{\beta+\frac{1}{r}} \xi\left(\frac{1}{n+1}\right) \right], 1 \le r < \infty.$$

This completes the proof of the our theorem.

7. APPLICATIONS

The following corollaries can be derived from the theorem:

Corollary 7.1: If β=0 and ξ(t)= t^α ,0 <α ≤ 1 , then the W (Λ^ρ,ξ(τ)) class, $r \ge 1$, reduces to Lip(α,r) class and the degree of approximation of a function $f(x)$, 2π- periodic function f ∈ Lip(α,r), $\frac{1}{r}$ < α ≤ 1 is given by

$$\| \tau_n - f(x) \|_r = O\left(\frac{1}{(n+1)^{\alpha-\frac{1}{r}}} \right).$$

Corollary7.2: If r→∞ in corollary 7.1, for 0<α<1,then the Lip(α, r) class reduces to Lip α class and the degree of approximation of a function f(x) , 2π- periodic function f ∈ Lip α, 0< α < 1 is given by

$$\| \tau_n - f(x) \|_\infty = O\left(\frac{1}{(n+1)^{\alpha}} \right), \quad 0 < \alpha < 1.$$

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