International Journal of Mathematical Archive-4(8), 2013, 289-292
IJMA Available online through www.ijma.info ISSN 2229 – 5046

T-stabilities of Mann Ishikawa Iterations and Multistep Iteration for Non expansive Mapping

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(Received on: 08-05-13; Revised & Accepted on: 07-08-13)

ABSTRACT

We show that the T-stability of Mann, Ishikawa iterations are equivalent to the T-stability of a multistep iteration.

Mathematical Subject Classification: *47H10.*

Keywords: *Mann, Ishikawa, Noor and Multistep iterations; T-stability.*

1. INTRODUCTION AND PRELIMINARIES

Let E be a Banach space, K a nonempty, convex subset of E, and T a self map of K. Three most popular iteration procedures for obtaining fixed points of T, if they exist, are Mann iteration [1], defined by

$$u_1 \in K,\ u_{n+1} = (l-\alpha_n)u_n + \alpha_n Tu_n,\ n \geq 1, \tag{1.1}$$

Ishikawa iteration [2], defined by

$$z_1 \in K,\ z_{n+1} = (l-\alpha_n)z_n + \alpha_n Ty_n,$$

$$y_n = (l-\beta_n)z_n + \beta_n Tz_n,\ n \geq 1, \tag{1.2}$$

Noor iteration [10], defined by

$$v_1 \in K,\ v_{n+1} = (l-\alpha_n)v_n + \alpha_n Tw_n,$$

$$w_n = (l-\beta_n)v_n + \beta_n Tt_n,$$

$$t_n = (l-\gamma_n)v_n + \gamma_n Tv_n,\ n \geq 1, \tag{1.3}$$

for certain choices of $\{\alpha_n\}$,$\{\beta_n\}$ and $\{\gamma_n\} \subset [0,l]$.

The multistep iteration [9], arbitrary fixed order $p \geq 2$, defined by

$$x_{n+1} = (l-\alpha_n)x_n + \alpha_n Ty'_n,$$

$$y'_n = (l-\beta'_n)x_n + \beta'_n Ty_n^{i+1},\ i = 1, 2,....,p-2 \tag{1.4}$$

$$y_n^{p-1} = (l-\beta_n^{p-1})x_n + \beta_n^{p-1} Tx_n,$$

where the sequence $\{\alpha_n\}$ is such that for all $n \in N$

$$\{\alpha_n\} \subset (0, 1)\ \lim_{n\to\infty} \alpha_n = 0, \sum_{n=1} \alpha_n = \infty \tag{1.5}$$

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and for all $n \in N$

$\{\beta_n^i\} \subset [0, 1),\ 1 \le i \le p - 1,\ \lim_{n\to\infty} \beta_n^1 = 0.$ (1.6)

In the above taking p = 3 in (1.4) we obtain iteration (1.3). Taking p = 2 in (1.4) we obtain iteration (1.2).

Let K be a closed convex bounded subset of normed linear space $E = (E, \|.\|)$ and T self-mappings of E. Then T is called nonexpansive on K if

$$\|Tx - Ty\| \le \|x - y\| \tag{1.7}$$

for all $x, y \in K$. Let $F(T):=\{x \in K: Tx = x\}$ be denoted as the set of fixed points of a mapping T.

Let X be a normed space and T a nonexpansive selfmap of X. Let x_0 be a point of X, and assume that $x_{n+1} = f(T, x_n)$ is an iteration procedure, involving T, which yields a sequence $\{x_n\}$ of point from X. Suppose $\{x_n\}$ converges to some $x^* \in F(T):=\{x \in K:Tx = x\} \neq \phi$. Let $\{\xi_n\}$ be an arbitrary sequence in X, and we consider $\Omega_n = \|\xi_{n+1} - f(T, \xi_n)\|$ for n = 1, 2, 3...

Definition 1.1: [3] If $((\lim_{n\to\infty} \Omega_n = 0) \Rightarrow (\lim_{n\to\infty} \xi_n = P))$ then the iteration procedure $x_{n+1} = f(T, x_n)$ is said to be T -stable with respect to T.

