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SOME PROPERTIES OF DUAL OF BANACH LATTCES

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ABSTRACT

In this note, we want to study some properties of dual of Banach lattices and some relationship between Banach lattice and its topoplogical dual.

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1. INTRODUCTION

Let (E,τ) is a topological space. The dual topological E' of E is the vector space consisting of all linear and τ −continuous functional on E. A Banach lattice is a Banach space (E,$\|.\|$) such that E is a vector lattice and its norm satisfies the following property: for each x, y ∈ E such that $|x| \leq |y|$, we have $\|x\| \leq \|y\|$. As an example, l^1 , l^∞, c_0 are Banach lattice.

If E is a Banach lattice, its topological dual E', endowed with the dual norm, is also a Banach lattice.

Definition 1.1: A Banach lattice E is said to be an AM-space if for each x, y ∈ E such that inf(x, y) = 0 we have $\|x + y\| = \max\{\|x\|, \|y\|\}$. As an example, l^∞ is an AM-space.

Definition 1.2: A Banach lattice E is said to be an AL-space if for each x, y ∈ E^+ such that inf(x, y) = 0 we have $\|x + y\| = \|x\| + \|y\|$. As an example, l^1 is an AL-space.

Theorem 1.3: A Banach lattice E is an AL-space (resp. an AM-space) if and only if E' is an AM-space (resp. an AL-space).

Proof: We show first that if E is an AL-space, then E' is an AM-space. To this end, assume that E is an AL-space, and let $x' \wedge y' = 0 \; in \; E'$. Put m = max $\{\|x'\|, \|y'\|\}$, and note that m ≤ $\|x' + y'\|$ holds triviality. Now let $\varepsilon > 0$. Choose some $x \in E^+$ with $\|x\| = 1$ and $\|x' + y'\| \leq (x' + y')(x) + \varepsilon$. Since $x' \wedge y'(x) = 0$, there exist u,v ∈ E^+ with u+v=x and $x'(u) + y'(v) < \varepsilon$. From $(u - v \wedge u) \wedge (v - v \wedge u)$=0, $0 \leq u + v - 2(u \wedge v) \leq x$, and the fact that E is an AL-space, it follows that

$$\|u - v \wedge u\| + \|v - v \wedge u\| = \|u + v - 2(u \wedge v)\| \leq \|x\| \leq 1,$$

And so

$$\begin{aligned}\|x' + y'\| &\leq x'(x) + y'(y) + \varepsilon = x'(v) + y'(u) + x'(u) + y'(v) + \varepsilon \\ &\leq x'(v) + y'(u) + 2\varepsilon \leq x'(v - v \wedge u) + y'(u - v \wedge u) + 3\varepsilon \\ &\leq m\,(\|v - v \wedge u\| + \|u - v \wedge u\|) + 3\varepsilon \leq m + 3\varepsilon.\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, $\|x' + y'\| \leq m$ also holds, and hence

$\|x' \vee y'\| = \|x' + y'\| = \max\{\|x'\|, \|y'\|\}$, as desired.

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Next we show that the topological dual of an AM-space is an AL-space. To this end let E be an AM-space, let $x' \wedge y' = 0$ *in* E', *and let* $\varepsilon > 0$. By Lemma 12.21 of [1] there exist x, y $\in U$ *with*

$$x \wedge y = 0, \ \|x'\| \leq x'(x) + \varepsilon, \quad \|y'\| \leq y'(y) + \varepsilon.$$

Now since E is an AM-space, we have $\|x + y\| \max\{\|x\|, \|y\|\} \leq 1,$ *and so*

$$\|x'\| + \|y'\| \leq x'(x + y) + y'(x + y) + 2\varepsilon \leq \|x' + y'\| \, \|x + y\| + 2\varepsilon$$

$$\leq \|x' + y'\| + 2\varepsilon \leq \|x'\| + \|y'\| + 2\varepsilon$$

holds for all $\varepsilon > 0$. Therefore, $\|x' + y'\| = \|x'\| + \|y'\|$ holds. Thus, E' is an AL-space.

To complete the proof, note that if E' is an AL-space, then E'' is an AM-space, and hence the closed sublattice E of E'' is likewise an AM-space. A similar observation is true E' is an AM-space.

