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ON NONCYCLIC VECTORS FOR CERTAIN BACKWARD SHIFTS

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ABSTRACT

Let $\{\beta(n)\}_n$ *be a sequence of positive numbers with* $\beta(0)=1$, *and let* $p>1$. *By the space* $H^p(\beta)$, *we mean the set of all formal power series* $f(z)=\sum_{n=0}^{\infty}\hat{f}(n)z^n$ *for which* $\sum_{n=0}^{\infty}|\hat{f}(n)|^p\beta(n)^p<\infty$. *We give some sufficient conditions under which the set of noncyclic vectors for the backward shift operator on* $H^p(\beta)$ *is a countable union of nowhere dense sets.*

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INTRODUCTION

Let x be a vector in a Banach space X, and T be an operator on X. The orbit of x under T is defined by $orb(T,x)=\{T^n x : n=0,1,2,\ldots\}$.

We recall that a vector x in a separable Banach space X is *cyclic* for an operator T on X if the closed linear span of $orb(T,x)$ is equal to X. Let $\{\beta(n)\}_n$ be a sequence of positive numbers with $\beta(0)=1$, and take $1<p<\infty$. Consider the space of all sequences $f=\{\hat{f}(n)\}_{n=0}^{\infty}$ such that

$$\|f\|^p=\|f\|_\beta^p=\sum_{n=0}^{\infty}|\hat{f}(n)|^p\beta(n)^p<\infty.$$

The notation $f(z)=\sum_{n=0}^{\infty}\hat{f}(n)z^n$, which is called a formal power series, shall be used whether or not the series converges for any value of z. Denote $H^p(\beta)$ the spaces with the norm $\|.\|_\beta$. Furthermor, the dual of $H^p(\beta)$ is $H^q(\beta^{p/q})$ where q is the conjugate exponent of p and $\beta^{p/q}=\{\beta(n)^{p/q}\}_n$. Where $p=2$, the Hardy, Bergman and Dirichlet spaces can be viewed in this way, respectively, by considering $\beta(n)=1, \beta(n)=(n+1)^{-1/2}$ and $\beta(n)=(n+1)^{1/2}$. For more information on the space $H^p(\beta)$ one can see [1]-[5].

Let $\hat{f}_k(n)=\delta_{n_k}$. So $f_k(z)=z^k$ and then $\{f_k\}_k$ is a basis such that $\|f_k\|=\beta(k)$. Now consider M_z, the operator of multiplication by z on $H^p(\beta)$, given by $(M_z f)(\xi)=\xi f(\xi)$. Clearly, M_z shifts the basis $\{f_k\}_k$. The operator M_z is bounded on $H^p(\beta)$ if and only if $\sup_n \beta(n+1)/\beta(n)<\infty$. In fact, in [2] it is shown that $\|M_z^n\|=\sup_k(n+k)/\beta(k)$. We define the weighted backward shif $\widetilde{B}$ on $H^p(\beta)$

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$$\widetilde{B}(\sum_{n=0}^{\infty}\hat{f}(n)z^n)=\sum_{n=0}^{\infty}\hat{f}(n+1)\left(\frac{\beta(n+1)}{\beta(n)}\right)^2 z^n.$$

A vector $x\in X$ is called a *supercyclic vector* for a bounded operator on a Banach space X if the set of all scalar multiples of the elements of $orb(T,x)$ is dense in X. Note that for an operator T, every supercylic vector is a cyclic vector.

Noncyclic Vectors

The author, has given, in [2], conditions for the supercyclicity of a vector f in $H^p(\beta)$, for the operator $\tilde{B}$. we will show that under the same conditions the set of all noncyclic vectors for $\widetilde{B}$ is not so large. First we bring a lemma.

Lemma 1: *If* $\sup_n \dfrac{\beta(n)}{\beta(n-1)}<\infty$, *then the operator* $\widetilde{B}$ *is bounded on* $H^p(\beta)$. *Indeed,* $\|\widetilde{B}\|=\sup_n \dfrac{\beta(n)}{\beta(n-1)}$.

