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PART I: SYSTEMATIC APPROACH TO BASIC CONCEPTS OF POLYNOMIALS (SABCP)

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ABSTRACT

This paper, explain new way to approach the standard mathematical concepts of algebra like factor theorem, remainder theorem, extended multiplicity and rational root test and fundamental theorems of reducible polynomials over a field of rational numbers $\mathbb{Q}$ through their order of polynomials.

Keyword and Phrases: *Order of polynomials.*

INTRODUCTION

Based on constructivists' perspective, in this study I might attempt to develop new approach to come up with the basic concepts of polynomials. Initially, this paper focuses on the relationships between the cumulative sum of coefficients of polynomials with their roots, factors and remainder of polynomials. If we consider, anyone who has accustomed to follow a factor theorem, to find the roots of $g(x) = x^3 + 20x^2 + 124x + 240$, (1). He/she might directly consider possible factors of $a_0 = 240$, like $\pm1, \pm2, \pm3, \pm4, \pm5, \pm6, \pm8, \pm10, \pm12, \pm15, \pm16, \pm24, \pm30, \pm60, \pm24, \pm30, \pm60, \pm120$ and ± 240 and then check the roots of g(x), by inserting the factors in place of x. But, this is complicated, when we come to the ideas in this paper, we consider the finite sums of the coefficients of g(x), the so called order of g(x). Let it be $u = \sum_{j=0}^{3} a_j = 385$ and taking the possible factors $u = 385 = 5x7x11$. Then using the relation $r_j = -k_j + 1$, for some $j = z$. If k_j for some j=z, a possible factor of $\beta = 385$, if and only if $r_j = -k_j + 1$, for some j=z, must be a possible factor of $a_0 = 240$. So, if we take $u = 385 = 5x7x11$, $k_1 = 5, k_2 = 7$ and $k_3 = 11$, but $r_1 = -k_1 + 1 = -4$, $r_2 = -k_2 + 1 = -6$ and $r_3 = -k_3 + 1 = -10$ such that $r_1 = -4, r_2 = -6$ and $r_3 = -10$ all divide $a_0 = 240$. Due to the fact that, it might reduce and overcome the challenging steps in constructing the above basic concepts of polynomials.

Therefore, systematic approach to the basic concepts of polynomials will expand the previous approach to factor theorem, remainder theorem and reducible and irreducible criterions of polynomials. Moreover, in the next series of this journal, it might put extra contribution for some challenging and unsolved problems under irreducibility of polynomials like primitive polynomials of the form $g(x) = x^{p-1} + x^{p-2} + x^{p-3} + \cdots + x^2 + x + 1$; composition function of the form $h(x) = x^8 + 98x^4 + 1$ and polynomials of the form $k(x) = x^3 + 8x^2 + x + 1$, $g(x) = x^3 + 4x^2 + 3x - 1$ and $u(x) = x^4 - x^3 + x^2 + x + 1$ etc.

REDUCIBLE POLYNOMIALS

Definition: 1 Let define polynomial P(x) of degree n;
$P(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots \ldots a_1 x + a_0$, for $a_n \neq 0$, (1), A polynomial P(x) is said to be reducible over $\mathbb{Q}$, if it has at least k possible factors over $\mathbb{Q}$, for $1 \leq k \leq n$.

Definition: 2 let $f(x) \in \mathbb{Z}[x]$ be a polynomial of degree n with integer coefficients said to be completely reducible over rational field $\mathbb{Q}$, if it can be written as the product of n linear factors of f(x) over $\mathbb{Q}$. Similarly, if f(x) is said to be partially reducible over $\mathbb{Q}$, if it can be written as the product of polynomials of lower degree than f(x) with integer coefficients over $\mathbb{Q}$.

Definition: 3 An order β_n of nth degree polynomial is defined by the sum of its coefficients;

$$\beta_n = \sum_{j=0}^{n} \frac{a_j}{a_n}, \ (1)$$

where a_j, for $j = 0,1,2, \ldots n$ are integer coefficients

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Definition: 4 Let P(x) be polynomial of degree n with integer coefficients, then there exists a unique order of P(x) represented by β_n and defined by;

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1 x + a_0, (1)$$

where $\beta_n = \sum_{j=0}^{n} \frac{a_j}{a_n}$ order of P(x). Then, a polynomial of degree n has a unique order representation of β_n.

