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ON A GENERALIZED HYPERBOLIC MEASURE OF FUZZY ENTROPY

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ABSTRACT

In the present communication a brief review of generalized measures of fuzzy information corresponding to already existing probabilistic measures of entropy is presented. Some axiomatic characterizations of fuzzy entropy are presented. A new generalized hyperbolic measure of fuzzy entropy is proposed.

Key Words: *Fuzzy set, Hyperbolic fuzzy entropy, Additive measure.*

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1. INTRODUCTION

The fuzzy set theory, proposed by Zadeh [20] gained vital interdisciplinary importance in many fields such as pattern recognition, image processing, fuzzy aircraft control, feature selection, Bioinformatics etc. Fuzzy sets play a significant role in many deployed systems because of their capability to model non statistical imprecision.

Uncertainty and fuzziness are basic elements of the human perspective and of many real world objectives. The main use of information is to remove the uncertainty and fuzziness. The measure of uncertainty removed is information measure while the measure of vagueness is measure of fuzziness. The uncertainty arises due to linguistic impression or vagueness is fuzzy uncertainty. It deals with the situations where the set boundaries are not sharply defined. Fuzzy entropy is a basic concept in fuzzy set theory; many researchers studied it from the point of view of the theory or from the point of view of applications.

De-Luca and Termini [3] introduced a measure of fuzziness corresponding to the information theoretic entropy of Shannon [15]. Renyi [13] generalized Shannon's entropy to overcome its non suitability in certain situations. Bhandari and Pal [1] proposed generalized measure of fuzzy entropy corresponding to Renyi [13] entropy. In last two decade many measures of fuzzy entropy proposed corresponding to probabilistic measures of entropy.

2. AXIOMATIC DEVELOPMENT OF FUZZY INFORMATION MEASURE OR MEASURE OF FUZZY ENTROPY

In fuzzy set theory, the entropy is measure of fuzziness which expresses the amount of average ambiguity /difficulty in taking a decision weather an element belong to a set or not. A measure of fuzziness H (A) of a fuzzy set A should have the at least following four properties [3].

E_1 (sharpness): H (A) is minimum if and only if A is crisp set, that is, $\mu_A(X)$ =0 or 1 for all $x \in X$.

E_2 (Maximality): H (A) is maximum if and only if A is most fuzzy set, that is, $\mu_A(X)$ =0.5 for all $x \in X$.

E_3 (Resolution): H (A) ≥H (A*), where A* is the sharpened version of A.

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E_4 (symmetry): H (A) =H(Ā), where Ā is the complement set of A, that is, $\mu_{\bar{A}}(x) = 1 - \mu_A(x)$ for all $x \in X$.

Ebanks [4] proposed one more axiom as essential condition for validity of a measure of fuzzy entropy

E_5 (Valuation): $H(A \cup B) + H(A \cap B) = H(A) + H(B)$

DeLuca and termini [3] introduced a measure of fuzziness analogous to the information theoretic entropy of Shannon[15] as

$$H(A) = -\frac{1}{n}\sum_{i=1}^{n}\left\{ \mu_A(x_i) \log \mu_A(x_i) + (1-\mu_A(x_i)) \log(1-\mu_A(x_i)) \right\} \tag{2.1}$$

One parametric generalized measure of fuzzy entropy corresponding to Renyi's [13] probabilistic measure proposed by Bhandari and Pal [1] as follows:

$$H_\alpha(A) = \frac{1}{1-\alpha}\log\left[\sum_{i=1}^{n}\left\{(\mu_A(x_i))^\alpha + (1-\mu_A(x_i))^\alpha\right\}\right], \quad \alpha > 0, \quad \alpha \neq 1 \tag{2.2}$$

Kapur [10] suggested generalized measure of fuzzy entropy corresponding to probabilistic measures proposed by Havrda and Charvat [7], Kapur [9], Sharma and Taneja [17] respectively by Eq.(2.3-2.5) .

