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THE CUBIC RATE OF CONVERGENCE OF GENERALIZED EXTRAPOLATED NEWTON – RAPHSON METHOD FOR SOLVING NONLINEAR EQUATIONS

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ABSTRACT

In this paper, we present the rate of convergence of Generalized Extrapolated Newton – Raphson (GEN-R) Method to find the multiple roots of non-linear equations. Numerical examples are given to illustrate the performance of the presented methods.

Key Words: *N – R Method, Iterative method.*

AMS Classification: *65 H 10.*

SECTION – 1: INTRODUCTION

One of the most important problems in Numerical Analysis is solving non-linear equations. To solve these equations, we can use iterative methods such as Newton-Raphson (NR) Method and its variants. In this paper, we consider iterative methods to find the multiple root of a non-linear equation

$$f(x) = 0 \tag{1.1}$$

where f may be algebraic, transcendental or combined of both.

Let η be a root of the equation (1.1) with multiplicity m and let $m>1$. Then,

$$f(\eta) = f^{|}(\eta) = \ldots = f^{(m-1)}(\eta) = 0 \text{ and } f^{m}(\eta) \neq 0 \tag{1.2}$$

and also the equation (1.1) can be expressed as

$$f(x) = (x-\eta)^{m}\ k(x) = 0 \tag{1.3}$$

where $k(x)$ is bounded and $k(\eta) \neq 0$.

As noted by Jain *et. al* [1] when the equation (1.1) has a multiple root, most of the methods exists for solving a simple root of $f(x)=0$ have only linear rate of convergence. If η be a root of (1.1) with multiplicity m, then the classical Generalized Newton - Raphson method (GN-R) is defined as

$$x_{n+1} = x_n - m\frac{f(x_n)}{f^{|}(x_n)} \quad (n=0, 1, 2\ldots) \tag{1.4}$$

This is a powerful and well-known iterative method known to convergence quadratically.

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The Generalized Extrapolated Newton – Raphson Method (GEN-R) considered by Vatti VBK *et al.* [3] for the multiple root of equation (1.1) is given as

$$x_{n+1} = x_n - m\,\alpha_n \frac{f(x_n)}{f^{|}(x_n)} \tag{1.5}$$

As it is well-known that any iterative method of the form $x_{n+1} = \phi(x_n)$ converges if $|\phi^{|}(x_n)| < 1$ for all x in I. Hence the method (1.5) converges under the condition

$$\mu = |1 - m\alpha_n + m\alpha_n\,\omega_n| < 1 \text{ for all } x \text{ in } I \tag{1.6}$$

$$\text{where } \omega_n = \frac{f(x_n)\,f^{||}(x_n)}{[f^{|}(x_n)]^2} \tag{1.7}$$

The function $f(x)$, (1.3) in the immediate neighborhood of $x = \eta$, can be written as

$$f(x) = k.\,(x - \eta)^{m}$$

where $k \simeq k(\eta)$ is effectively constant.

Then $\quad f^{|}(x) = k.\,m\,(x - \eta)^{m-1}$

$$f^{||}(x) = k.\,m\,(m-1)\,(x - \eta)^{m-2}$$

Thus, we have

$$\frac{ff^{||}}{[f^1]^2} = \frac{k.(x-\eta)^m.m(m-1)k(x-\eta)^{m-2}}{[mk(x-\eta)^{m-1}]^2}$$

$$\frac{ff^{||}}{[f^1]^2} = \frac{m-1}{m} \tag{1.8}$$

We need to find a real value of α_n for each iteration, which minimizes μ of (1.6). Since ω_n of (1.7) is positive and real for all x, in the immediate vicinity of η with the fact (1.8), we have in general

$$\frac{m-1}{m} \le \omega_n \le \frac{f(x_n)\,f^{||}(x_n)}{[f^1(x_n)]^2} \tag{1.9}$$

The process of minimizing μ of (1.6) keeping in view of (1.8) with respect to α_n using the procedure given in Young [5] and in Kumar and Koneru [4], is given the optimal choice for α_n as

$$\alpha_n(opt) = \frac{2}{2 - \left(\frac{m-1}{m} + \frac{ff^{||}}{[f^1]^2}\right)}$$

$$= \frac{2}{m+1-m\,\omega_n} \tag{1.10}$$

With the optimal value of α_n, μ of (1.10) takes the form

$$\mu = \left|\frac{1-m\,(1-\omega_n)}{1+m\,(1-\omega_n)}\right| \tag{1.11}$$

which is always less than unity as long as $\omega_n < 1$ and hence the convergence of the GEN-R method (1.5) is assured.

