International Journal of Mathematical Archive-4(10), 2013, 93-100
IJMA Available online through www.ijma.info ISSN 2229 – 5046

CONE METRIC SPACE AND COMMON FIXED POINT THEOREMS FOR GENERALIZED JAGGI AND DASS-GUPTA CONTRACTIVE MAPPINGS

S. K. Tiwari[1*], R. P. Dubey[1] and A. K. Dubey[2]

[1]*Department of Mathematics, Dr. C. V. Raman University, Bilaspur, India.*

[2]*Department of Mathematics, Bhilai Institute of Technology, Durg, India.*

(Received on: 30-09-13; Revised & Accepted on: 15-10-13)

ABSTRACT

The purpose of this paper is to establish the generalization of Jaggi and Dass – Gupta contraction as such contractive type mapping on cone metric spaces satisfying rational expression and obtained common fixed point. Our results unify, generalize and complement the comparable results of R. Uthayakumar and G. Arockia Prabhakar [2] and [14].

Key words: *Cone metric spaces, Common fixed point, contraction mapping, contractive type mapping, rational expression.*

1. INTRODUCTION

The fixed point theorems in metric space are playing a major role to construct methods in mathematics to solve problems in applied mathematics and sciences. So the attraction of metric spaces to a large numbers of mathematicians is understandable. Some generalizations of the notion of a metric space have been proposed by some authors. Impact of the fixed point theory in different branches of mathematics and its applications is immense. The first result on fixed points for contractive type mapping was the much celebrated Banach contraction principle with rational expressions have been expanded and some fixed and common fixed point theorems have been obtained in [1], [3], [5] [11], and [12].

Quiet recently, Huang and Zhang [4] initiated cone metric spaces, which is a generalization of metric spaces, by substituting the real numbers with ordered Banach spaces. They have investigated convergence in cone metric spaces, introduced completeness of cone metric spaces, and proved a Banach contraction mapping theorem, and some other fixed point theorems involving contractive type mappings in cone metric spaces using the normality condition.

Later, many authors have proved some common fixed point theorems with normal and non-normal cones in these spaces; see for instance [6], [7], [8], [9] and [10]. Author [11] and [13] have introduced almost Jaggi and Dass-Gupta contraction in partially ordered metric space to prove the fixed point theorem. In [14] introduced the notion of complex valued metric space and obtained common fixed point.

So, in this paper we prove common fixed point theorem in cone metric spaces for generalized Jaggi and Dass-Gupta contractive type mapping satisfying rational inequalities [2] and also generalized the results of [14].

2. PRELIMINARY NOTES

Definition 2.1 [4]: Let E be a real Branch space and P a subset of E. Then P is called a cone if it is satisfied the following conditions,

(a) P is closed, non-empty and $P \neq \{0\}$;
(b) $ax + by \in P$ for all $x, y \in P$ and non negative real numbers $a, b \in R$;
(c) $x \in P$ and $-x \in P \Rightarrow x = 0$.

Corresponding author: S. K. Tiwari[1]*
[1]*Department of Mathematics, Dr. C. V. Raman University, Bilaspur, India.*

For a given cone $P \subset E$, we define a Partial ordering $\leq$ on E with respect to P by $x \leq y$ if and only if $y - x \in P$. we shall write $x \ll y$ if $y - x \in$ int P, int P denotes the interior of P. The cone P is called normal if there is a number $K > 0$ such that for all $x, y \in E, 0 \leq x \leq y$ implies

$\|x\| \leq K\|y\|$.

The least positive number satisfying above is called the l constant of P. In following we always suppose E is a Banach space. P is a cone in E with $intP \neq \phi$ and $\leq$ is partial ordering with respect to P.

Definition 2.2 [4]: Let X be a non- empty set. Suppose that the mapping $d: X\, x\, X \rightarrow E$ satisfies,
(a) $0< d(x,y)$ for all $x, y \in X$ and $d(x,y) = 0$ if and only if $x = y$;
(b) $d(x,y) = d(y,x)$ for all $x, y \in X$;
(c) $d(x,y) =(x,z) + (y,z)$ for all $x, y \in X$.

Then d is called a cone metric on X and (X, d) is called a cone metric space.

Example 2.3 [4]: Let $E = R^2$, $P = \{(x,y) \in E: x,y \geq 0\}$, $X = R$ and $d: X \times X \rightarrow E$ defined by $d(x,y) = (|x-y|, \propto |x-y|)$ where $\propto \geq 0$ is a constant. Then (X,d) is a cone metric space.

