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ON CERTAIN TOPOLOGICAL STRUCTURES OF ORLICZ SPACE $(S((X, ||.||), \Phi, \bar{\alpha}, \bar{u}), F)$ OF VECTOR VALUED SEQUENCES

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ABSTRACT

The aim of this paper is to introduce and study a new class $S((X, ||.||), \Phi, \bar{\alpha}, \bar{u})$ of vector valued sequences defined by using Orlicz function Φ as a generalization of the basic space of converging sequences. Besides the investigation of linear topological structures of the class $(S((X, ||.||), \Phi, \bar{\alpha}, \bar{u}), F)$ when topologized it with suitable natural paranorm, our primarily interest is to explore the conditions pertaining to the containment relation of the class $S((X, ||.||), \Phi, \bar{\alpha}, \bar{u})$ in terms of $\bar{\alpha}$ and $\bar{u}$.

Keywords and Phrases: *Paranormed Space, Sequence Space, Orlicz Function, Orlicz Space, Solid Space, GK –, GC–space*

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1. INTRODUCTION AND PRELIMINARIES

The concept of paranorm is closely related to linear metric space, (see, Wilansky [15]) and its studies on sequence spaces were initiated by Maddox [10] and many others.

A paranormed space (X, F) is a linear space X with zero element θ together with a function F: $X \to \boldsymbol{R}_+$ (called a paranorm on X) which satisfies the following axioms:

For all $\xi, \eta \in X$, $F(\theta) = 0$; $F(\xi) = F(-\xi)$; $F(\xi + \eta) \le F(\xi) + F(\eta)$ and scalar multiplication is continuous.

We note that the continuity of scalar multiplication is equivalent to
(i) if $F(\xi_n) \to 0$ and $\gamma_n \to \gamma$ as $n \to \infty$, then $F(\gamma_n \xi_n) \to 0$ as $n \to \infty$, and
(ii) if $\gamma_n \to 0$ as $n \to \infty$ and ξ be any element in X, then $F(\gamma_n \xi) \to 0$, (see, Wilansky [15]).

In particular, various types of paranormed sequence spaces were further investigated by several workers, (see [1], [2], [6], [11]).

Let X be a normed space over $\boldsymbol{C}$, the field of complex numbers. Let $\omega(X)$ denote the inear space of all sequences $\bar{\xi} = (\xi_k)$ with $\xi_k \in X$, $k \ge 1$ with usual coordinate wise operations i.e., $\bar{\xi} + \bar{\eta} = (\xi_k + \eta_k)$ and $\alpha \bar{\xi} = (\alpha\xi_k)$,for each $\bar{\xi}, \bar{\eta} \in \omega(X)$ and $\alpha \in \boldsymbol{C}$.

We shall denote $\omega(\boldsymbol{C})$ by ω. Further, $\bar{\lambda} = (\lambda_k) \in \omega$ and $\bar{\xi} \in \omega(X)$ we shall write

$$\bar{\lambda}\,\bar{\xi} = (\lambda_k \xi_k).$$

Further by a vector (scalar) valued sequence space we mean a linear subspace of $\omega(X)$.

A function $\Phi : [0, \infty) \to [0, \infty)$ is called an *Orlicz function*, if it is continuous, non-decreasing and convex with $\Phi(0) = 0$, $\Phi(\xi) > 0$ for $\xi > 0$ and $\Phi(\xi) \to \infty$ as $\xi \to \infty$.

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An Orlicz function Φ can be represented in the following integral form

$$\Phi(x) = \int_0^x q(t)\, dt$$

where q, known as the kernel of Φ, is right-differentiable for $t \geq 0$, $q(0) = 0$, $q(t) > 0$ for $t > 0$, q is non decreasing, and $q(t) \to \infty$ as $t \to \infty$, (see, Krasnosel'skiî and Rutickiî [8]).

An Orlicz function Φ is said to satisfy Δ_2 - condition for all values of t, if there exists a constant $K > 0$ such that

$$\Phi(2t) \leq K\, \Phi(t), \text{ for all } t \geq 0.$$

The Δ_2-condition is equivalent to the satisfaction of inequality

$$\Phi(Lt) \leq K\, L\, \Phi(t)$$

for all values of t for which $L>1$, (see, Krasnosel'skiî and Rutickiî [8]).

Lindenstrauss and Tzafriri [9] used the idea of Orlicz function to construct the sequence space ℓ_Φ of scalars (ξ_k) such that

$$\ell_\Phi = \left\{ \bar{\xi} = (\xi_k) \in \omega : \sum_{k=1}^{\infty} \Phi\left(\frac{|\xi_k|}{r}\right) < \infty \text{ for some } r > 0 \right\}.$$

The space ℓ_Φ with the norm

$$\| \bar{\xi} \|_\Phi = \inf \left\{ r > 0 : \sum_{k=1}^{\infty} \Phi\left(\frac{|\xi_k|}{r}\right) \leq 1 \right\}$$

becomes a Banach space which is called an *Orlicz sequence space*. The space ℓ_Φ is closely related to the space ℓ_p which is an Orlicz sequence space with

$$\Phi(\xi) = \xi^p : 1 \leq p < \infty.$$

Subsequently, various algebraic and topological properties of sequence spaces using Orlicz function have been studied in [2], [3],[4], [5], [6], [7],[11], [12], [13], [14], as the generalization of various well known sequence spaces.

A sequence space S is said to be *solid* if $\bar{\xi} = (\xi_k) \in S$ and $\bar{\lambda} = (\lambda_k)$ a sequence of scalars with $|\lambda_k| \leq 1$, for all $k \geq 1$, then $\bar{\lambda}\, \bar{\xi} = (\lambda_k\, \xi_k) \in S$.