Remark 1.2: [3] In practice, such a sequence $\{\xi_n\}$ could arise in the following way. Let x_0 be a point in X. Set $x_{n+1} = f(T, x_n)$. Let $\xi_0 = x_0$. Now $x_l = f(T, x_o)$. Because of rounding or discretization in the function T, a new value ξ_l approximately equal to x_l might be obtained instead of the true value of $f(T, x_o)$. Then to approximate ξ_2, the value $f(T, \xi_1)$ is computed to yields ξ_2, an approximation of $f(T, \xi_1)$. This computation is continued to obtain $\{\xi_n\}$ an approximate sequence of $\{x_n\}$.

A reasonable conjecture is that the Ishikawa iteration and the corresponding Mann iteration are equivalent for all maps for which either method provides convergence to a fixed point. In an attempt to verify this conjecture the authors, in a series of papers [4-9] have shown the equivalence for several classes of maps.

In [11], Rhoades and Soltuz considered the equivalence between T -stabilities of (1.1) and (1.2). More precisely, they proved the following theorem.

Theorem (Rhoades and Soltuz[11]): Let X be a normed space and $T: X\rightarrow X$ a map. Then the following are equivalent:
(i) for all $\{\alpha_n\} \subset (0,1)$, $\{\beta_n\} \subset [0,1)$ satisfying (1.5) and (1.6), the Ishikawa iteration (1.2) is T -stable,
(ii) for all $\{\alpha_n\} \subset (0,1)$, satisfying (1.5), the Mann iteration (1.1) is T-stable.

In this paper, we shall prove the equivalence between T -stabilities of (1.4) and (1.2). Consequently, we shall prove the equivalence between T -stabilities of (1.4) and (1.1). Throughout this paper, we shall assume that both Mann, Ishikawa and multistep iterations converge to a fixed point of T.

2. THE T –STABILITIES

Let $\{u_n\}$ be the Mann iteration, $\{z_n\}$ be the Ishikawa iteration and $\{x_n\}$ be the Multistep iteration. Let $\{x_n\}$, $\{z_n\}$ and $\{u_n\} \subset X$ be such that $x_o=z_o=u_o$ and let $(\alpha_n)_n \subset(0,1)$, $(\beta_n)_n \subset[0,1)$ and $(\beta_n^i)_n \subset[0,1)$ satisfy (1.5) and (1.6), and

$$x_{n+1} = (1-\alpha_n) x_n + \alpha_n T y_n^i,$$

$$y_n^i = (1-\beta_n^i) x_n + \beta_n^i T y_n^{i+1},\ i = 1, 2, \ldots, p - 2 \tag{2.1}$$

$$y_n^{p-1} = (1 - \beta_n^{p-1}) x_n + \beta_n^{p-1} T x_n.$$

We consider the following nonnegative sequences, for all $n \in N$:

$$\epsilon_n = \|x_{n+1} - ((1 - \alpha_n) x_n + \alpha_n T y_n^i)\|, \tag{2.2}$$

$$\delta_n = \|z_{n+1} - ((1 - \alpha_n) z_n + \alpha_n T y_n)\|, \tag{2.3}$$

$$\gamma_n = \|u_{n+1} - ((1 - \alpha_n) u_n + \alpha_n T u_n)\| \tag{2.4}$$

Definition 2.1: Definition 1.1 for (2.2), (2.3) and (2.4) gives:

(i) If $\lim_{n\to\infty} \epsilon_n = 0$ implies that $\lim_{n\to\infty} x_n = x^*$, then the Multistep iteration (1.4), is said to be T -stable.

(ii) If $\lim_{n\to\infty} \delta_n = 0$ implies that $\lim_{n\to\infty} z_n = x^*$ then the Ishikawa iteration (1.2) is said to be T -stable.

(iii) If $\lim_{n\to\infty} \lambda_n = 0$ implies that $\lim_{n\to\infty} u_n = x^*$ then the Mann iteration (1.1) is said to be T -stable.