Example 2.4: Let E= l^1, $P = \left\{\{x_n\}n \geq 1E: x_n \geq 0, for\ all\ n\right\}$ (X,d) a metric space and $d: X \times X \rightarrow E$, defined by $d(x,y) = \left\{\frac{d(x,y)}{2^n}\right\}$ $n \geq 1$.Then (X,d) is a cone metric space.

Definition 2.5[9]: Let (X,d) be a cone metric space. Let $\{\boldsymbol{x}_n\}_{n\geq 1}$ be a sequence in X and $\boldsymbol{x} \in X$. Then
(a) $\{\boldsymbol{x}_n\}_{n\geq 1}$ converges to x whenever for every c $\in$E with $0 \ll c$ there is a natural number N such that $d(\boldsymbol{x}_n, \boldsymbol{x}) \ll c$ for all $n \geq N$
(b) $\{\boldsymbol{x}_n\}_{n\geq 1}$ is said to Cauchy sequence for every c $\in$ E with $0 \ll c$ there is a natural number N such that $d(\boldsymbol{x}_n, \boldsymbol{x}_m) \ll c$ for $all\ n\ m \geq N$.
(c) (X,d) is called a complete cone metric space, if every Cauchy sequence is convergent in X.

Lemma 2.6[4]: Let (X, d) be a cone metric space, $P \subset E$ a normal cone with normal constant K.
Let $\{x_n\}$, $\{y_n\}$ be a sequences in X and $x, y \in X$. Then,
(i) $\{x_n\}$ converges to x if and only if $\lim_{n\rightarrow\infty} d(x_n, x) = 0$;
(ii) If $\{x_n\}$ converges to x and $\{x_n\}$ converges to y then $x = y$. That is the limit of $\{x_n\}$ is unique;
(iii) If $\{x_n\}$ converges to x, then $\{x_n\}$ is Cauchy sequence.
(iv) $\{x_n\}$ is a Cauchy sequence if and only if $\lim_{n, m\rightarrow\infty} d(x_n, x_m) = 0$;
(v) If $x_n \rightarrow x$ and $\{y_n\}$ is another sequence in X such that $y_n \rightarrow y$ $(n \rightarrow \infty)$ then $d(x_n, y_n) \rightarrow d(x, y)$

3. MAIN RESULTS

The following which will give, are generalization of theorems 2.1, 2.2.and 2.3 of [2].

[A]. GENERALIZED JAGGI CONTRACTION

Definition 3.1: [**13**] Let (X,d) be a cone metric space. A self mapping $T: X \rightarrow X$ is called generalized Jaggi contraction if it satisfies the following condition:

$d(Tx,Ty) \leq \frac{\alpha d(x,Tx)d(y,Ty}{d(x,y)} + \beta d(x,y) + Lmin\{d(x,Ty), d(y,Tx\}$ for all x. $y \in X$, where $L \geq 0$ and $\alpha, \beta \in [0,1]$ with $\alpha + \beta < 1$. (1)

Theorem 3.2: Let (X,d) be a complete cone metric space and P be a normal cone with normal constant K. Suppose the mappings $T_1, T_2 : X \rightarrow X$ be any two generalized Jaggi contractions, by definition [3.1] ,satisfies the following condition

$d(T_1x, T_2y) \leq \frac{\alpha d(x,T_1x)d(y,T_2y)}{d(x,y)} + \beta d(x,y) + Lmin\{d(x,T_2y), d(y,T_2y)\}$ for all $x, y \in X$ where $L \geq 0$ and α, β [0.1] with $\alpha + \beta < 1$. Then T_1 and T_2 have a unique common fixed point in X.

Proof: Let x_0 be an arbitrary element and define the sequences $\{x_{2n}\}$ and $\{x_{2n+1}\}$ such that $x_{2n+1} = T_1 x_{2n}$ and $x_{2n+2} = T_2 x_{2n+1}$ for each n = 0, 1, 2,…∞.

© 2013, IJMA. All Rights Reserved

We have

$d(x_{2n}, x_{2n+1}) = d(T_1 x_{2n-1}, T_2 x_{2n})$

$$\leq \frac{\alpha d(x_{2n-1}, T_1 x_{2n-1})\, d(x_{2n}, T_2 x_{2n})}{d(x_{2n-1}, x_{2n})} + \beta d(x_{2n-1}, x_{2n}) + L\min\{d(x_{2n-1}, x_{2n+1}), d(x_{2n}, T_1 x_{2n-1})\}$$

$$\leq \frac{\alpha d(x_{2n-1}, x_{2n})\, d(x_{2n}, x_{2n+1})}{d(x_{2n-1}, x_{2n})} + \beta d(x_{2n-1}, x_{2n}) + L\min\{d(x_{2n-1}, x_{2n+1}), d(x_{2n}, T_1 x_{2n-1})\}$$

$$\leq \alpha d(x_{2n}, x_{2n+1}) + \beta d(x_{2n+1}, x_{2n})$$

$(1-\alpha) d(x_{2n}, x_{2n+1}) \leq \beta d(x_{2n-1}, x_{2n})$

$d(x_{2n}, x_{2n+1}) \leq \frac{\beta}{1-\beta}(x_{2n-1}, x_{2n})$

$= k d(x_{2n-1}, x_{2n})$ where $k = \frac{\beta}{1-\beta}$, $\alpha + \beta < 1, 0 < K < 1$.