Now, corresponding to K, AK, AD properties of a scalar sequence space, see [9], we have the following definition

A normed space X - valued topological sequence space $(S(X), \Im)$ (i.e., $S(X)$ equipped with the linear topology $\Im$ is said to be a GK-space if the map

$$P_k : S(X) \to X,\ P_k(\bar{x}) = x_k$$

is continuous for each k. A GK-space is called a GC – space if the map

$$R_k : X \to S(X),\ R_k(x) = \delta_k(x),\ x \in X$$

is continuous for each $k \geq 1$, where $\delta_k(x) = (\theta, \theta, \theta, \ldots, \theta, x, \theta, \ldots)$, x is at k^{th} place.

Following inequality will also be used in this paper:

$$|\xi + \eta|^{u_k} \leq D\, \{|\xi|^{u_k} + |\eta|^{u_k}\},$$

where $\xi, \eta \in \boldsymbol{C}$, $0 < u_k \leq \sup_k u_k = L$, and $D = \max(1, 2^{L-1})$.

2. THE CLASS $S((X, \|.\|), \Phi, \bar{\alpha}, \bar{u})$ OF VECTOR SEQUENCES

Let $\bar{u} = (u_k)$ and $\bar{v} = (v_k)$ be any sequences of strictly positive real numbers and $\bar{\alpha} = (\alpha_k)$ and $\bar{\gamma} = (\gamma_k)$ be sequences of non zero complex numbers. We now introduce the following class of Banach space valued sequences

$$S((X, \|.\|), \Phi, \bar{\alpha}, \bar{u}) = \{\bar{\xi} = (\xi_k) : \xi_k \in X,\ k \geq 1 \text{ and } \Phi\left(\frac{\|\alpha_k\, \xi_k\|^{u_k}}{r}\right) \to 0 \text{ as } k \to \infty, \text{ for some } r > 0\} \quad (2.1)$$

Further when $u_k = 1$ for all k, then $S((X, \|.\|), \Phi, \bar{\alpha}, \bar{u})$ will be denoted by $S((X, \|.\|), \Phi, \bar{\alpha})$ and when $\alpha_k = 1$ for all k, then $S((X, \|.\|), \Phi, \bar{\alpha}, \bar{u})$ will be denoted by $S((X, \|.\|), \Phi, \bar{u})$. If $u_k = \alpha_k = 1$ for all k, then the class $S((X, \|.\|), \Phi, \bar{\alpha}, \bar{u})$ will be denoted by $S((X, \|.\|), \Phi)$.

If in the definition (2.1), the phrase 'for some $r > 0$' is replaced by 'for every $r > 0$' then we denote this class by $\bar{S}((X, \|.\|), \Phi, \bar{\alpha}, \bar{u})$. Thus

$\bar{S}((X, \|.\|), \Phi, \bar{\alpha}, \bar{u}) = \{\bar{\xi} = (\xi_k): \xi_k \in, \Phi\left(\frac{\|\alpha_k \xi_k\|^{u_k}}{r}\right) \to 0$ as $k \to \infty$ for every $r > 0\}$.

In this section, we characterize some linear topological structures of the class $S((X, \|.\|), \Phi, \bar{\alpha}, \bar{u})$ by endowing it with suitable natural paranorm and then investigate some of the inclusion relations between the classes $S((X, \|.\|), \Phi, \bar{\alpha}, \bar{u})$ arising in terms of different $\bar{\alpha}$ and $\bar{u}$. Throughout, we shall denote $\frac{v_k}{u_k} = r_k$ and $\left|\frac{\alpha_k}{\gamma_k}\right|^{u_k} = s_k$, $\sup_k u_k = L$ for all $k \geq 1$ and for scalar α, $A[\alpha] = \max(1, |\alpha|)$.

But when the sequences u_k and v_k occur, then to distinguish L we use the notations $L(u)$ and $L(v)$ respectively.

Theorem 2.1: $S((X, \|.\|), \Phi, \bar{\alpha}, \bar{u})$ forms a vector space over the field of complex numbers if and only if $\sup_k u_k$ is bounded above.

Proof: For the necessity, suppose that $S((X, \|.\|), \Phi, \bar{\alpha}, \bar{u})$ is a linear space over C but $\sup_k u_k = \infty$. Then there exists a sequence $(k(n))$ of positive integers satisfying $1 \leq k(n) < k(n+1)$, $n \geq 1$, for which $u_{k(n)} > n$, for each $n \geq 1$.

Now, corresponding to $z \in X$ with $\|z\| = 1$, we define a sequence $\bar{\xi} = (\xi_k)$ by

$$\xi_k = \begin{cases} \alpha_{k(n)}^{-1} n^{-2/u_{k(n)}} z, & \text{if } k = k(n), n \geq 1, \text{and} \\ \theta, & \text{otherwise.} \end{cases} \tag{2.2}$$

Let $r > 0$. Then for $k = k(n)$, $n \geq 1$ and in view of (2.2), using convexity of Φ we have

$$\Phi\left(\frac{\|\alpha_k \xi_k\|^{u_k}}{r}\right) = \Phi\left(\frac{\|n^{-2/u_{k(n)}} z\|^{u_{k(n)}}}{r}\right)$$
$$= \Phi\left(\frac{1}{n^2 r}\right) \leq \Phi\left(\frac{1}{r}\right) \frac{1}{n^2}$$

and $\Phi\left(\frac{\|\alpha_k \xi_k\|^{u_k}}{r}\right) = 0$, otherwise.