$$H_\alpha(A) = \frac{1}{1-\alpha}\log\left[\sum_{i=1}^{n}\left\{(\mu_A(x_i))^\alpha + (1-\mu_A(x_i))^\alpha - 1\right\}\right], \quad \alpha > 0, \quad \alpha \neq 1 \tag{2.3}$$

$$\begin{aligned} H_\alpha(A) = {} & \frac{1}{n}\sum_{i=1}^{n}\left\{ \mu_A(x_i) \log \mu_A(x_i) + (1-\mu_A(x_i)) \log(1-\mu_A(x_i)) \right\} \\ & + \frac{1}{\alpha}\sum_{i=1}^{n}\left\{(1+\alpha\,\mu_A(x_i))\log(1+\alpha\,\mu_A(x_i))\right\} \\ & + \frac{1}{\alpha}\sum_{i=1}^{n}\left\{(1+\alpha-\alpha\,\mu_A(x_i))\log(1+\alpha-\alpha\,\mu_A(x_i))\right\} \\ & + \frac{1}{\alpha}\sum_{i=1}^{n}\left\{(1+\alpha)\log(1+\alpha)\right\}, \ \alpha > 0, \quad \alpha \neq 1 \end{aligned} \tag{2.4}$$

$$\begin{aligned} H_{\alpha,\beta}(A) = {} & \frac{1}{1-\alpha}\log\left[\sum_{i=1}^{n}\left\{(\mu_A(x_i))^\alpha + (1-\mu_A(x_i))^\alpha - (\mu_A(x_i))^\beta + (1-\mu_A(x_i))^\beta\right\}\right], \\ & \alpha \geq 1, \quad \beta \leq 1 \quad or \quad \alpha \leq 1, \quad \beta \geq 1 \end{aligned} \tag{2.5}$$

$where \quad \alpha = \beta$ if and only if each is equal to 1.

Hooda [8] suggested one and two parametric measures of fuzzy entropy corresponding to probabilistic measures proposed by Sharma and Mittal [16].

3. SOME CHARACTERIZATIONS THEOREMS OF MEASURE OF FUZZY ENTROPY

Ebank [4] suggested the following necessary and sufficient condition on functions that satisfies axioms E1-E5 given in second section for discrete fuzzy sets.

Theorem 3.1: Let $H : F(X) \to R^+$. Then H satisfies E1-E5 iff. H has the form

$$H_E(A) = \sum_{i=1}^{n} g(\mu_A(x_i)) \tag{3.1}$$

for some function $g : [0,1] \to R^+$ that satisfies:

G1: $g(0) = g(1) = 0; g(t) > 0$ for all $t \in (0,1)$

G2: $g(t) < g(0.5) = 0$ for all $t \in (0,1) - \{0.5\}$

G3: g is non decreasing on [0,0.5] and non increasing on (0.5,1].

G4: $g(t) = g(1-t)$ for all $t \in [0,1]$

Remark: To formulate a valid measure of entropy, it is sufficient to determine a function *g* with properties G1-G4.

Theorem 3.2: (Pal and Bezdek [11]**)** Let $f : [0,1] \to R^+$ be concave increasing function on [0, 1]; that is, $f'(t) > 0$ and $f''(t) < 0$ for all $t \in [0,1]$. Define $\hat{g}(t) = f(t)\, f(1-t);$ with $\hat{g}$ defined by $g(t) = \hat{g}(t) - \min_{0 \le t \le 1}.\{\hat{g}(t)\};$ Finally, for $A \in F(X)$, define

$$H_*(A) = K \sum_{i=1}^{n} g(\mu_A(x_i)) \tag{3.2}$$

where $K \in R^+$ is normalizing constant.

Remark: $H_*(A)$ is called a multiplicative measure of fuzzy entropy.

Theorem 3.3: (Pal and Bezdek [11]**)** Let $f : [0,1] \to R^+$ be concave function on [0, 1]; that is, $f''(t) < 0$ for all $t \in [0,1]$. Define $\hat{g}(t) = f(t) + f(1-t);$ with $\hat{g}$ defined by

$g(t) = \hat{g}(t) - \min_{0 \le t \le 1}.\{\hat{g}(t)\};$ Finally, for $A \in F(X)$, define

$$H_+(A) = K \sum_{i=1}^{n} g(\mu_A(x_i)) \tag{3.3}$$

where $K \in R^+$ is normalizing constant.

Remark: $H_+(A)$ is called an additive measure of fuzzy entropy.

Definition 3.4: Let $X = \{x_1, x_2, ..., x_n\}$ be a universe of discourse, the cardinality of fuzzy set A on X is defined as

$$M(A) = \sum_{i=1}^{n} \mu_A(x_i) \tag{3.4}$$

Obviously, $M(X) = n, \quad M(A_F) = \frac{n}{2},$ where A_F is most fuzzy set.