Hence by induction

$d(x_{2n}, x_{2n+1}) = k d(x_{2n-1}, x_{2n}) \leq k^2 d(x_{2n-2}, x_{2n-1}) \leq \ldots\ldots\ldots k^{2n} d(x_0, x_1)$

Now for $m > n$, we have

$d(x_{2n}, x_{2m}) \leq d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2}) \ldots + d(x_{2n+2m-1}, x_{2m})$

$\leq (k^{2n} + k^{2n+1} + \ldots + k^{2n+2m-1}) d(x_0, x_1)$

$\leq \frac{k^{2n}}{1-k} d(x_0, x_1)$

We get $\|d(x_{2n}, x_{2m}\| \leq K \frac{k^{2n}}{1-k} \|d(x_0, x_1)\|$ which implies that $d(x_{2n}, x_{2m}) \to 0$ as $n \to \infty$.

Hence $\{x_{2n}\}$ is a Cauchy sequence in X. Since (X, d) be a complete cone metric space. So there exist $x^* \in X$ such that $x_{2n} \to x^* (n \to \infty)$.

Since $d(x^*, T_1 x^*) \leq d(x^*, x_{2n+1}) + d(x_{2n+1}, T_1 x^*)$

$\leq d(x^*, x_{2n+1}) + d(T_1 x_{2n}, T_1 x^*)$

$$\leq d(x^*, x_{2n+1}) + \frac{\alpha d(x_{2n}, T_1 x_{2n})\, d(x^*, T_1 x^*)}{d(x_{2n}, x^*)} + \beta d(x_{2n}, x^*) + L\min\{d(x_{2n}, T_1 x^*), d(x^*, T_1 x_{2n})\}$$

$$\leq d(x^*, x_{2n+1}) + \frac{\alpha d(x_{2n}, x_{2n+1})\, d(x^*, x^*)}{d(x_{2n}, x^*)} + \beta d(x_{2n}, x^*) + L\min\{d(x_{2n}, x^*), d(x^*, x_{2n+1})\}$$

$$\leq d(x^*, x_{2n+1}) + \beta d(x_{2n}, x^*) + L\min\{d(x_{2n}, x^*), d(x^*, x_{2n+1})\}$$

So, using the condition of normality of cone

$\| d(x^*, T_1 x^*)\| \leq K(k\|d(x^*, x_{2n+1})\| + \beta\|d(x_{2n}, x^*)\| + L\min\|d(x_{2n}, x^*), d(x^*, x_{2n+1})\| \to 0.$

Hence $\| d(x^*, T_1 x^*)\| = 0$. This implies that $x^* = T_1 x^*$. So, x^* is a fixed point of T_1. Now if y^* is another fixed point of T_1. Then we have

$d(x^*, y*) = d(T_1 x^*, T_1 y^*)$

$$\leq K\left[\frac{\alpha d(x^*, T_1 x^*)\, d(y^*, T_1 y^*)}{d(x^*, y^*)} + \beta d(x^*, y^*) + L\min\{d(x^*, T_1 y^*) d(y^*, T_1 x^*\}\right]$$

$$\leq K\left[\alpha d(\frac{x^*, T_1 x^*)\, d(y^*, y^*)}{d(x^*, y^*)} + \beta d(x^*, y^*) + L\min\{d(x^*, y^*) d(y^*, T_1 x^*\}\right]$$

Hence $\|d(x^*, y*)\| = 0$ and $x^* = y^*$. Therefore the fixed point of T_1 is unique.

© 2013, IJMA. All Rights Reserved

Similarly, it can be established that $T_2x^* = x*$. Hence $T_1x^* = x^* = T_2x^*$. Thus x^* is common fixed point of T_1 and T_2. This completes the proof.

Corollary 3.3: Let (X, d) be a complete cone metric space and P be a normal cone with normal constant *K*. Suppose the mappings $T_1, T_2 : X \to X$ be any two generalized Jaggi contractions, by definition [3.1], satisfies the following condition

$d(T_1x, T_2y) \leq \frac{\alpha d(x,T_1x) d(y,T_2y)}{d(x,y)} + \beta d(x,y)$for all $x, y \in X$whereL ≥ 0 and α, β [0.1] with $\alpha + \beta < 1$. Then T_1 and T_2 have a unique common fixed point in *X*.