Which shows that $\Phi\left(\frac{\|\alpha_k \xi_k\|^{u_k}}{r}\right) \to 0$ as $n \to \infty$ and so $\bar{\xi} \in S((X, \|.\|), \Phi, \bar{\alpha}, \bar{u})$. On the other hand for any $r > 0$ and scalar $\nu = 4$, we find that for $k = k(n)$, $n \geq 1$

$$\Phi\left(\frac{\|\alpha_k \nu \xi_k\|^{u_k}}{r}\right) = \Phi\left(\frac{\|4\, n^{-2/u_{k(n)}} z\|^{u_{k(n)}}}{r}\right)$$
$$= \Phi\left(\frac{4^n}{n^2 r}\right) \geq \Phi\left(\frac{1}{r}\right).$$

This shows that $\nu \bar{\xi} \notin S((X, \|.\|), \Phi, \bar{\alpha}, \bar{u})$, which contradicts our assumption.

For the sufficiency, assume that $\sup_k u_k = L < \infty$. Let $\bar{\xi} = (\xi_k)$, $\bar{\eta} = (\eta_k) \in S((X, \|.\|), \Phi, \bar{\alpha}, \bar{u})$ and $\beta, \nu \in C$. Then there exist $r_1 > 0$ and $r_2 > 0$ such that

$$\Phi\left(\frac{\|\alpha_k \xi_k\|^{u_k}}{r_1}\right) \to 0 \text{ and } \Phi\left(\frac{\|\alpha_k \eta_k\|^{u_k}}{r_2}\right) \to 0, \text{ as } k \to \infty.$$

We now choose $r_3 > 0$ such that $2D\, r_1 \max(1, |\beta|^L) \le r_3$ and $2D\, r_2 \max(1, |\nu|^L) \le r_3$.

For such r_3, using non decreasing and convex properties of Φ, we have

$$\Phi\left(\frac{\|\alpha_k(\beta\,\xi_k + \nu\eta_k)\|^{u_k}}{r_3}\right) \le \Phi\left(\frac{D\|\beta\alpha_k\,\xi_k\|^{u_k} + D\|\nu\alpha_k\,\eta_k\|^{u_k}}{r_3}\right)$$

$$= \Phi\left(\frac{D|\beta|^{u_k}\|\alpha_k\,\xi_k\|^{u_k}}{r_3} + \frac{D|\nu|^{u_k}\|\alpha_k\,\eta_k\|^{u_k}}{r_3}\right)$$

$$\le \Phi\left(\frac{D\,max(1,|\beta|^L)\,\|\alpha_k\,\xi_k\|^{u_k}}{r_3} + \frac{D\,max(1,|\nu|^L)\,\|\alpha_k\,\eta_k\|^{u_k}}{r_3}\right)$$

$$\le \Phi\left(\frac{1}{2\,r_1}\|\alpha_k\,\xi_k\|^{u_k} + \frac{1}{2\,r_2}\|\alpha_k\,\eta_k\|^{u_k}\right)$$

$$\le \frac{1}{2}\,\Phi\left(\frac{\|\alpha_k\,\xi_k\|^{u_k}}{r_1}\right) + \frac{1}{2}\,\Phi\left(\frac{\|\alpha_k\,\eta_k\|^{u_k}}{r_2}\right) \to 0, \text{ as } k \to \infty.$$

This implies that $S((X,\|\,.\,\|), \Phi, \bar{\alpha}, \bar{u})$ forms a vector space over $\boldsymbol{C}$.

This completes the proof.

Theorem 2.2: $S((X, \|\,.\,\|),\Phi, \bar{\alpha}, \bar{u})$ forms a paranormed space with respect to

$$F(\bar{\xi}) = \inf\{r > 0: \sup_k \Phi\left(\frac{\|\alpha_k\,\xi_k\|^{u_k/L}}{\varepsilon}\right) \le 1\} \tag{2.3}$$

Proof: Let $\bar{\xi} = (\xi_k) \in S((X,\|\,.\,\|), \Phi, \bar{\alpha}, \bar{u})$ and $\bar{\theta}$ be the zero elements of $S((X,\|\,.\,\|), \Phi, \bar{\alpha}, \bar{u})$.

Obviously, $F(\bar{\theta}) = 0$ and $F(-\bar{\xi}) = F(\bar{\xi})$. Let

$$A(\bar{\xi}) = \{r > 0: \sup_k \Phi\left(\frac{\|\alpha_k\,\xi_k\|^{u_k/L}}{r}\right) \le 1\}, \text{ for } \bar{\xi} \in S((X,\|\,.\,\|, \Phi, \bar{\alpha}, \bar{u}) \tag{2.4}$$

Now for $\bar{\xi} = (\xi_k)$, $\bar{\eta} = (\eta_k) \in S((X,\|\,.\,\|), \Phi, \bar{\alpha}, \bar{u})$, consider $r_1 \in A(\bar{\xi})$ and $r_2 \in A(\bar{\eta})$.

Then clearly by the convexity of Φ we have

$$\Phi\left(\frac{\|\alpha_k(\xi_k + \eta_k)\|^{u_k/L}}{r_1 + r_2}\right) \le \frac{r_1}{r_1 + r_2}\sup_k \Phi\left(\frac{\|\alpha_k\,\xi_k\|^{u_k/L}}{r_1}\right) + \frac{r_2}{r_1 + r_2}\sup_k \Phi\left(\frac{\|\alpha_k\,\eta_k\|^{u_k/L}}{r_2}\right) \le 1.$$