FUNDAMENTAL THEOREM OF REDUCIBLE POLYNOMIALS

Definition: 5 Let f(x) and g(x) monic integer coefficients polynomials of degree n and m with order β_n and α_m respectively. Then polynomials h(x), k(x), z(x) defined as the product, sum and division of f(x) and g(x) respectively, $h(x) = f(x)g(x)$, (1); $k(x) = f(x) + g(x)$, (2) and $z(x) = \frac{f(x)}{g(x)}$, for $g(x) \neq 0$, …(3). Then the following properties hold;

1. An order φ_{n+m} of h(x); $\varphi_{n+m} = \beta_n\ \alpha_m$, (4)
2. An order ω_n of k(x); $\omega_n = \beta_n + \alpha_m$, for $n \geq m$, (5)
3. An order μ_n of z(x); $\mu_n = \frac{\beta_n}{\alpha_m}$, for $\alpha_m \neq 0$ and $n \geq m$, (6)

Lemma: 1 An order β_n of reducible polynomial $P(x) \epsilon Z[x]$ is reducible.

Proof: A monic polynomial P(x) of degree n with order β_n is reducible, then P(x) can be written as the product of polynomials of lower degree than P(x), with integer coefficients. Let f(x), g(x) and m(x) be polynomials of orders φ_f, β_g and α_h respectively with integer coefficients and their degrees lower than P(x) such that;

$$P(x) = f(x)g(x)h(x), (1)$$

Then by definition (5), order β_n of P(x) written as the product of orders φ_f, β_g and α_h such that $\beta_n = \varphi_f\ \beta_g\ \alpha_h$. Moreover, let the number of possible factors of β_n be z and less than degree of P(x). Therefore, β_n is reducible in to at most z possible factors and can be denoted by $\beta_n = \prod_{j=1}^{z} k_j$ for $z \leq n$, (2)

Lemma: 2 Let g(x) and $f(x) \neq 0$ monic integer coefficients polynomials of degree n and m with order β_n and α_m respectively and if $f(x)|g(x)$, then $\alpha_m|\beta_n$, (1)

Proof: Supposef(x)|g(x), then there exists non zero polynomial $u(x) \in \mathbb{Z}[x]$ such that $g(x) = u(x)f(x)$, (1). Let φ_k be an order of u(x) and from the given hypothesis β_n and α_m are orders of g(x) and f(x) respectively. Then from (1) and lemma (1); $\beta_n = \varphi_k \alpha_m$. Therefore, $\alpha_m|\beta_n$

Theorem: 1 let P(x) a monic integer coefficients reducible polynomial of degree n with order β_n. Then there exist unique integers $m_j \epsilon \mathbb{Z}$, and $m_j|\beta_n$, for $j = 1,2,3,..k \leq n$ such that $\beta_n = \prod_{j=1}^{k} m_j$, for $k \leq n$. If P(x) is completely reducible, then we have $\beta_n = \prod_{j=1}^{n} m_j$.

Proof: suppose P(x) is reducible, then it has at most k rational roots $r_j \neq 1$, for j=1, 2,.., k for $k < n$ and can be written as;

$$P(x) = G(x) \prod_{j=1}^{k} (x - r_j), \text{for } k < n, (1)$$

It implies β_n is reducible, then by lemma (2), their exist linear factors $m_j \epsilon \mathbb{Z}$ and $w \epsilon \mathbb{Z}$ for non-linear factors G(x) such that $m_j|\beta_n$ and $w|\beta_n$, for $j = 1,2, \ldots, k$, so we need to show $\beta_n = \prod_{j=1}^{k+1} m_j$, for $k < n$ its uniqueness. From (1) and since β_n sum of its coefficients, we have;

$$\beta_n = P(x)|_{x=1} = G(1) \prod_{j=1}^{k} (1 - r_j), \text{for } k < n, (2)$$

From (2), there is an arranged index so that the left and the right to be equal. So, let it be $m_j = (1 - r_j)$, for $j = 1,2, \ldots, k$ and since p(x) has at most k possible roots $r_j \neq 1$, for j=1,2,..,k and $k < n$, implies $\beta_n = w \prod_{j=1}^{k} (1 - rj)$, for k<n. Therefore, there exist unique integers mk+1=G1 and mjϵ $\mathbb{Z}$, such that mj|βn, for j=1,2,...,k+1 and $\beta_n = \prod_{j=1}^{z} m_j$, for $z = k + 1 \leq n$.

Hence the results

Lemma: 3 An order β_n of reducible polynomial P(x) has at most n possible factors $k_j \neq 0$, for j=1,2,..,n; provided that $\theta_n = \text{Min} \sum_{j=1}^{n} k_j$, (1), then the possible value of θ_n given by; $\theta_n = \text{Min} \sum_{j=1}^{n} k_j = \frac{na_n}{a_n} + \frac{a_{n-1}}{a_n} = n + \frac{a_{n-1}}{a_n}$, (2). Where sum is taken over the minimum number of possible factors of β_n and θ_n is the criterion of reducible polynomials.