Proof: The proof of corollary immediately follows by putting L= 0 in previous theorem in 3.1.

Theorem 3.4: Let (X, d) be a complete cone metric space and P be a normal cone with normal constant *K*. suppose $T, R, S : X \to X$ be three self maps which generalized Jaggi contractions, by definition [3.1], satisfies the following condition

$d(TRx, TSy) \leq \frac{\alpha d(Tx,TRx) d(Ty,TSy)}{d(Tx,Ty)} + \beta d(Tx,Ty) + L\min\{d(Tx,TSy), d(Ty,TRx)\}$ for all $x, y \in X$where L ≥ 0 and α, β [0.1] with $\alpha + \beta < 1$. Then T, R and S have a unique common fixed point in *X*.

Proof: Let x_0 be an arbitrary element and define the sequences $\{x_{2n}\}$ and $\{x_{2n+1}\}$ such that $x_{2n+1} = R\, x_{2n}$ and $x_{2n+2} = S\, x_{2n+1}$ for each n = 0, 1, 2,...∞.

Then we have

$$d(Tx_{2n+1}, Tx_{2n+2}) = d(TRx_{2n}, TSx_{2n+1})$$

$$\leq \frac{\alpha d(Tx_{2n},TRx_{2n}) d(Tx_{2n+1},TSx_{2n+1})}{d(Tx_{2n},Tx_{2n+1})} + \beta d(Tx_{2n}, Tx_{2n+1}) + L\min\{d(Tx_{2n}, TSx_{2n+1}), d(Tx_{2n+1}, TRx_{2n})\}$$

$$\leq \frac{\alpha d(Tx_{2n},Tx_{2n+1}) d(Tx_{2n+1},Tx_{2n+2})}{d(Tx_{2n},Tx_{2n+1})} + \beta d(Tx_{2n}, Tx_{2n+1}) + L\min\{d(Tx_{2n}, Tx_{2n+2}), d(Tx_{2n+1}, Tx_{2n+1})\}$$

$$\leq \alpha d(Tx_{2n+1}, Tx_{2n+2}) + \beta d(Tx_{2n}, Tx_{2n+1})$$

$$(1-\alpha)\, d(Tx_{2n+1}, Tx_{2n+2}) \leq \beta d(Tx_{2n}, Tx_{2n+1})$$

$$d(Tx_{2n+1}, Tx_{2n+2}) \leq \frac{\beta}{1-\alpha} d(Tx_{2n}, Tx_{2n+1})$$

$d(Tx_{2n+1}, Tx_{2n+2}) \leq K\, d(Tx_{2n}, Tx_{2n+1})$. where $K = \frac{\beta}{1-\alpha}$, $\alpha + \beta < 1$.

So, for $m, n \in N$, with $m > n$. we have

$$d(Tx_{2n+1}, Tx_{2m+2}) \leq d(Tx_{2n+1}, Tx_{2n+2}) + d(Tx_{2n+2}, Tx_{2n+3}) + \ldots + d(Tx_{2m}, Tx_{2m+1})$$

$$\leq (k^{2n+1} + k^{2n+2} + k^{2n+3} + \cdots + k^{2n})\, d(Tx_0, Tx_1)$$

$$\leq \frac{k^{2n}}{1-k}\, d(Tx_0, Tx_1)$$

We get$\|d(Tx_{2n+1}, Tx_{2m+2})\| \leq \frac{k^{2n}}{1-k} K \|d(Tx_0, Tx_1)\|$ .This implies that $d(Tx_{2n+1}, Tx_{2m+2}) \to 0$ $(n, m \to \infty)$. Hence $\{Tx_{2n+1}\}$ is a Cauchy sequence. Since (X, d) is a complete cone metric space. So, there exist $x* \in X$ such that $Tx_{2n+1} \to x^*$ $(n \to \infty)$.