This shows that $r_1 + r_2 \in A(\bar{\xi} + \bar{\eta})$. Thus $F(\bar{\xi}+\bar{\eta}) \le r_1+r_2$ for each $r_1 \in A(\bar{\xi})$ and $r_2 \in A(\bar{\eta})$ implies that

$$F(\bar{\xi}+\bar{\eta}) \le F(\bar{\xi}) + F(\bar{\eta}).$$

Finally we show the continuity of scalar multiplication. Let $\bar{\xi}^{(n)} = (\xi_k^{(n)})$ be a sequence in $S((X,\|\,.\,\|), \Phi, \bar{\alpha}, \bar{u})$ such that $F(\bar{\xi}^{(n)}) \to 0$ as $n \to \infty$ and (γ_n) a sequence of scalars such that $\gamma_n \to \gamma$. Then in view of (2.3), we have

$$F(\gamma_n\,\bar{\xi}^{(n)}) = \inf\left\{r : \sup_k \Phi\left(\frac{\|\alpha_k\gamma_n\,\xi_k^{(n)}\|^{u_k/L}}{r}\right) \le 1\right\}$$

$$\le \inf\left\{r : \sup_k \Phi\left(\frac{H^{u_k/L}\,\|\alpha_k\,\xi_k^{(n)}\|^{u_k/L}}{\rho}\right) \le 1\right\}$$

where $H = \sup_n|\gamma_n|$ and $|\gamma_n|^{u_k/L} \le H^{u_k/L}$ is used. Thus for $t = \max(1, H)$, we get

$$F(\gamma_n\,\bar{\xi}^{(n)}) \le \inf\left\{r : \sup_k \Phi\left(\frac{t\,\|\alpha_k\,\xi_k^{(n)}\|^{u_k/L}}{r}\right) \le 1\right\}.$$

Now if we take $\frac{r}{t} = s$, then

$$F(\gamma_n \bar{\xi}^{(n)}) \le \inf \left\{ st : \sup_k \Phi\left(\frac{\|\alpha_k \xi_k^{(n)}\|^{u_k/L}}{s}\right) \le 1 \right\}$$
$$= t \quad g(\bar{\xi}^{(n)})$$

implies that

$F(\gamma_n \bar{\xi}^{(n)}) \to 0$, as $F(\bar{\xi}^{(n)}) \to 0$ as $n \to \infty$.

Next let $\gamma_n \to 0$ as $n \to \infty$ and $\bar{\xi}$ be any element in $S((X,||.||), \Phi, \bar{\alpha}, \bar{u})$. We show that $F(\gamma_n \bar{\xi}) \to 0$.

Now for $0 < \varepsilon < 1$, we can find a positive integer N such that $|\gamma_n| \le \varepsilon$ for all $n \ge N$. But taking $\inf_k u_k = l > 0$, we get $|\gamma_n|^{u_k/L} \le \varepsilon^{l/L}$. This shows that for each $n \ge N$, we have

$$\Phi\left(\frac{\|\gamma_n \alpha_k \xi_k\|^{u_k/L}}{r}\right) \le \Phi\left(\frac{\varepsilon^{l/L} \|\alpha_k \xi_k\|^{u_k/L}}{r}\right)$$

and consequently in view of (2.4) we get

$$A(\varepsilon^{l/L}\bar{\xi}) \subseteq A(\gamma_n \bar{\xi}).$$

Hence we have

$$\inf\{r: r \in A(\gamma_n \bar{\xi})\} \le \inf\{r: r \in A(\varepsilon^{l/L} \bar{\xi})\}$$
$$= \varepsilon^{l/L} \inf\{r: r \in A(\bar{\xi})\}$$

which shows that

$F(\gamma_n \bar{\xi}) \le \varepsilon^{l/L} F(\bar{\xi})$ for all $n \ge N$, i.e., $F(\gamma_n \bar{\xi}) \to 0$ as $n \to \infty$.

Hence $S((X, ||.||), \Phi, \bar{\alpha}, \bar{u})$ forms a total paranormed space. This completes the poof.

Theorem 2.3: If $\bar{u} = (u_k)$ is bounded, then $(S((X,||.||), \Phi, \bar{\alpha}, \bar{u}), F)$ forms a GK – and GC – space.

Proof: Let $\bar{\xi} = (\xi_k) \in S((X,||.||), \Phi, \bar{\alpha}, \bar{u})$. Then as defined in (2.3) of paranorm F, we see that

$$\sup_k \Phi\left(\frac{\|\alpha_k \xi_k\|^{u_k/L}}{F(\bar{\xi})}\right) \le 1 \text{ and hence } \Phi\left(\frac{\|\alpha_k \xi_k\|^{u_k/L}}{F(\bar{\xi})}\right) \le 1.$$

Let k_0 be a fixed positive real number such that $\Phi(k_0) \ge 1$, then

$$\Phi\left(\frac{\|\alpha_k \xi_k\|^{u_k/L}}{F(\bar{\xi})}\right) \le \Phi(k_0).$$

Since Φ is non-decreasing therefore

$$\|\alpha_k \xi_k\|^{u_k/L} < k_0 F(\bar{\xi})$$

or $\|\xi_k\| < |\alpha_k|^{-1} [k_0 F(\bar{\xi})]^{L/uk}$

and so

$$\|P_k(\bar{\xi})\| = \|\xi_k\| < |\alpha_k|^{-1} [k_0 F(\bar{\xi})]^{L/uk}$$

shows that $P_k : S((X,||.||), \Phi, \bar{\alpha}, \bar{u}) \to X$, where $P_k(\bar{\xi}) = \xi_k$ for each $\xi_k \in X$, $k \ge 1$, is continuous and hence $(S((X,||.||), \Phi, \bar{\alpha}, \bar{u}), F)$ forms a GK- space.

To prove $(S((X,||\,.\,||),\Phi,\bar{\alpha},\bar{u}),F)$ a GC –space, let $R_n : X \to S((X,||\,.\,||),\Phi,\bar{\alpha},\bar{u})$, defined by

$$R_n(\xi) = \delta_n(\xi),\ n \geq 1,\ \xi \in X,$$

where $\delta_n(\xi) = (\theta, \theta, \theta, \ldots, \theta, \xi, \theta, \theta, \ldots)$, ξ at n^{th} place. We prove the continuity of R_n at origin.