Definition 3.5: (Rudas and Kayanak [14]**)** Let A be a fuzzy set on X, we define then

$$\tilde{A}(x) = \begin{cases} \mu_A(x), & \mu_A(x) \le \frac{1}{2} \\ 1 - \mu_A(x) & \mu_A(x) \ge \frac{1}{2} \end{cases} \tag{3.5}$$

Theorem 3.6: (Rudas and Kayanak [14]**)** Let $f : R \to R$ be a monotonic increasing real function such that $f(0) = 0$ then $f(M(\tilde{A}))$ is entropy.

Remark: A reasonable relation between fuzzy cardinality and entropy of a fuzzy set A is that the crisper the fuzzy set, the crisper the fuzzy cardinality, i.e. the lower the entropy, the crisper (the clearer) the fuzzy cardinality. Obviously, the reverse relation the crisper the fuzzy cardinality, the lower the entropy must also hold. This could allow us to measure the entropy of a fuzzy set by measuring the entropy of its fuzzy cardinality.

Theorem 3.7: (Lun Fan, Liang Ma and Xin Xie [5]**)** Let $H(A)$ be entropy on $F(X)$, then $H_1(A) = \dfrac{H(A)}{2 - H(A)}$ is also an entropy on $F(X)$.

Definition 3.8: (Xuechang [19]) A fuzzy entropy *H(A)* is called a $\sigma -$entropy on $F(X)$, if *H(A)* satisfies $H(A) = H(A \cap D) + H(A \cap \overline{D})$ for all $D \in P(X)$, the power set of *X*.

Theorem 3.9: (Xuechang [19]) Let $H(A)$ be entropy on $F(X)$, then *H(A)* is called a $\sigma-$entropy on $F(X)$, iff. for all $A, B \in F(X)$

$H(A \cap B) + H(A \cup B) = H(A) + H(B)$

Theorem 3.10: (Xuechang [19] Let $A \in F(X)$ be arbitrary fuzzy set in X. If fuzzy entropy *H(A)* is a $\sigma-$entropy on $F(X)$ and $\hat{A}$ is fold set of A, then

$$H(\hat{A}) = H(A)$$

4. NEW GENERALIZED HYPERBOLIC MEASURE OF FUZZY ENTROPY

Bhatia and Singh [2] proposed a new parametric measure of fuzzy entropy of order α. Corresponding to this generalized probabilistic measure of entropy, we propose a new hyperbolic measure of fuzzy entropy as follows:

$$H_\alpha(A) = -\frac{1}{\sinh(\alpha)}\left[\sum_{i=1}^{n}\mu_A(x_i)\sinh(\alpha\log\mu_A(x_i)) + \sum_{i=1}^{n}(1-\mu_A(x_i))\sinh(\alpha\log(1-\mu_A(x_i)))\right] \tag{4.1}$$

The real number α (0< α<1) is associated with the non extensiveness of the system.

Theorem 4.1: $H_\alpha(A)$ is valid measure of fuzzy entropy.

Proof: Consider the function $g:[0,1] \to R$ defined by

$$g(t) = -\frac{1}{\sinh(\alpha)}\left[t\sinh(\alpha\log t) + (1-t)\sinh(\alpha\log(1-t))\right] \tag{4.2}$$

Clearly, $H_\alpha(A) = \sum_{i=1}^{n} g(\mu_A(x_i))$ (4.3)

From the Eq. (4.2) it is clear that

$g(0) = g(1) = 0$

$$\frac{dg(t)}{dt} = -\frac{1}{\sinh(\alpha)}\left[t^\alpha - (1-t)^\alpha\right]\left[(\alpha+1)t^\alpha(1-t)^\alpha + 1 - \alpha\right]$$

Let $t \in [0, 0.5]$, we have,

$\frac{dg(t)}{dt} > 0$ when 0< α < 1

Similarly for $t \in [0.5, 1]$,

$\frac{dg(t)}{dt} < 0$ when 0< α < 1

Thus $g(t)$ is non decreasing on [0, 0.5] and non increasing on (0.5, 1] and maximum value is obtained at *t*= 0.5
Also,

$g(t) = g(1-t)$ for all $t \in [0,1]$

Therefore, in view of theorem (3.1) , $H_\alpha(A)$ is valid measure of fuzzy entropy.

Theorem 4.2: $H_\alpha(A)$ is an additive measure of fuzzy entropy.