Remark 2.2: Let X be a normed space and T: X $\to$ X a nonexpansive map. The following are equivalent:

(i) for all $\{\alpha_n\} \subset (0,1)$, $\{\beta_n\} \subset [0,1)$ and $\{\beta_n^i\} \subset [0,1)$ satisfying (1.5) and (1.6), the Multistep iteration is T -stable,

(i*) for all $\{\alpha_n\} \subset (0,1)$, $\{\beta_n\} \subset [0,1)$ and $\{\beta_n^i\} \subset [0,1)$ satisfying (1.5) and (1.6), $\forall \{x_n\} \subset X$:

$$\lim_{n\to\infty} \epsilon_n = \lim_{n\to\infty} \|x_{n+1} - ((1 - \alpha_n) x_n + \alpha_n T y_n^i)\| = 0 \Rightarrow \lim_{n\to\infty} x_n = x^*, \tag{2.5}$$

Remark 2.3: Let X be a normed space and T: X $\to$ X a nonexpanslve map. The following are equivalent:
(ii) for all $\{\alpha_n\} \subset (0,1)$, $\{\beta_n\} \subset [0,1)$ satisfying (1.5) and (1.6), the Ishikawa iteration is T -stable,
(ii*) for all $\{\alpha_n\} \subset (0,1)$, $\{\beta_n\} \subset [0,1)$ satisfying (1.5) and (1.6), $\forall \{z_n\} \subset X$:

$$\lim_{n\to\infty} \delta_n = \lim_{n\to\infty} \|z_{n+1} - ((1 - \alpha_n) z_n + \alpha_n T y_n)\| = 0 \Rightarrow \lim_{n\to\infty} z_n = x^*, \tag{2.6}$$

Remark 2.4: Let X be a normed space and T: X$\to$ X a nonexpansive map. The following are equivalent:
(iii) for all $\{\alpha_n\} \subset (0,1)$ satisfying (1.5), the Mann iteration is T -stable,
(iii*) for all $\{\alpha_n\} \subset (0,l)$ satisfying (1.5), $\forall \{u_n\} \subset X$:

$$\lim_{n\to\infty} \gamma_n = \lim_{n\to\infty} \|u_{n+1} - ((1 - \alpha_n) u_n + \alpha_n T u_n)\| = 0 \Rightarrow \lim_{n\to\infty} u_n = x^*, \tag{2.7}$$

Theorem 2.5: Let X be a normed space and T: X $\to$ X a nonexpanslve map. Then the following are equivalent:

(i) for all $\{\alpha_n\} \subset (0,1)$, $\{\beta_n\} \subset [0,1)$ and $\{\beta_n^i\} \subset [0,1)$ satisfying (1.5) and (1.6), the Multistep iteration is T -stable,

(ii) for all $\{\alpha_n\} \subset (0,1)$, $\{\beta_n\} \subset [0,l)$ satisfying (1.5) and (1.6), the Ishikawa iteration is T -stable,

(iii) for all $\{\alpha_n\} \subset (0,1)$ satisfying (1.5), the Mali1 iteration is T -stable.

Proof: Let $\lim_{n\to\infty} x_n = x^*$. Since the Mann, Ishikawa and multistep iterations converge, M < ∞. Remarks 2.2, 2.3 and 2.4 assure that (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) is equivalent to (i*) $\Leftrightarrow$ (ii*) $\Leftrightarrow$ (iii*). (ii*) $\Rightarrow$ (iii*) proved in [11]. We shall prove that (i*) $\Leftrightarrow$ (ii*). Therefore, we shall prove that (i*) $\Leftrightarrow$ (ii*) $\Leftrightarrow$ (iii*).

We prove (i*) $\Rightarrow$ (ii*). The proof is complete if we consider

$$\begin{aligned}
0 \le \|\delta_n\| &= \|z_{n+1} - ((1 - \alpha_n) z_n + \alpha_n T y_n)\| \\
&\le \|z_{n+1} - x^*\| + \|-((1 - \alpha_n) z_n + \alpha_n T y_n) + x^*\| \\
&\le \|z_{n+1} - x^*\| + (1 - \alpha_n)\|z_n - x^*\| + \alpha_n \|((1 - \beta_n)(z_n - x^*) + \beta_n(z_n - x^*)\| \\
&\le \|z_{n+1} - x^*\| + (1 - \alpha_n)\|z_n - x^*\| + \alpha_n \|((1 - \beta_n)(z_n - x^*) + \alpha_n\beta_n\|z_n - x^*\| \\
&= \|z_{n+1} - x^*\| + \|z_n - x^*\| \to 0 \text{ as } n \to \infty.
\end{aligned} \tag{2.8}$$

Thus, for a $\{z_n\}$ satisfying $\lim_{n\to\infty} \delta_n = \lim_{n\to\infty} \|z_{n+1} - ((1 - \alpha_n) z_n + \alpha_n T y_n)\| = 0$, we have shown that $\lim_{n\to\infty} z_n = x^*$.