2. MAIN RESULT

Definition 2.1: A norm $\|.\|$ of a Banach lattice E is order continuous if for each net (x_α) such that $x_\alpha \downarrow 0$ in E, the net (x_α) converges to 0 for the norm $\|.\|$. As an example, every L_P-space E has order continuous norm.

Definition 2.2: A Banach lattice E is said to be a KB-space whenever every increasing norm bounded sequence of E^+ is norm convergent. As an example, each AL-space is a KB-space.

Theorem 2.3: The topological dual E' of a Banach lattice E is a KB-space if and only if E' has order continuous norm.

Proof: Assume that E' has order continuous norm and that $0 \leq x'_n \uparrow$ holds in E' with sup $\{\|x'_n\|\} < \infty$. Then $x'(x) = \lim_n x'_n(x)$ exists in $\mathbb{R}$ for each x $\in E^+$, and moreover this formula defines a positive linear functional on E. Since $x'_n \uparrow x'$ holds in E', we see that $\{x'_n\}$ is a norm convergent sequence.

Conversely, obvious.

Definition 2.4: A weak Cauchy sequence $\{x_n\}$ in a Banach space X is said to satisfy property (u) whenever there exists a sequence $\{y_n\}$ of X such that

a. $\sum_{n=1}^{\infty} |x'(y_n)| < \infty$ *holds for all* $x' \in X'$; *and*
b. $x_n - \sum_{i=1}^{n} y_i \xrightarrow{w} 0.$

If every weak Cauchy sequence in a Banach space X satisfies property (u), then X itself is said to have property (u).

Theorem 2.5: Let Banach lattice E is a $\sigma - dedekind$ complete with order continuous norm. Then, E has property (u).

Proof: Let $\{x_n\}$ be a weak Cauchy sequence of E. Then $x_n \xrightarrow{w^*} x''$ *holds in* E''. Consider the element $x = \sum_{n=1}^{\infty} 2^{-n} |x_n| \in E^+$, and let B denote the band generated by x in E''. From Theorem 13.14 of [1], it follows that $x'' \in B$. *Now let* $v_n = (x'')^+ \wedge nx$ *and* $u_n = (x'')^- \wedge nx, n = 1,2, \ldots$. Since $u_n, v_n \in [0, nx]$ and E is an ideal of E'', we see that $\{v_n\}$ *and* $\{u_n\}$ are both sequence of E^+. In addition, since $v_n \uparrow (x'')^+$ *and* $u_n \uparrow (x'')^-$, we see that

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$$v_n \xrightarrow{w^*} (x'')^+ \ \text{ and } \ u_n \xrightarrow{w^*} (x'')^-$$

hold in E'', *and so* $x_n - (v_n - u_n) \xrightarrow{w} 0$ holds in E. Next, for each n put $y_n = (v_n - v_{n-1}) - (u_n - u_{n-1}) \in E$, and note that $\sum_{i=1}^{n} y_i = v_n - u_n$. Therefore, $x_n - \sum_{i=1}^{n} y_i \xrightarrow{w} 0$ holds in E. On the other hand, if $x' \in E'$, then for each k we have

$$\sum_{n=1}^{k} |x'(y_n)| \leq \sum_{n=1}^{\infty} |x'|(v_n - v_{n-1}) + \sum_{n=1}^{k} |x'|(u_n - u_{n-1})$$

$$\leq |x'|(v_k) + |x'|(u_k) \leq |x''|(|x'|) < \infty,$$

and so $\sum_{n=1}^{\infty} |x'(y_n)| < \infty$ *holds for each* $x' \in E'$. *Thus,* E *has property* (u).

Definition 2.6: The Banach space X has the Dunford-pettis property whenever $x_n \xrightarrow{w} 0$ in X and $x_n' \xrightarrow{w} 0$ in X' imply $\lim x_n'(x_n) = 0$, and we say that an operator $T: X \to Y$ between two Banach space is a Dunford-pettis operators whenever $x_n \xrightarrow{w} 0$ in X implies $\lim_n \|Tx_n\| = 0$.

Theorem 2.7: Every AL-space and every AM-space has the Dunford-pettis property.

Proof: Since the topological dual of an AL-space is an AM-space with unit, by Theorem 19.5 of [1] it is enough to establish the result when E is an AM-space with unit.