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Proof: suppose a monic polynomial P(x) of degree n is reducible, then its order β_n is also reducible. From given hypothesis an order β_n is reducible in to at most n possible factors $k_j \neq 0$, for j=1, 2,.., n. Since P(x) is reducible, assume that there exist at most n possible roots r_j, for $j = 1,2,3..,n-1,n$ and P(x) can be written;

$$P(x) = a_n \prod_{j=1}^{n}(x - r_j), (3)$$

From the above theorem (1), we have obtained an order β_n has possible factors of the form; $\beta_n = \prod_{j=1}^{n}(1 - r_j)$, (4). From possible factors $k_j \neq 0$ of β_n, for j=1,2,..,n, then for each possible factors in (4) there exist an order $k_j = 1 - r_j$, for $j = 1,2,3..,n-1,n$. Let $r_j = -(k_j - 1)$, for $j = 1,2,3..,n-1,n$, and then substitute in (3);

$$P(x) = a_n \prod_{j=1}^{n}(x + (k_j - 1)), (5)$$

After expanding equation (5), we have obtained each coefficients of P(x) in terms of the possible factors $k_j \neq 0$, for j=1, 2,.., n of β_n. Since P(x) is monic, therefore we can obtain that the coefficients of the tem x^{n-1} as the sum of n possible factors as $\sum_{j=1}^{n} k_j - n = \frac{a_{n-1}}{a_n}$, (6). Then we obtain reducible criterion of polynomial; $\theta_n = \text{Min} \sum_{j=1}^{n} k_j = n + \frac{a_{n-1}}{a_n}$, (7)

Definition: 6 Let f(x) and g(x) are monic polynomials of degree n and m with order β_n and α_m respectively. Let θ_n and ω_m be their respective criterions of reducibility. A polynomial h(x) defined as the product of f(x) and g(x) by $h(x) = f(x)g(x)$, ..(1). Then order φ_{n+m} and criterion τ_{n+m} of h(x) of degree (n+m) is given by;

$$\varphi_{n+m} = \beta_n \alpha_m \text{ and } \tau_{n+m} = \theta_n + \omega_m, (2)$$

Therefore, (2) is called reducibility criterions

Lemma: 4 Let k_j, for j=1,2,..,$k \leq n$ be a linear factors of an order β_n of reducible polynomials such that k_j could be a unique linear factor of β_n, if and only if, $k_j | \beta_n$ and $-k_j + 1 | a_0$, for all $j = 1,2,..,k \leq n$. Otherwise, factors$-k_j + 1$ that does not divide the constant coefficient a_0 could not be the linear factor of P(x).

Proof: Since k_j, for j=1,2,..,$k \leq n$ be a linear factors of an order β_n of reducible polynomial P(x) such that every factors k_j, for j=1,2,..,$k \leq n$ of β_n can be expressed as $k_j = -r_j + 1$, for j=1,2,..,$k \leq n$, in terms of the roots of P(x). Then the constant coefficient of P(x) can be approximated by;

$$a_0 = (-1)^n \prod_{j=1}^{n} r_j, \text{for } r_j = -k_j + 1 = (-1)^n \prod_{j=1}^{n}(-k_j + 1), (1)$$

Therefore, for a unique linear factor k_j of β_n, we have $-k_j + 1 | a_0$, for all $j = 1,2,..,k \leq n$.

Theorem: 2 A monic polynomial P(x) of degree n is completely reducible over $\mathbb{Q}$, if and only if an order β_n of P(x) is completely reducible and has at most n possible factors k_j, for j=1,2,..,n, with $\theta_n = \text{Min} \sum_{j=1}^{n} k_j = \frac{na_n}{a_n} + \frac{a_{n-1}}{a_n}$ (1). Moreover, there exist polynomials of order k_j, for j=1,2,..,n possible factors of P(x).

Proof: (→) Suppose an order β_n is reducible over $\mathbb{Q}$, into at most z possible factors k_j, for j=1,2,..,z≤ n such that $\beta_n = \prod_{j=1}^{z} k_j$, which satisfy the criterion $\theta_n = \text{Min} \sum_{j=1}^{n} k_j = \frac{na_n}{a_n} + \frac{a_{n-1}}{a_n}$, so P(x) is reducible over $\mathbb{Q}$ into at most z≤ n possible factors. Conversely, suppose P(x) is reducible over $\mathbb{Q}$ then by lemma (1) an order β_n of a reducible polynomial is reducible into at most z possible factors k_j, for j=1,2,..,z≤ n with criterion $\theta_n = \text{Min} \sum_{j=1}^{n} k_j = \frac{na_n}{a_n} + \frac{a_{n-1}}{a_n}$, (1).