Since $d(Tx^*, TRx^*) \leq d(Tx^*, Tx_{2n+1}) + d(Tx_{2n+1}, TRx^*)$

$$\leq d(Tx^*, Tx_{2n+1}) + d(TRx_{2n}, TRx^*)$$

© 2013, IJMA. All Rights Reserved

$$\leq d(Tx^*, Tx_{2n+1}) + \frac{\alpha d(Tx_{2n}, TRx_{2n})\, d(Tx^*, TRx^*)}{d(Tx_{2n}, T x^*)} + \beta d(Tx_{2n}, Tx^*)$$
$$+ Lmin\{d(Tx_{2n}, TRx^*), d(Tx^*, Tx_{2n+1})\}$$

$$\leq d(Tx^*, Tx_{2n+1}) + \frac{\alpha d(Tx_{2n},\ Tx_{2n+1})\, d(Tx^*, Tx^*)}{d(Tx_{2n}, Tx^*)} + \beta d(Tx_{2n}, Tx^*)$$
$$+ Lmin\{d(Tx_{2n}, Tx^*), d(Tx^*, Tx_{2n+1})\}$$

$$\leq d(Tx^*, Tx_{2n+1}) + \beta d(Tx_{2n}, Tx^*) + Lmin\{d(Tx_{2n}, Tx^*), d(Tx^*, Tx_{2n+1})\}$$

So, using the condition of normality of cone, we get

$$\|d(Tx^*, TRx^*)\| \leq K(\|d(Tx^*, Tx_{2n+1})\| + \beta\|d(Tx_{2n}, Tx^*)\| + Lmin\|d(Tx_{2n}, Tx^*), d(Tx^*, Tx_{2n+1})\| \ \rightarrow\ 0$$

Hence $\|d(Tx^*, TRx^*)\| = 0$ .This implies that $Tx^* = TRx^*$. So $x^* = Rx^*$ is fixed point of R.

Now if $y^*$is another fixed point of R. Then we have $\|d(Tx^*, Ty *)\| = 0$.

This implies that $Tx^* = Ty*$.Therefore $x^* = y^*$ the fixed point of R.

Similarly it can be established that $TSx^* = Tx^*$. Hence $TRx^* = x^* = TSx *$.Thus$Tx * = x *$ is common fixed point of R and S. This completes the proof.

Corollary 3.5: Let (X, d) be a complete cone metric space and P be a normal cone with normal constant *K*. suppose $T, R, S : X \rightarrow X$ be three self maps which generalized Jaggi contractions, by definition [3.1], satisfies the following condition

$d\,(TRx,\ TSy) \leq \frac{\alpha d(Tx, TRx) d\ (Ty, TSy)}{d(Tx, Ty)} + \beta d\ (Tx, Ty)$ for all $x, y \in X$where L ≥ 0 and α, β [0.1] with $\alpha + \beta < 1$.

Then T, *R* and *S* have a unique common fixed point in *X*.

Proof: The proof of corollary immediately follows by putting L= 0 in previous theorem in 3.3.

[B] GENERALIZED DASS AND GUPTA CONTRACTION

Definition 3.6: [13] Let (X, d) be a cone metric space. A self mapping$T: X \rightarrow X$ is called generalized Dass and Gupta contraction, if it satisfies the following condition:

$d(Tx, Ty) \leq \frac{\alpha d(y, Ty)[1 + d(x, Tx)]}{1 + d(x, y)} + \beta d(x, y) + Lmin\,\{d(x, Tx), d(x, Tx), d(y, Ty)\}$ for all $x.\ y \in X$, where L≥ 0 and $\alpha, \beta \in [0,1]$ with $\alpha + \beta < 1$. (2)

Theorem 3.7: Let (X, d) be a complete cone metric space and P be a normal cone with normal constant *K*. suppose the mappings $T_1, T_2 : X \rightarrow X$ be any two generalized Dass and Gupta contractions, by definition [3.6], satisfies the following condition

$d\,(T_1x,\ T_2y) \leq \frac{\alpha d(y, T_2y)[1 + d\ (x, T_1x)]}{1 + d(x, y)} + \beta d\ (x, y) + Lmin\,\{d(x, T_1x), d(x, T_2y) d(y, T_2y)\}$ for all $x, y \in X$ where L ≥ 0 and α, β [0.1] with $\alpha + \beta < 1$. Then T_1 and T_2 have a unique fixed point in *X*.

Proof: Let x_0 be an arbitrary element and define the sequences $\{x_{2n}\}$ and $\{x_{2n+1}\}$ such that $x_{2n+1} = T_1\, x_{2n}$ And $x_{2n+2} = T_2\, x_{2n+1}$ for each n = 0, 1, 2, … … ∞.

