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MAZUR-ULAM THEOREM AND TWO-ISOMETRIC MAPS

K. Hedayatian*

Department of Mathematics, College of Sciences, Shiraz University, Shiraz 71454, Iran.

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ABSTRACT

A map f from the real normed space χ into itself is called a two-isometry if

$$\| f^2(x) - f^2(y) \|^2 - 2\| f(x) - f(y) \|^2 + \| x - y \|^2 = 0$$

for all x and y in χ. It is shown that every surjective two-isometry is affine, that is,

$$f((1-t)x + ty) = (1-t)f(x) + tf(y)$$

for all x and y in χ and $0 \le t \le 1$.

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INTRODUCTION

Let χ be a real normed space. A map $f : \chi \to \chi$ is an isometry, if $\| f(x) - f(y) \| = \| x - y \|$ for all $x, y \in \chi$. It is called a two-isometry, if

$$| f^2(x) - f^2(y) \|^2 - 2\| f(x) - f(y) \|^2 + \| x - y \|^2 = 0 \tag{1}$$

for all $x, y \in \chi$. Also, f is an affine map, if

$$f((1-t)x + ty) = (1-t)f(x) + tf(y)$$

for all $x, y \in \chi$ and $0 \le t \le 1$. Observe that f is affine if and only if the map $T : \chi \to \chi$ defined by $T(x) = f(x) - f(0)$ is linear. The Mazur-Ulam theorem states that every bijective (equivalently, surjective) isometry is affine. This result was proved by Mazur and Ulam in [3]; their proof is also brought in the books [1] and [2]. A simple proof of this theorem is given in [4] which is based on the ideas in [5]. In this note we see that this theorem holds for surjective two-isometries. Let $\chi = \ell^2$ and $\{e_n : n \ge 0\}$ be the standard basis for χ. It is easily seen that the unilateral weighted shift S on χ defined by $Se_n = \sqrt{\frac{n+2}{n+1}}\, e_{n+1}$ is a two-isometry but not an isometry.

MAIN RESULTS

Theorem 1: *Every surjective two-isometric map is an affine map.*

Proof: Suppose that $f : \chi \to \chi$ is a two-isometry. Substituing x by $f^k(x)$ and y by $f^k(y)$ in (1) we get

$$\| f^{k+2}(x) - f^{k+2}(y) \|^2 - \| f^{k+1}(x) - f^{k+1}(y) \|^2 \le \| f^{k+1}(x) - f^{k+1}(y) \|^2 - \| f^k(x) - f^k(y) \|^2 .$$

Corresponding author: K. Hedayatian*
Department of Mathematics, College of Sciences, Shiraz University, Shiraz 71454, Iran.
E-mail: hedayati@shirazu.ac.ir

Therefore,

$$\begin{aligned} 0 \le \| f^n(x) - f^n(y) \|^2 &= \sum_{k=1}^{n} [\| f^k(x) - f^k(y) \|^2 - \| f^{k-1}(x) - f^{k-1}(y) \|^2] + \| x - y \|^2 \\ &\le n(\| f(x) - f(y) \|^2 - \| x - y \|^2) + \| x - y \|^2 \\ &= n \| f(x) - f(y) \|^2 + (1-n) \| x - y \|^2 \end{aligned}$$

which implies that $\frac{n-1}{n} \| x - y \|^2 \le \| f(x) - f(y) \|^2$. Now, let $n \to \infty$ to obtain

$$\| f(x) - f(y) \| \ge \| x - y \| \tag{2}$$

for all $x, y \in \chi$. Since f is one to one and surjective it follows from (1) that

$$\| f^{-2}(x) - f^{-2}(y) \|^2 - 2 \| f^{-1}(x) - f^{-1}(y) \|^2 + \| x - y \|^2 = 0$$

for all $x, y \in \chi$. Therefore, f^{-1} is a two-isometry and by the above argument

$$\| f^{-1}(x) - f^{-1}(y) \| \ge \| x - y \| \tag{3}$$

for all $x, y \in \chi$. Now (2) and (3) imply that f is an isometry and by the Mazur -Ulam theorem f is affine.

For $m \ge 1$, a map $f : \chi \to \chi$ is an m-isometry, if

$$\sum_{k=0}^{m} (-1)^k \binom{m}{k} \| f^{m-k}(x) - f^{m-k}(y) \|^2 = 0$$

for all $x, y \in \chi$. Observe that 1-isometry is, indeed, an isometry and every $m-1$ - isometry is an m-isometry. A natural question which arise runs as follows:

Question: Is every surjective m - isometric map an affine map?

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