Conversely, we prove (ii*) $\Rightarrow$ (i*). The proof is complete if we consider

$$0 \le \epsilon_n = \|x_{n+1} - ((1 - \alpha_n) x_n + \alpha_n T y_n^i)\|$$

$\leq \|x_{n+1} - x^*\| + \|x^* - (1 - \alpha_n) x_n + \alpha_n T y_n^i)\|$

$\leq \|x_{n+1} - x^*\| + (1 - \alpha_n) \|x_n - f\| + \alpha_n \|(1 - \beta_n^i) x_n + \beta_n^i T y_n^2 - f\|$

$\leq \|x_{n+1} - x^*\| + (1 - \alpha_n) \|x_n - f\| + \alpha_n \|(1 - \beta_n^i)(x_n - f) + \beta_n^i [P(1 - \beta_n^2) x_n + \beta_n^2 T x_n) - f]\|$

$\leq \|x_{n+1} - x^*\| + (1 - \alpha_n) \|x_n - f\| + \alpha_n \|(1 - \beta_n^i)(x_n - f) + \beta_n^i [(1 - \beta_n^2) x_n + \beta_n^2 T x_n) - f]\|$

$\leq \|x_{n+1} - x^*\| + (1 - \alpha_n) \|x_n - f\| + \alpha_n (1 - \beta_n^i) \| x_n - f\| + \alpha_n \beta_n^i \|(1 - \beta_n^2) x_n + \beta_n^2 T x_n) - f\|$

$\leq \|x_{n+1} - x^*\| + (1 - \alpha_n) \|x_n - f\| + \alpha_n (1 - \beta_n^i) \| x_n - f\| + \alpha_n \beta_n^i \|(1 - \beta_n^2)(x_n - f) + \beta_n^2 (x_n - f)\|$

$\leq \|x_{n+1} - x^*\| + (1 - \alpha_n) \|x_n - f\| + \alpha_n (1 - \beta_n^i) \| x_n - f\| + \alpha_n \beta_n^i \|(1 - \beta_n^2)\| x_n - f\| + \alpha_n \beta_n^1 \beta_n^2 \|x_n - f\|$

$= \|x_{n+1} - x^*\| + \|x_n - f\| \to 0$ as $n \to \infty$. (2.9)

Thus, for a $\{x_n\}$ satisfying $\lim_{n\to\infty} \epsilon_n = \lim_{n\to\infty} \|x_{n+1} - ((1 - \alpha_n) x_n + \alpha_n T y_n^i)\| = 0$, we have shown that $\lim_{n\to\infty} x_n = x^*$. Then this complete the proof of Theorem 2.5.

Set in (1.1), (1.2) and (1.4), $T: = T^n$ to obtain the modified Mann, modified Ishikawa and modified multistep iterations. We suppose that both modified Mann and modified Ishikawa also modified multistep iterations converge to a fixed point of T. Note that Definition 2.1, Remarks 2.2 and 2.3, and Theorem 2.5 hold in this case too.

Corollary 2.6: Let X be a normed space and $T: X \to X$ a nonexpansive map. The following are equivalent:

(i) for all $\{\alpha_n\} \subset (0,1)$, $\{\beta_n\} \subset [0,1)$ and $\{\beta_n^i\} \subset [0,1)$ satisfying (1.5) and (1.6), the modified Multistep iteration is T -stable,

(ii) for all $\{\alpha_n\} \subset (0,1)$, $\{\beta_n\} \subset [0,1)$ satisfying (1.5) and (1.6), the modified Ishikawa iteration is T -stable,

(iii) all $\{\alpha_n\} \subset (0,1)$ satisfying (1.5), the Mann modified iteration is T -stable.

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Source of support: Nil, Conflict of interest: None Declared