We have
$d(x_{2n}, x_{2n+1}) = d(T_1x_{2n-1}, T_2x_{2n})$

$$\leq \frac{\alpha d(x_{2n}, T_2x_{2n})\ [1 + d(x_{2n-1}, T_1x_{2n-1})]}{1 + d(x_{2n-1}, x_{2n})} + \beta d(x_{2n-1}, x_{2n})$$

$$+ Lmin\{d(x_{2n-1}, T_1x_{2n-1}), d(x_{2n-1}, T_2x_{2n}) d(x_{2n}, T_1x_{2n-1})\}$$

$$\leq \frac{\alpha d(x_{2n},\ x_{2n+1})[1 + d(x_{2n-1}, x_{2n})}{1 + d(x_{2n-1}, x_{2n})} + \beta d(x_{2n-1}, x_{2n}) + Lmin\{d(x_{2n-1}, x_{2n}), d(x_{2n-1}, x_{2n+1}) d(x_{2n}, x_{2n})\}$$

$\leq \alpha d(x_{2n}, x_{2n+1}) + \beta d(x_{2n+1}, x_{2n})$

$(1-\alpha) d(x_{2n}, x_{2n+1}) \leq \beta d(x_{2n-1}, x_{2n}$

$d(x_{2n}, x_{2n+1}) \leq \frac{\beta}{1-\beta}(x_{2n-1}, x_{2n})$

$= kd(x_{2n-1}, x_{2n})$.

where $k = \frac{\beta}{1-\beta}$, $\alpha + \beta < 1, 0 < K < 1$.

Hence by induction

$d(x_{2n}, x_{2n+1}) = kd(x_{2n-1}, x_{2n}) \leq k^2 d(x_{2n-2}, x_{2n-1}) \leq \ldots k^{2n} d(x_0, x_1)$

Now for $m > n$, we have

$d(x_{2n}, x_{2m}) \leq d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2}) \ldots\ldots\ldots + d(x_{2n+2m-1}, x_{2m})$

$\leq (k^{2n} + k^{2n+1} + \ldots\ldots + k^{2n+2m-1}) d(x_0, x_1)$

$\leq \frac{k^{2n}}{1-k} d(x_0, x_1)$

We get $\|d(x_{2n}, x_{2m}\| \leq K \frac{k^{2n}}{1-k} \|d(x_0, x_1)\|$ which implies that$d(x_{2n}, x_{2m}) \to 0$ as$n, m \to \infty$.

Hence $\{x_{2n}\}$ is a Cauchy sequence in *X*. Since (*X*, *d*) be a complete cone metric space. So there exist $x^* \in X$ such that $x_{2n} \to x^*(n \to \infty)$.

Since $d(x^*, T_1x^*) \leq d(x^*, x_{2n+1}) + d(x_{2n+1}, T_1x^*)$

$\leq d(x^*, x_{2n+1}) + d(T_1x_{2n}, T_1x^*)$

$\leq d(x^*, x_{2n+1}) + \frac{\alpha d(x^*, T_1x^*)[1+d(x_{2n}, T_1x_{2n})]}{1+d(x_{2n}, x^*)} + \beta d(x_{2n}, x^*)$
$+ Lmin\{d(x_{2n}, T_1x_{2n}), d(x_{2n}, T_1x^*) d(x^*, T_1x_{2n})\}$

$\leq d(x^*, x_{2n+1}) + \frac{\alpha d(x^*, x^*)[1+d(x_{2n}, x_{2n+1})]}{1+d(x_{2n}, x^*)} + \beta d(x_{2n}, x^*)$
$+ Lmin\{d(x_{2n}, x_{2n+1}), d(x_{2n}, x^*), d(x^*, x_{2n+1})\}$

$\leq d(x^*, x_{2n+1}) + \beta d(x_{2n}, x^*) + Lmin\{d(x_{2n}, x_{2n+1}), d(x_{2n}, x^*), d(x^*, x_{2n+1})\}$

So, using the condition of normality of cone

$\| d(x^*, T_1x^*)\| \leq K(k\|d(x^*, x_{2n+1})\| + \beta\|d(x_{2n}, x^*)\| + Lmind(x_{2n}, x_{2n+1}), d(x_{2n}, x^*), d(x^*, x_{2n+1}) \to 0$.

Hence $\| d(x^*, T_1x^*)\| = 0$. This implies that $x^* = T_1x^*$. So, x^* is a fixed point of T_1.Now if y^* is another fixed point of T_1. Then we have

$\|d(x^*, y*)\| = 0$ and $x^* = y^*$ .Therefore the fixed point of T_1 is unique.

Similarly, it can be established that $T_2x^* = x*$. Hence $T_1x^* = x^* = T_2x^*$. Thus x^* is common fixed point of T_1 and T_2. This completes the proof.

Corollary 3.8: Let (X, d) be a complete cone metric space and *P* be a normal cone with normal constant *K*. suppose the mappings $T_1, T_2 : X \to X$ be any two generalized Dass and Gupta contractions, by definition [3.5], satisfies the following condition

$d(T_1x, T_2y) \leq \frac{\alpha d(y, T_2y)[1+d(x, T_1x)]}{1+d(x,y)} + \beta d(x, y)$ for all *x*, *y*∈ *X* where L ≥ 0 and α, β [0, 1] with $\alpha + \beta < 1$.