Clearly $\delta_n(\xi) \in S((X,||\,.\,||),\Phi,\bar{\alpha},\bar{u})$. Now for given $\varepsilon > 0$, we choose $\xi \in X$ such that

$$\Phi\left(\frac{||\alpha_n \xi||^{u_n/L}}{\varepsilon}\right) < 1. \tag{2.5}$$

Then in view of (2.5), we have

$$F(R_n(\xi)) = F(\delta_n(\xi)) = \inf\{r > 0 : \sup_n \Phi\left(\frac{||\alpha_n \xi||^{u_n/L}}{r}\right) \leq 1\} \leq \varepsilon,$$

and therefore R_n is continuous for each $n \geq 1$. Hence $(S((X,||\,.\,||),\Phi,\bar{\alpha},\bar{u}),F$ is a GC space.

Theorem 2.4: $S((X,||\,.\,||),\Phi,\bar{\alpha},\bar{u})$ forms a solid space.

Proof: Let $\bar{\xi} = (\xi_k) \in S((X,||\,.\,||),\Phi,\bar{\alpha},\bar{u})$. So that

$$\Phi\left(\frac{||\alpha_k \xi_k||^{u_k}}{r}\right) \to 0 \text{ as } k \to \infty \text{ for some } r > 0.$$

Let (β_k) be a sequence of scalars such that $|\beta_k| \leq 1$ for all $k \geq 1$. Since Φ is non-decreasing, we have

$$\Phi\left(\frac{||\alpha_k \beta_k \xi_k||^{u_k}}{r}\right) = \Phi\left(\frac{|\beta_k|^{u_k} ||\alpha_k \xi_k||^{u_k}}{r}\right)$$
$$\leq \Phi\left(\frac{||\alpha_k \xi_k||^{u_k}}{r}\right) \to 0 \text{ as } k \to \infty.$$

and hence $(\beta_k \xi_k) \in S((X,||\,.\,||),\Phi,\bar{\alpha},\bar{u})$. So $S((X,||\,.\,||),\Phi,\bar{\alpha},\bar{u})$ is solid.

Theorem 2.5: $S((X,||\,.\,||),\Phi,\bar{\alpha},\bar{u}) \subseteq \bar{S}((X,||\,.\,||),\Phi,\bar{\alpha},\bar{u})$, if Φ satisfies Δ_2 condition.

Proof: Let $\bar{\xi} \in S((X,||\,.\,||),\Phi,\bar{\alpha},\bar{u})$. Then for some $r > 0$, $\Phi\left(\frac{||\alpha_k \xi_k||^{u_k}}{r}\right) \to 0$ as $k \to \infty$.

Let us consider an arbitrary $r_1 > 0$. If $r > r_1$, so that $\frac{r}{r_1} > 1$ then by using Δ_2 condition of Φ, we get

$$\Phi\left(\frac{||\alpha_k \xi_k||^{u_k}}{r_1}\right) = \Phi\left(\frac{r}{r_1}\frac{||\alpha_k \xi_k||^{u_k}}{r}\right)$$
$$\leq K.\frac{r}{r_1}\Phi\left(\frac{||\alpha_k \xi_k||^{u_k}}{r}\right) \to 0 \text{ as } k \to \infty,$$

where K is the number involved in Δ_2 - condition .

On the other hand, if $r \leq r_1$, then obviously we have

$$\Phi\left(\frac{||\alpha_k \xi_k||^{u_k}}{r_1}\right) \leq \Phi\left(\frac{||\alpha_k \xi_k||^{u_k}}{r}\right) \to 0 \text{ as } k \to \infty,$$

and hence we get $\bar{\xi} \in \bar{S}((X,||\,.\,||),\Phi,\bar{\alpha},\bar{u})$. Hence $\bar{\xi} \in \bar{S}((X,||\,.\,||),\Phi,\bar{\alpha},\bar{u})$.

Theorem 2.6: For any $\bar{\alpha} = (\alpha_k)$, $S((X,||\,.\,||),\Phi,\bar{\alpha},\bar{u}) \subset S((X,||\,.\,||),\Phi,\bar{\alpha},\bar{v})$ if and only if $\liminf_k r_k > 0$.

Proof: For the necessity**,** assume that $S((X,||\,.\,||),\Phi,\bar{\alpha},\bar{u}) \subset S((X,||\,.\,||),\Phi,\bar{\alpha},\bar{v})$ but $\liminf_k \frac{v_k}{u_k} = 0$. Then there exists a sequence $(k(n))$ of positive integers such that $1 \leq k(n) < k(n+1)$ for which

$$n\, v_{k(n)} < u_{k(n)}, \text{ for each } n \geq 1. \tag{2.6}$$

Now, corresponding to $z \in X$ with $\| z \| = 1$, we define a sequence $\bar{\xi} = (\xi_k)$ by

$$\xi_k = \begin{cases} \alpha_{k(n)}^{-1} n^{-1/u_{k(n)}} \; z \text{ , if } k = k(n),\ n \geq 1, \text{and} \\ \theta, \text{ otherwise.} \end{cases} \tag{2.7}$$

Let $r > 0$. Then for each $k = k(n)$, $n \geq 1$, we have

$$\Phi\left(\frac{\| \alpha_k \ \xi_k \|^{u_k}}{r}\right) = \Phi\left(\frac{\| n^{-1/u_{k(n)}} \ z \|^{u_{k(n)}}}{r}\right) \leq \frac{1}{n} \ \Phi\left(\frac{1}{r}\right)$$

and $\Phi\left(\frac{\| \alpha_k \ \xi_k \|^{u_k}}{r}\right) = 0$, for $k \neq k(n)$, $n \geq 1$,

showing that $\bar{\xi} \in S((X ,\| . \|) , \Phi, \bar{\alpha} , \bar{u})$. But for each $k = k(n)$, $n \geq 1$ and in view of (2.6) and (2.7), we have

$$\Phi\left(\frac{\| \alpha_k \ \xi_k \|^{v_k}}{r}\right) = \Phi\left(\frac{\| n^{-1/ u_{k(n)}} \ z \|^{v_{k(n)}}}{r}\right)$$
$$\geq \Phi\left(\frac{1}{r\, n^{1/n}}\right) \geq \Phi\left(\frac{1}{r\sqrt{e}}\right),$$

This shows that $\bar{x} \notin S((X,\| . \|), \Phi, \bar{\alpha}, \bar{v})$, a contradiction.