To this end, let E be an AM-space with unit e, let $x_n \xrightarrow{w} 0$ *in E, and let* $x_n' \xrightarrow{w} 0$ in E'. Pick some $M > 0$ such that $\|x_n\| \leq M$ holds for all n. Let $\varepsilon > 0$. Now the set $\{x_1', x_2', \dots\}$ is a relatively weakly compact subset of E', and so by Theorem 13.10 of [1] there exists some $0 \leq y' \in E'$ satisfying

$$\|(|x_n'| - y')^+\| < \varepsilon/M$$

for all n. Since the lattice operations of E are weakly sequentially continuous (see Theorem 12.30 of [1]), we have $|x_n| \xrightarrow{w} 0$, and thus there exists some k satisfying $y'(|x_n|) < \varepsilon$ for all $n \geq k$. In particular for $n \geq k$ *we have*

$$|x_n'(x_n)| \leq |x_n'|(|x_n|) = (|x_n'| - y')^+(|x_n|) + |x_n'| \wedge y'(|x_n|)$$

$$\leq M. \|(|x_n'| - y')^+\| + y'(|x_n|) < M.\varepsilon/M + \varepsilon = 2\varepsilon,$$

which shows that $x_n'(x_n) \to 0$ holds, as required.

Theorem 2.8: If the topological dual X' of a Banach space X has the Dunford-pettis property, then X itself has the Dunford-pettis property.

Proof: Let X' have the Dunford-pettis property, and consider two weakly compact operators $Z \xrightarrow{S} X \xrightarrow{T} Y$ (where Y and Z are Banach spaces). Taking adjoints we have $Y' \xrightarrow{T'} X' \xrightarrow{S'} Z'$ with s' and T' weakly compact. Since X' has the Dunford-pettis property, the Theorem 19.4 of [1] shows that S' is a Dunford-pettis operator, and so $S'T' = (TS)'$ is a compact operator. Thus, TS is a compact operator, and by Theorem 19.3 of [1] the operator T must be a Dunford-pettis operator. Now by Theorem 19.4 the Banach space X must have the Dunford-pettis property.

3. CONCLUSION

Definition 3.1: A vector lattice E is said to have property (b) if A$\subset E$ is order bounded whenever A is order bounded in$(E')'$.

Definition 3.2: A continuous operator $T: E \to X$ from a Banach lattice into a Banach space is said to be b-weakly compact whenever T carries each b-order bounded subset of E into relatively weakly compact subset of X.

Definition 3.3: A continuous operator $T: E \to X$ from a Banach lattice into a Banach space is order weakly compact whenever T [0,x] is a relatively weakly compact subset of X for each $x \in E^+$.

In the following, we establish some properties of topological dual of Banach lattices and relationships between Banach lattice and its dual topological.

i) We see that if E is a Banach lattice, its topological dual E' is also a Banach lattice.
ii) A Banach lattice E is an AL-space (resp. an AM-space) if and only if E' is an AM-space (resp. an AL-space).
iii) If Banach lattice E is KB-space, E' is not KB-space in general. As an example l^1 is KB-space but l^∞ is not KB-space. If Banach lattice E' is KB-space, E is not KB-space in general. As an example l^1 is a KB-space but c_0 is not KB-space.
iv) If Banach lattice E has order continuous norm, E' has not order continuous norm in general. As an example, l^1 has order continuous norm but $(l^1)' = l^\infty$ by Theorem 2.3 has not continuous norm.
v) If Banach lattice E has property (u), E' has not property (u) in general. As an example, by Theorem 2.5 l^1 has property (u) but by example 14.8 of [1] l^∞ has not property (u).
vi) If Banach lattice E' has property (u), E has not property (u) in general. As an example $(c_0)'$ has property (u) but c_0 by Theorem 14.7 of [1] and example 14.8 of [1] has not property (u).
vii) By Theorems 1.3 and 2.7 E is a AM-space (resp. AL-space) with of Duonford-pettice property if and only if E' is a AL-space (resp. AM-space) with property of Duonford-pettice.
viii) By Theorem 2.8 if E' has the Dunford-pettis property then, E has the property Duonford-pettice property.

ix) Every perfect vector lattice and therefore every topological dual has property (b).

x) Recall from [3] if E and F are Banach lattices, then each b-weakly compact operator (order weakly compact operator) $T: E \rightarrow F$ admits a b-weakly compact (order weakly compact) adjoint T' if and only if E' or F' is a KB-space. (Theorems 3.1. and 3.3)

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