Then T_1 and T_2 have a unique fixed point in *X*.

Proof: The proof of corollary immediately follows by putting *L*= 0 in previous theorem in 3.7.

Next, we shall generalize the theorem 2.1 of [14] in cone metric space for generalized contractive mappings with rational expression.

Theorem 3.9: Let (X, d) be a complete cone metric space and P be a normal cone with normal constant *K*. suppose $T, R, S : X \to X$ be three self maps contractions satisfies the following condition

$d\,(TRx,\ TSy) \le \frac{\alpha[d(Tx,TRx)d\,(Tx,TSy)+d(Ty,TSy)d(Ty,TRx)}{d(Tx,TSy)+d(Ty,TRx)}$ all *x*, $y \in X$where and $\alpha \in [0.1]$ with Then T, R and S have a unique common fixed point in *X*.

Proof: Let x_0 be an arbitrary element and define the sequences $\{x_{2n}\}$ and $\{x_{2n+1}\}$ such tha t $x_{2n+1} = R\,x_{2n}$ And $x_{2n+2} = S\,x_{2n+1}$ for each *n* = 0, 1, 2,- - --- ---∞.

Then we have

$$d(Tx_{2n+1}, Tx_{2n+2}) = d(TRx_{2n}, TSx_{2n+1})$$

$$\le \frac{\alpha[d(Tx_{2n},TRx_{2n})d(Tx_{2n},TSx_{2n+1})+d(Tx_{2n+1},TSx_{2n+1})d(Tx_{2n+1},TRx_{2n})}{d(Tx_{2n},TSx_{2n+1})+d(Tx_{2n+1},TRx_{2n})}$$

$$= \frac{\alpha[d(Tx_{2n},Tx_{2n+1})d(Tx_{2n},Tx_{2n+2})+d(Tx_{2n+1},Tx_{2n+2})d(Tx_{2n+1},Tx_{2n+1})}{d(Tx_{2n},Tx_{2n+2})+d(Tx_{2n+1},Tx_{2n+1})}$$

$$\le \alpha d(Tx_{2n,}Tx_{2n+1})$$

Hence by induction

$$d(x_{2n}, x_{2n+1}) \le \alpha d(Tx_{2n,}Tx_{2n+1}) \le \alpha^2 d(x_{2n+1}, x_{2n+2}) \le \ldots\ldots \le \alpha^{2n} d(Tx_0, Tx_1)$$

Now for $m \ge n$, we have

$$d(Tx_{2m}, Tx_{2n}) \le d(Tx_{2n}, Tx_{2n+1}) + d(Tx_{2n+1}, Tx_{2n+2}) + \ldots + d(Tx_{2m-1}. Tx_{2m})$$

$$\le (\alpha^{2n} + \alpha^{2n+1} + \cdots + \alpha^{2m-1}) d(Tx_0. Tx_1)$$

$$\le \frac{\alpha^{2n}}{1-\alpha}\ d(Tx_0, Tx_1).$$

We get$\|d(Tx_{2m}, Tx_{2n})\| \le \frac{\alpha^{2n}}{1-\alpha} K \|d(Tx_0, Tx_1)\|$.

This implies that$d(Tx_{2m}, Tx_{2n}) \to 0 (m, n \to \infty)$. Hence$\{Tx_{2n}\}$ is a Cauchy sequence. Since (X, d) is a complete cone metric space. So, there exist$x^* \in X$ such that $Tx_{2n} \to x^* (n \to \infty)$.

Since $d(Tx^*, TRx^*) \le d(Tx^*, Tx_{2n+1}) + d(Tx_{2n+1}, TRx^*)$

$$\le d(Tx^*, Tx_{2n+1}) + d(TRx_{2n}, TRx^*)$$

$$\le d(Tx^*, Tx_{2n+1}) + \frac{\alpha[d(Tx_{2n},TRx_{2n})d(Tx_{2n},TRx^*)+d(Tx^*,TRx^*)d(Tx^*,TRx_{2n})}{d(Tx_{2n},TRx^*)+d(Tx^*,TRx_{2n})}$$

$$= d(Tx^*, Tx_{2n+1}) + \frac{\alpha[d(Tx_{2n},Tx_{2n+1})d(Tx_{2n},Tx^*)+d(Tx^*,Tx^*)d(Tx^*,Tx_{2n+1})]}{d(Tx_{2n,Tx^*})+d(Tx^*,Tx_{2n+1})}$$