For the sufficiency, assume that $\liminf_k \ r_k > 0$. Then there exists a $m > 0$ such that $v_k > m\, u_k$ for all sufficiently large values of k. Let $\bar{\xi} = (\xi_k) \in S((X,\| . \|), \Phi, \bar{\alpha}, \bar{u})$. Then for some $r > 0$,

$$\Phi\left(\frac{\| \alpha_k \ \xi_k \|^{u_k}}{r}\right) \to 0 \text{ as } k \to \infty.$$

Hence for a given $\varepsilon > 0$, if we choose $0 < \eta < 1$ satisfying $\eta^m \ \Phi\left(\frac{1}{r}\right) < \varepsilon$, then we have

$$\Phi\left(\frac{\| \alpha_k \ \xi_k \|^{u_k}}{r}\right) < \Phi\left(\frac{\eta}{r}\right)$$

for all sufficiently large values of k.

Since Φ is non decreasing, therefore for all large values of k

$\| \alpha_k \ \xi_k \|^{u_k} < \eta < 1$ and so $\| \alpha_k \ \xi_k \| < 1$.

Hence using the convexity of Φ, we have

$$\Phi\left(\frac{\| \alpha_k \ \xi_k \|^{v_k}}{r}\right) \leq \Phi\left(\frac{[\| \alpha_k \ \xi_k \|^{u_k}]^m}{r}\right)$$
$$\leq \Phi\left(\frac{\eta^m}{r}\right) \leq \eta^m \ \Phi\left(\frac{1}{r}\right) < \varepsilon,$$

for all sufficiently large values of k. This implies that $\bar{x} \in S((X,\| . \|), \Phi, \bar{\alpha}, \bar{v})$ and hence.

$$S((X ,\| . \|) , \Phi, \bar{\alpha} , \bar{u}) \subset S((X ,\| . \|) , \Phi, \bar{\alpha} , \bar{v}) .$$

This completes the proof.

Theorem 2.7: For any $\bar{\alpha} = (\alpha_k)$, $S((X,\| . \|), \Phi, \bar{\alpha}, \bar{v}) \subset S((X,\| . \|), \Phi, \bar{\alpha}, \bar{u})$ if and only if $\limsup_k r_k < \infty$.

Proof: For the necessity, suppose that the inclusion holds but $\limsup_k \frac{v_k}{u_k} = \infty$. Then there exists a sequence $(k(n))$ of positive integers such that $1 \leq k(n) < k(n+1)$, $n \geq 1$, for which

$$v_{k(n)} > n\, u_{k(n)} \text{ for all } n \geq 1. \tag{2.8}$$

Corresponding to $z \in X$ with $\| z \| = 1$, we define a sequence $\bar{\xi} = (\xi_k)$ by

$$\xi_k = \begin{cases} \alpha_{k(n)}^{-1} n^{-1/v_{k(n)}} \; z \text{ , if } k = k(n),\ n \geq 1 \text{,and} \\ \theta, \text{ otherwise.} \end{cases} \tag{2.9}$$

Let $r > 0$. Then for each $k = k(n)$, $n \geq 1$, we have

$$\Phi\left(\frac{\| \alpha_k \; \xi_k \|^{v_k}}{r}\right) = \Phi\left(\frac{\| n^{-1/v_{k(n)}} \; z \|^{v_{k(n)}}}{r}\right)$$
$$= \Phi\left(\frac{\| z \|^{v_{k(n)}}}{nr}\right) \leq \frac{1}{n} \Phi\left(\frac{1}{r}\right)$$

and $\Phi\left(\frac{\| \alpha_k \; \xi_k \|^{v_k}}{r}\right) = 0$, for $k \neq k(n)$, $n \geq 1$, showing that $\bar{\xi} \in S((X ,\| . \|), \Phi, \bar{\alpha} , \bar{v})$. But for each $k = k(n)$, $n \geq 1$ and in view of (2.8) and (2.9), we have

$$\Phi\left(\frac{\| \alpha_k \; \xi_k \|^{u_k}}{r}\right) = \Phi\left(\frac{\| n^{-1/v_{k(n)}} \; z \|^{u_{k(n)}}}{r}\right)$$
$$\geq \Phi\left(\frac{1}{r\, n^{1/n}}\right) \geq \Phi\left(\frac{1}{r\sqrt{e}}\right),$$

This shows that $\bar{x} \notin S((X,\| . \|), \Phi, \bar{\alpha}, \bar{u})$, a contradiction.

For the sufficiency, assume that $\limsup_k \frac{v_k}{u_k} < \infty$. Then there exists $L > 0$ such that $v_k < L\, u_k$ for all sufficiently large values of k. Then analogous to the Theorem 2.6 we can easily show that
$S((X,\| . \|), \Phi, \bar{\alpha}, \bar{v}) \subset S((X,\| . \|), \Phi, \bar{\alpha}, \bar{u})$.

This completes the proof.

On combining the Theorems 2.6 and 2. 7, one obtain

Theorem 2.8: For any $\bar{\alpha} = (\alpha_k)$, $S((X,\| . \|), \Phi, \bar{\alpha}, \bar{u}) = S((X,\| . \|), \Phi, \bar{\alpha}, \bar{v})$ if and only if

$0 < \liminf_k r_k \leq \limsup_k r_k < \infty$.