Proof: For every $f\in H^p(\beta)$, we have

$$\begin{aligned}\|\widetilde{B}f\|_p^p&=\sum_{n=0}^{\infty}|(\widetilde{B}f)^{\wedge}(n)|^p\ \beta(n)^p\\&=\sum_{n=0}^{\infty}|\ \hat{f}(n+1)\frac{\beta(n+1)^2}{\beta(n)^2}|^p\ \beta(n)^p\\&=\sum_{n=1}^{\infty}|\ \hat{f}(n)|^p\ (\frac{\beta(n)}{\beta(n-1)})^{2p}\beta(n-1)^p\\&\le\sup_n(\frac{\beta(n)}{\beta(n-1)})^p\sum_{n=1}^{\infty}|\ \hat{f}(n)|^p\ \beta(n)^p\\&\le\sup_n(\frac{\beta(n)}{\beta(n-1)})^p\ \|\ f\|_p^p\ .\end{aligned}$$

Thus, $\widetilde{B}$ is a bounded operator and $\|\widetilde{B}\|\le\sup_n\beta(n)/\beta(n-1)$. On the other hand,

$$\widetilde{B}(z^n)=(\beta(n)/\beta(n-1))^2 z^{n-1},$$

which implies that $(\beta(n)/\beta(n-1))^2\,\|z^{n-1}\|\le\|\tilde{B}\|\,\|z^n\|$.

Hence $\sup_n\beta(n)/\beta(n-1)\le\|\widetilde{B}\|$ and the result holds.

In the following theorem, we give some sufficient conditions under which the set of noncyclic vectors for the backward shift operator $\widetilde{B}$ on $H^p(\beta)$ is a countable union of nowhere dense sets.

Theorem 1: *Suppose that* $\beta(i+1)\beta(i-1)\le\beta(i)^2\le 1$ *for all* $i\ge 1$, *and* $\{\beta(i)/\beta(i-1)\}_{i=1}^{\infty}\in\ell^p$. *Then the set of all noncyclic vectors for* $\widetilde{B}$ *is a countable union of nowhere dense sets.*

Proof: Let $f(z)=\sum_{i=0}^{\infty}\hat{f}(i)z^i$ be in $H^p(\beta)$. It is shown [2, Theorem 3.1] that if $f(z)$ is not a polynomial then it is a supercyclic vector, and so a cyclic vector, for $\widetilde{B}$. On the other hand, if $f(z)$ is a polynomial then $(\widetilde{B})^n f=0$ for a sufficiently large n, which implies that $f(z)$ is not cyclic for $\widetilde{B}$. Therefore, the set of noncyclic vectors for $\widetilde{B}$

is the set of polynomials which we denote by *P*. Suppose that $f(z)=\sum_{i=0}^{m}\hat{f}(i)z^i$ is a polynomial with $\hat{f}(m)\neq 0$ and put $M=\vee\{(\tilde{B})^k f : k\geq 0\}$.

Now, an easy computation shows that

$$((\tilde{B})^n f)(z)=\sum_{i=0}^{m-n}\hat{f}(i+n)\frac{\beta(i+n)^2}{\beta(i)^2}z^i,\quad 0\leq n\leq m.$$

So the equality $((\tilde{B})^m f)(z)=\hat{f}(m)\beta(m)^2$ states that M contains the constants. Moreover, since

$$((\tilde{B})^{m-1} f)(z)=\hat{f}(m-1)\beta(m-1)^2+\hat{f}(m)(\beta(m)^2/\beta(1)^2)z,$$

We see that $z\in M$. Continuing this process, we obtain $z^i\in M; i=0,1,\ldots,m$.

Hence $M_m=\vee\{z^i;\ 0\leq i\leq m\}\subseteq M$. On the other hand, it is clear that $M\subseteq M_m$. So $M=M_m$.

Let $\{p_i(z)\}_{i=1}^{\infty}$ be a countable dense subset of polynomials where coefficients have rational coordinates. Clearly, for every integer $m\geq 0$, there is a polynomial $p_i(z)$ of degree m; so the above argument shows that $\mathrm{P}=\bigcup_{i=1}^{\infty}N_i$, where $N_i=\vee\{\tilde{B})^k p_i : k\geq 0\}$. If we show that each N_i is nowhere dense, the proof will be over. If g_i is an interior point of N_i then there is an $\varepsilon_i>0$ such that $\{g\in H^p(\beta):\|g-g_i\|_\beta<\varepsilon_i\}$ is a subset of N_i. Therefore, if $\|g\|_\beta<\varepsilon_i$ and $g+g_i\in N_i$ we conclude that $g\in N_i$. Hence, $N_i=H^p(\beta)$ which is a contradiction.

Example 1: *Consider* $\beta(1)$ *as a fixed number in the interval (0,1), and let* $\beta(i)=\beta(1)/(i-1)!$, $i>1$. *If* $p=2$ *then we can use Theorem 1.*

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