So, using the condition of normality of the cone

$$\|d(Tx^*, TRx^*)\| \le K(\|d(Tx^*, Tx_{2n+1})\| + \frac{\alpha[\|d(Tx_{2n},TRx_{2n})d(Tx_{2n},TRx^*)\|+\|d(Tx^*,Tx^*)d(Tx^*,Tx_{2n+1})\|]}{\|d(Tx_{2n,Tx^*})+d(Tx^*,Tx_{2n+1})\|})$$

As $n \to 0$ we have $\|d(Tx^*, TRx^*)\| \le 0$, a contradiction. So $\|d(Tx^*, TRx^*)\| = 0$. This implies that $Tx^* = TRx^*$. So $x^* = Rx^*$ is fixed point of R. Now if $y^*$is another fixed point of R. Then we have

$$d(Tx^*, Ty^*) = d(TRx^*, TRy^*)$$

$$\le K\ \frac{\alpha[d(Tx^*,TRx^*)d(Tx^*,TRy*)+d(Ty^*,TRy^*)d(Ty^*,TRx^*)}{d(Tx^*,TRy^*)+d(Ty^*,TRx^*)}$$

© 2013, IJMA. All Rights Reserved

So, $\|d(Tx^*, Ty^*)\|$ = 0. Implies that $Tx^* = Ty^*$. Therefore $x^* = y^*$ is unique the fixed point R.

Similarly, it can be established that $TSx^* = x^*$. Hence $TRx^* = x^* = TSx^*$. Thus x^* is the common fixed point R and S. This completes the proof.

Corollary 4.0: Let (X, d) be a complete cone metric space and P be a normal cone with normal constant *K*. suppose $T, R, S : X \to X$ be three self maps contractions satisfies the following condition

$d\,(TR^n x,\ TS^n y) \leq \frac{\alpha[d(Tx,TR^n x)d\,(Tx,TS^n y)+d(Ty,TS^n y)d(Ty,TR^n x)}{d(Tx,TS^n y)+d(Ty,TR^n x)}$ all x, $y \in X$where and $\alpha \in [0.1]$ and $n \in N$. Then T, R and S have a unique common fixed point in *X*.

REFERENCES

1. B. Fisher, Common fixed points and constant mappings satisfying rational inequality, Math. Sem. Notes, Univ. Kobe, (1978)
2. R. Uthaya Kumar and G.A. Prabhakar, Common fixed point theorems in cone metric space for rational contractions, Int. J. of. Anal. And appl.vol.3(2)(2013),112-118
3. B. Fisher, Fixed points, Common fixed points and cons and constant mappings, Studia Science Math. Hungar. 11 (1978) 467-470
4. Huang Long–Gaung, Zhang Xian, Cone metric spaces and Fixed Point Theorems of Contractive mappings. J. Math. Anal. Appl. 332 (2007) 1468-1476.
5. N. Bajaj, Some maps on unique common fixed points, Indian j. Pure and Appl. Math. 15(8) (1984) 843-848.
6. Sh. Rezapour, R. Hamlbarani, some notes on the paper “Cone metric spaces and Fixed Point Theorems of Contractive mappings. J. math. Anal. Appl. 345(2), 2008, 719-724.
7. Xiao an Sun, Ian Zhao, Guotao Wang, New common fixed point theorems for maps on cone metric spaces, Appl. Math. Letter 23(2010) ,1033-1037
8. Mehdi Asadi, S. Mansour vaezpour, Vladimir Rakocevic, Billy E. Rhoades, Fixed point theorems for contractive mapping in cone metric spaces, Math.Commun.16 (2011)147-155.
9. Mahpeyker Ozturkon, Metin Basarr, Some common fixed point theorems with rational expressions on cone metric spaces over Banach algebra, Hacettepe j. math and stat. 41(2)(2012) 211-222.
10. J. O. Olaleru, some generalizations of fixed point theorems in cone metric spaces, Fixed point theory and Applications Vol. 2009,Article ID 65791
11. B. K. Dass and S. Gupta, An extension of Banach contraction principle through rational expressions, Indian J. Pure Appl. Math. 6(1975) 1455-1458
12. D. S. Jaggi, Some Unique fixed point theorems, Indian J. Pure Appl. Math. 8 (1977) 223-230.
13. Muhammad Arshad Erdal Karapinar Jamsaid Ahmad, Some Unique fixed point theorems for rational contractions in partially ordered metric spaces, J. of Ineq. And Appl.2013, 2013:248 DOI: 10.1148|1029-242X-2013-248.
14. Sandeep Bhatt, Shruti Chaukiyal, R. C. Dimri, Common fixed point of mapping rational inequality in complex valued metric spaces, Int. j. Pure and Appl. Math, 73(2) (2011) 159- 164.

Source of support: Nil, Conflict of interest: None Declared