Corollary 2.9: For any $\bar{\alpha} = (\alpha_k)$,
(i) $S((X,\| . \|), \Phi, \bar{\alpha}) \subset S((X,\| . \|), \Phi, \bar{\alpha}, \bar{u})$ if and only if $\liminf_k u_k > 0$;
(ii) $S((X,\| . \|), \Phi, \bar{\alpha}, \bar{u}) \subset S((X,\| . \|), \Phi, \bar{\alpha})$ if and only if $\limsup_k u_k < \infty$;
(iii) $S((X,\| . \|), \Phi, \bar{\alpha}, \bar{u}) = S((X,\| . \|), \Phi, \bar{\alpha})$ if and only if $0 < \liminf_k u_k \leq \limsup_k u_k < \infty$.

Proof: The proof follows by taking $u_k = 1$ for all k and is $\bar{v}$ replaced by $\bar{u}$ in the Theorems 2.6, 2.7 and 2.8.

Theorem 2.10: For any $\bar{u} = (u_k)$, $S((X,\| . \|), \Phi, \bar{\alpha}, \bar{u}) \subset S((X,\| . \|), \Phi, \bar{\gamma}, \bar{u})$ if and only if $\liminf_k s_k > 0$.

Proof: For the necessity**,** assume that $S((X,\| . \|), \Phi, \bar{\alpha}, \bar{u}) \subset S((X,\| . \|), \Phi, \bar{\gamma}, \bar{u})$ holds but $\liminf_k \left|\frac{\alpha_k}{\gamma_k}\right|^{u_k} = 0$. So we can find a sequence $(k(n))$ of integers such that

$1 \leq k(n) < k(n+1)$, $n \geq 1$

for which

$$n^2 |\alpha_{k(n)}|^{u_{k(n)}} < |\gamma_{k(n)}|^{u_{k(n)}}, \text{ for all } n \geq 1. \tag{2.10}$$

Now corresponding to $z \in X$ with $\| z \| = 1$, we define a sequence $\bar{\xi} = (\xi_k)$ by

$$\xi_k = \begin{cases} \alpha_{k(n)}^{-1} n^{-2/u_{k(n)}} \; z \text{ , if } k = k(n),\ n \geq 1, \text{and} \\ \theta, \text{ otherwise.} \end{cases} \tag{2.11}$$

Let $r > 0$. Then for $k = k(n)$, $n \geq 1$, using convexity of Φ we have

$$\Phi\left(\frac{\| \alpha_k \xi_k \|^{u_k}}{r}\right) = \Phi\left(\frac{\| n^{-2/u_{k(n)}} \; z \|^{u_{k(n)}}}{r}\right) \leq \Phi\left(\frac{1}{r}\right) \frac{1}{n^2}$$

and $\Phi\left(\frac{\| \alpha_k \xi_k \|^{u_k}}{r}\right) = 0$,otherwise.

Which shows that $\Phi\left(\frac{\| \alpha_k \xi_k \|^{u_k}}{r}\right) \to 0$ as $n \to \infty$ and so $\bar{x} \in S((X,\| . \|), \Phi, \bar{\alpha}, \bar{u})$.

But on the other hand for any $r > 0$ and $k = k(n)$, $n \geq 1$ and in view of (2.10) and (2.11), we have

$$\Phi\left(\frac{\| \gamma_k \xi_k \|^{u_k}}{r}\right) = \Phi\left(\frac{\left\| \frac{\gamma_{k(n)}}{\alpha_{k(n)}} \; n^{-2/u_{k(n)}} \; z \right\|^{u_{k(n)}}}{r}\right)$$

$$= \Phi\left(\left|\frac{\gamma_{k(n)}}{\alpha_{k(n)}}\right|^{u_{k(n)}} \frac{1}{n^2 r}\right) \geq \Phi\left(\frac{1}{r}\right), \text{ showing that } \bar{\xi} \notin S((X, \| . \|), \Phi, \bar{\gamma}, \bar{u}), \text{ a contradiction.}$$

For the sufficiency, assume that $\liminf_k s_k > 0$. Then there exists $m>0$ such that $m\,|\gamma_k|^{u_k} < |\alpha_k|^{u_k}$ for all sufficiently large values of k. Let $\bar{\xi} = (\xi_k) \in S((X,\| . \|), \Phi, \bar{\alpha}, \bar{u})$, then for some $r > 0$,

$$\Phi\left(\frac{\| \alpha_k \xi_k \|^{u_k}}{r}\right) \to 0 \text{ as } k \to \infty.$$

Now we choose $r_1 > 0$ such that $r \leq m\, r_1$. Since Φ is non decreasing, we have

$$\Phi\left(\frac{\| \gamma_k \xi_k \|^{u_k}}{r_1}\right) \leq \Phi\left(\frac{\| \alpha_k \xi_k \|^{u_k}}{r}\right)$$

which clearly implies that $\bar{\xi} \in S((X,\| . \|), \Phi, \bar{\gamma}, \bar{u})$. Hence

$$S((X,\| . \|), \Phi, \bar{\alpha}, \bar{u}) \subset S((X,\| . \|), \Phi, \bar{\gamma}, \bar{u}).$$

This completes the proof.

Theorem 2.11: For any $\bar{u} = (u_k)$, $S((X,\| . \|), \Phi, \bar{\gamma}, \bar{u}) \subset S((X,\| . \|), \Phi, \bar{\alpha}, \bar{u})$ if and only if $\limsup_k s_k < \infty$.