In this paper, we develop the rate of convergence of Generalized Extrapolated Newton - Raphson Method for multiple roots of the equation (1.1) in section 2. And in section 3, we consider some examples for finding a multiple root of equation (1.1) using the methods discussed in this paper.

It is assumed throughout this paper that $f(x)$, $f^{|}(x)$, $f^{||}(x)$ are continuous in $I : a \leq x \leq b$ such that $f(a)\, f(b) < 0$ and $f(x)$ and $f^{||}(x)$ have the same sign for all successive approximations of x starting with x_0.

SECTION – 2: CONVERGENCE ANALYSIS OF GEN-R METHOD

Let η be the multiple root of the equation $f(x) = 0$ and ε_{n+1}, ε_n be errors when x_{n+1}, x_n are the $(n+1)^{th}$ and n^{th} approximates. Then, we have

$$X_{n+1} = \eta + \varepsilon_{n+1}$$

$$X_n = \eta + \varepsilon_n \tag{2.1}$$

Substituting these values of x_{n+1} and x_n in (1.1), we have

$$\varepsilon_{n+1} = \varepsilon_n - m\,\frac{2}{m+1-m\dfrac{ff^{||}}{[f^1]^2}}\;\frac{f}{f^1}$$

$$= \varepsilon_n - \frac{2ff^{|}}{\left(2-\dfrac{m-1}{m}\right)[f^1]^2 - ff^{||}}$$

$$= \varepsilon_n - \frac{2ff^{|}}{(2-k)[f^1]^2 - ff^{||}}$$

where $k = \dfrac{m-1}{m}$

$$\varepsilon_{n+1} = \varepsilon_n - \frac{2\left[f+\varepsilon_n f^{|}+\frac{\varepsilon_n^2}{2}f^{||}+\frac{\varepsilon_n^3}{6}f^{|||}+\frac{\varepsilon_n^4}{24}f^{|V}+\frac{\varepsilon_n^5}{120}f^{V}+\frac{\varepsilon_n^6}{720}f^{V|}+\frac{\varepsilon_n^7}{5040}f^{V||}+\ldots\right] \bullet \left[f^{|}+\varepsilon_n f^{||}+\frac{\varepsilon_n^2}{2}f^{|||}+\frac{\varepsilon_n^3}{6}f^{|V}+\frac{\varepsilon_n^4}{24}f^{V}+\frac{\varepsilon_n^5}{120}f^{V|}+\frac{\varepsilon_n^6}{720}f^{V||}+\ldots\right]}{(2-k)\left[f^{|}+\varepsilon_n f^{||}+\frac{\varepsilon_n^2}{2}f^{|||}+\frac{\varepsilon_n^3}{6}f^{|V}+\frac{\varepsilon_n^4}{24}f^{V}+\frac{\varepsilon_n^5}{120}f^{V|}+\frac{\varepsilon_n^6}{720}f^{V||}+\frac{\varepsilon_n^7}{5040}f^{V|||}\ldots\right]^2 - \left[\left[f+\varepsilon_n f^{|}+\frac{\varepsilon_n^2}{2}f^{||}+\frac{\varepsilon_n^3}{6}f^{|||}+\frac{\varepsilon_n^4}{24}f^{|V}+\frac{\varepsilon_n^5}{120}f^{V}+\frac{\varepsilon_n^6}{720}f^{V|}+\frac{\varepsilon_n^7}{5040}f^{V||}+\ldots\right] \bullet \left[\begin{array}{l} f^{||}+\varepsilon_n f^{|||}+\frac{\varepsilon_n^2}{2}f^{|V} \\ +\frac{\varepsilon_n^3}{6}f^{V}+\frac{\varepsilon_n^4}{24}f^{V|}+\frac{\varepsilon_n^5}{120}f^{V||}+\frac{\varepsilon_n^6}{720}f^{V|||}+\ldots \end{array}\right]\right]}$$