Proof: Let $f:[0,1] \to R^+$ be defined as

$$f(t) = -\frac{t\sinh(\alpha \log t)}{\sinh(\alpha)}$$

$f(t)$ can also be expressed as follows:

$$f(t) = -\frac{t}{\sinh(\alpha)}\left[\frac{t^{2\alpha}-1}{2t^{\alpha}}\right]$$

We have, $f(t) \geq 0$ for all $t \in [0,1]$

Also, we have,

$$f''(t) = -\frac{\alpha}{t^{\alpha+1}\sinh(\alpha)}\left[\alpha(t^{2\alpha}-1)+(t^{2\alpha}+1)\right]$$

The quantity in the bracket is positive for all $t \in [0,1]$ and for all α > 0. Consequently,

$$f''(t) < 0$$

Now, we have, $\hat{g}(t) = f(t) + f(1-t);$ with $\hat{g}$ defined by

$$g(t) = \hat{g}(t) - \min_{0\leq t\leq 1}.\{\hat{g}(t)\};$$

Here

$$\hat{g}(t) = -\frac{1}{\sinh(\alpha)}\left[t\sinh(\alpha \log t) + (1-t)\sinh(\alpha \log(1-t))\right]$$

$$\hat{g}(0) = \hat{g}(1) = 0$$

Therefore,

$$\min_{0\leq t\leq 1}.\{\hat{g}(t)\} = 0$$

Thus, we have

$$g(t) = \hat{g}(t) = -\frac{1}{\sinh(\alpha)}\left[t\sinh(\alpha \log t) + (1-t)\sinh(\alpha \log(1-t))\right]$$

Finally, for $A \in F(X)$, we have $H_\alpha(A) = \sum_{i=1}^{n} g(\mu_A(x_i))$

where $K = 1 \in R^+$ is normalizing constant.

Thus in view of Theorem (3.3), $H_\alpha(A)$ is an additive measure of fuzzy entropy.

Special Case: When α = 0, we have

$$H_\alpha(A) = H(A) = -\frac{1}{n}\sum_{i=1}^{n}\left\{\mu_A(x_i)\log\mu_A(x_i) + (1-\mu_A(x_i))\log(1-\mu_A(x_i))\right\}$$

which is classical measure of fuzzy entropy proposed by De-Luca and Termini.

Theorem 4.3: $H_\alpha(A)$ is a $\sigma-$entropy.

Proof: In view of Theorem 3.1 and Theorem 4.1 axioms E1-E5 are satisfied by $H_\alpha(A)$. By Theorem 3.9 axiom E5 is sufficient for a fuzzy entropy to be σ- entropy.

Theorem 4.4: $H_\alpha(A)$ is monotonic increasing function of α.

Proof: The determining function of $H_\alpha(A)$ is

$$h(x) = -\frac{1}{\sinh(\alpha)}\left[x\sinh(\alpha\log x) + (1-x)\sinh(\alpha\log(1-x))\right],\ x \in [0,1]$$

Graph of function h(x) is given in the fig.1, and from the fig. 1 it is clear that h(x) is monotonically increasing function of α.

Hence, $H_\alpha(A)$ is monotonic increasing function of α.

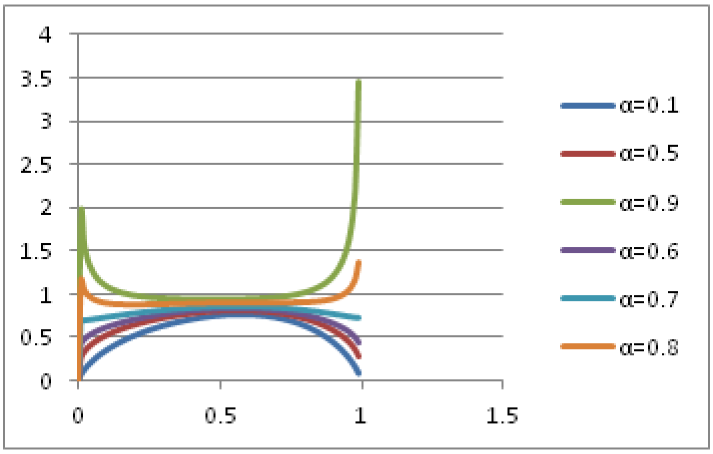

Figure 1 Monotonic Behavior of $H_\alpha(A)$

5. CONCLUSION

The generalized hyperbolic measure of fuzzy entropy proposed in this communication is additive, therefore its applicability to find the entropy of a fuzzy set in an extensive systems seems to be suitable. Two dimensional generalized fuzzy entropy is also suitable for image thresholding [6].Roughness of rough sets can also be measured using generalized fuzzy entropies[18]. Our future work will focus on development of more application oriented measures of fuzzy entropy and their characterizations.

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