Proof: For the necessity, suppose that

$$S((X,\| . \|), \Phi, \bar{\gamma}, \bar{u}) \subset S((X,\| . \|), \Phi, \bar{\alpha}, \bar{u})$$

holds but $\limsup_k \left|\frac{\alpha_k}{\gamma_k}\right|^{u_k} = \infty$. Then there exists a sequence $(k(n))$ of positive integers satisfying $1 \leq k(n) < k(n+1)$, $n \geq 1$,for which

$$|\alpha_{k(n)}|^{u_{k(n)}} > n^2\, |\gamma_{k(n)}|^{u_{k(n)}}, \text{ for all } n \geq 1. \tag{2.12}$$

Now, as written to Theorem 2.10, corresponding to $z \in X$ with $\| z \| = 1$, we define a sequence

$\bar{\xi} = (\xi_k)$ by

$$\xi_k = \begin{cases} \gamma_{k(n)}^{-1} n^{-2/u_{k(n)}} z \text{ , if } k = k(n),\ n \geq 1, \text{and} \\ \theta, \text{ otherwise.} \end{cases} \tag{2.13}$$

Then in view of (2.12) and (2.13), we can show that $\bar{\xi} \in S((X,||\,.\,||), \Phi, \bar{\gamma}, \bar{u})$, but $\bar{\xi} \notin S((X,||\,.\,||), \Phi, \bar{\alpha}, \bar{u})$, a contradiction.

For the sufficiency, assume that $\limsup_k s_k < \infty$.Then we can find a positive number L such that

$$L\,|\mu_k|^{u_k} > |\alpha_k|^{u_k}$$

for all sufficiently large values of k. Then analogous to the Theorem 2.10, the result follows. This completes the proof.

On combining the Theorems 2.10 and 2.11, we get

Theorem 2.12: For any $\bar{u} = (u_k)$, $S((X,||\,.\,||), \Phi, \bar{\alpha}, \bar{u}) = S((X,||\,.\,||), \Phi, \bar{\gamma}, \bar{u})$ if and only if

$$0 < \liminf_k s_k \le \limsup_k s_k < \infty.$$

Corollary 2.13: For any $\bar{u} = (u_k)$,

(i) $S((X,||\,.\,||), \Phi, \bar{\alpha}, \bar{u}) \subset S((X,||\,.\,||), \Phi, \bar{u})$ if and only if $\liminf_k |\alpha_k|^{u_k} > 0$;

(ii) $S((X,||\,.\,||), \Phi, \bar{u}) \subset S((X,||\,.\,||), \Phi, \bar{\alpha}, \bar{u})$ if and only if $\limsup_k |\alpha_k|^{u_k} < \infty$;

(iii) $S((X,||\,.\,||), \Phi, \bar{\alpha}, \bar{u}) = S((X,||\,.\,||), \Phi, \bar{u})$ if and only if $0 < \liminf_k |\alpha_k|^{u_k} \le \limsup_k |\alpha_k|^{u_k} < \infty$.

Proof: By taking $\gamma_k = 1$ for all k, in Theorems 2.10 , 2.11 and 2.12, the assertions (i), (ii) and (iii) follow.

Theorem 2.14: For any sequences $\bar{u} = (u_k)$, $\bar{v} = (v_k)$, $\bar{\alpha} = (\alpha_k)$ and $\bar{\gamma} = (\gamma_k)$, $S((X,||\,.\,||), \Phi, \bar{\alpha}, \bar{u}) \subset S((X,||.||), \Phi, \bar{\gamma}, \bar{v})$ if and only if

(i) $\liminf_k r_k > 0$, and

(ii) $\liminf_k s_k > 0$

Proof: Proof of the theorem follows immediately from the Theorems 2.6 and 2.10.

In the following example, we show that $S((X,||\,.\,||), \Phi, \bar{\alpha}, \bar{u})$ may strictly be contained in $S((X,||\,.\,||), \Phi, \bar{\gamma}, \bar{v})$ in spite of the conditions (i) and (ii) of Theorem 2.14 are satisfied.

Example: 2.15 Let X be a Banach space and consider a sequence $\bar{\xi} = (\xi_k)$ defined by

$$\xi_k = k^{-k} z, \text{ if } k = 1, 2, 3, \ldots \tag{2.14}$$

where $z \in X$ such that $||z|| = 1$. Further, let $u_k = k^{-1}$, if k is odd integer, $u_k = k^{-2}$, if k is even integer, $v_k = k^{-1}$ for all values of k, $\alpha_k = 3^k$, $\gamma_k = 2^k$ for all values of k. Then

$s_k = \left|\dfrac{\alpha_k}{\gamma_k}\right|^{u_k} = \dfrac{3}{2}$ or $\left(\dfrac{3}{2}\right)^{1/k}$ according as k is odd or even integers and hence $\liminf_k s_k > 1$.

Further, $r_k = 1$, if k is odd integers, $r_k = k$, if k is even integers.

Therefore $\liminf_k r_k > 0$. Hence the conditions (i) and (ii) of Theorem 2.14 are satisfied.

Let $r > 0$, then in view of (2.14), we have

$$\Phi\left(\frac{||\gamma_k \xi_k||^{v_k}}{r}\right) = \Phi\left(\frac{||2^k k^{-k} z||^{1/k}}{r}\right)$$

$$\le \frac{1}{k}\Phi\left(\frac{2}{r}\right) \to 0 \text{ as } k \to \infty,$$ showing that $\bar{x} \in S((X,||\,.\,||), \Phi, \bar{\gamma}, \bar{v})$. But for even integers k,

$$\Phi\left(\frac{||\alpha_k \xi_k||^{u_k}}{r}\right) = \Phi\left(\frac{||3^k k^{-k} z||^{1/k^2}}{r}\right) = \Phi\left(\frac{(3/k)^{1/k}}{r}\right)$$

$$> \Phi\left(\frac{1}{2r}\right).$$

This implies that $\bar{\xi} \notin S((X,||\,.\,||), \Phi, \bar{\alpha}, \bar{u})$. Thus, the containment of $S((X,||\,.\,||), \Phi, \bar{\alpha}, \bar{u})$ in $S((X,||\,.\,||), \Phi, \bar{\gamma}, \bar{v})$ is strict inspite of the satisfaction of the conditions (i) and (ii) of the Theorem 2.14.

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