$$= \varepsilon_n - \frac{2ff^{|}+2\left[\varepsilon_n (f^{|})^2+\varepsilon_n^2\left(\frac{3}{2}f^{|}f^{||}\right)+\varepsilon_n^3\left(\frac{(f^{||})^2}{2}+\frac{2}{3}f^{|}f^{|||}\right)+\varepsilon_n^4\left(\frac{5}{12}f^{||}f^{|||}+\frac{5}{24}f^{|}f^{|V}\right)+\ldots\right]}{(2-k)\left[\left(f^{|}\right)^2+\varepsilon_n\left(2f^{|}f^{||}\right)+\varepsilon_n^2\left(f^{|}f^{|||}+(f^{||})^2\right)+\varepsilon_n^3\left(\frac{1}{3}f^{|}f^{|V}+f^{||}f^{|||}\right)+\ldots\right] - \left[\varepsilon_n(f^{|}f^{||})+\varepsilon_n^2\left(\frac{(f^{||})^2}{2}+f^{|}f^{|||}\right)+\varepsilon_n^3\left(\frac{2}{3}f^{||}f^{|||}+\frac{1}{2}f^{|}f^{|V}\right)+\ldots\right]}$$

Simplifying further and neglecting the higher powers of ε_n, we obtained

(i) When $m = 1$

$$\varepsilon_{n+1} = \varepsilon_n^3 \left[\frac{1}{4}\left(\frac{f^{II}}{f^{I}}\right)^2 - \frac{1}{6}\left(\frac{f^{III}}{f^{I}}\right) \right]$$

(ii) When $m = 2$

$$\varepsilon_{n+1} = \varepsilon_n^3 \left[\frac{1}{24}\left(\frac{f^{III}}{f^{II}}\right)^2 - \frac{1}{24}\left(\frac{f^{IV}}{f^{II}}\right) \right]$$

(iii) When $m = 3$

$$\varepsilon_{n+1} = \varepsilon_n^3 \left[\frac{1}{2592}\left(\frac{f^{IV}}{f^{III}}\right)^2 - \frac{1}{2160}\left(\frac{f^{V}}{f^{III}}\right) \right]$$

In general, one can have

$$\varepsilon_{n+1} \simeq \varepsilon_n^3 . k$$

where, $k = k_1 \left[\frac{f^{(m+1)}}{f^m} \right]^2 - k_2 \left[\frac{f^{(m+2)}}{f^m} \right]$

where k_1 and k_2 are constants.

Therefore, $\varepsilon_{n+1} \propto \varepsilon_n^3$

Hence the GEN-R method has a cubic rate of convergence.

SECTION- 3: NUMERICAL EXAMPLES

We consider some examples given in Sastry[2] and Jain [1] for finding the multiple roots of an equation using the methods discussed in this paper and the successive approximations of the roots are tabulated below until the functional value becomes negligible.

Table 3.1

Finding double root of $f(x) = x^3 - x^2 - x + 1$ with $x_1 = 4$

	Method (1.4)	Method (1.5)
n	x_{n+1}	x_{n+1}
1	1.692707692	1.282229965
2	1.078870497	1.001516096
3	1.001468286	1.000000005
4	1.000000566	--

Table 3.2

Finding the triple root of $f(x) = x^4 - x^3 - 3x^2 + 5x - 2$ lies in (0, 2) with $x_1 = 0$.

	Method (1.4)	Method (1.5)
n	x_{n+1}	x_{n+1}
1	1.2	0.9375
2	1.004081633	0.999993656
3	1.000001932	---

CONCLUSIONS

It can be seen from the above tabulated results that the method (1.5), GEN-R takes less iterations than that of the (1.4), GN-R method and the presented method (1.5) converges more rapidly than the classical Generalized Newton - Raphson method (GN-R). Therefore, the method (1.5) has better convergence efficiency.

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