Let $g^{k_j}(x) = (x + (k_j - 1))$, for $j = 1,2,3, \dots \dots z-1, z$ be linear factors of P(x) with orders k_j, for $j = 1,2,..z-1, z$. To show the uniqueness by using lemma (4); if the linear factors $g^{k_j}(x)$ of order k_j, for j=1,2,..,$z \leq n$ could be a linear factor of P(x), if and only if, each factor of β_n must satisfy $k_j | \beta_n$ and $-k_j + 1 | a_0$, for all $j = 1,2,..,z \leq n$. Otherwise, factors$-k_j + 1$ that does not divide the constant coefficient a_0 could not be the linear factor of P(x).

Corollary: 1 Every linear factors $g^{k_j}(x)$ of orders k_j, for $j = 1,2, \dots, n$ and criterions θ_j, for $j = 1,2,..,n$ of reducible polynomials must satisfy;

$$k_j = \theta_j, \text{for } j = 1,2,..n-1,n, (1)$$

Proof: simply consider linear polynomials $g^{k_j}(x) = (x + (k_j - 1))$, for $j = 1,2,3, \dots, n$ and compute the corresponding orders and criterions θ_j, for $j = 1,2,..n-1,n$ such that $k_j = \theta_j$, for $j = 1,2,..n-1,n$ are fulfilled.

Theorem: 3 Let $g^{\beta_n}(x)$ reducible polynomial of degree n with order $\beta_n \neq 0$ and let k_j, for $j = 1,2,3, \dots . n$ possible factors of $\beta_n \neq 0$ and satisfy $\theta_n = \text{Min} \sum_{j=1}^{n} k_j = n + \frac{a_{n-1}}{a_n}$. Then $g^{k_j}(x) | g^{\beta_n}(x)$, if and only if $k_j | \beta_n$ for $j = 1,2, \dots . n$ and $\theta_n = n + \frac{a_{n-1}}{a_n}$, (1)

Proof: (→) let $g^{k_j}(x) | g^{\beta_n}(x)$, for $j = 1,2,3, \dots . n$. Then $g^{k_j}(x)$, for $j = 1,2, . . n$ factors of $g^{\beta_n}(x)$.

Let $g^{k_j}(x) = \left(x + \left(k_j - 1\right)\right)$ linear factors of $g^{\beta_n}(x)$, for $j = 1,2,3, \dots . n$, then

$$g^{\beta_n}(x) = a_n\left(x + (k_1 - 1)\right)\left(x + (k_2 - 1)\right) \dots . . (x + (k_{n-1} - 1))(x + (k_n - 1)), (2)$$

After expanding, then compute the finite sum of b_j, for $j = 0,1,2, . . n$ of $g^{\beta_n}(x)$ such that;

$$\beta_n = \sum_{j=0}^{n} b_j = \prod_{j=1}^{n} k_j. \text{ Similarly, the sum of the factors } k_j, \text{ for } j = 1,2, . . n \text{ of } \beta_n;$$

$$\beta_n = \prod_{j=1}^{n} k_j \text{ and } \theta_n = \sum_{j=1}^{n} k_j = n + \frac{a_{n-1}}{a_n}, (3)$$

Therefore, from (3), $k_j \backslash \beta_n$ for all $j = 1,2,3, \dots . n$. Conversely, suppose $k_j | \beta_n$ for $j = 1,2,3, \dots . n$ and $\theta_n = \text{Min} \sum_{j=1}^{n} k_j = n + \frac{a_{n-1}}{a_n}$, (4). Since reducible polynomial has at most n factors such that, for each k_j of β_n, then there exist factors $g^{k_j}(x)$ of $g^{\beta_n}(x)$ with orders k_j, for $j = 1,2,3, \dots . n$ such that $g^{\beta_n}(x)$ can be written as; $g^{\beta_n}(x) = \prod_{j=1}^{n} g^{k_j}(x)$, (5)

Therefore, we can simply conclude $g^{k_j}(x) | g^{\beta_n}(x)$, for all $j = 1,2,3, \dots . n$.

Theorem: **4** Suppose $g^{k_j}(x)$ of orders k_j, for $j = 1,2,3, \dots . n$ are factors of $P(x)$, if and only if $k_j \neq 0$, for $j = 1,2,3, \dots . n$ are the possible factors of β_n such that complete factorization of $g^{\beta_n}(x)$ is given by;

$$P(x) = a_n \prod_{j=1}^{n} g^{k_j}(x), \text{ iff} \beta_n = \prod_{j=1}^{n} k_j, (1), \text{ provided } \theta_n = \text{Min} \sum_{j=1}^{n} k_j = n + \frac{a_{n-1}}{a_n}$$

Where $g^{k_j}(x) = \left(x - r_j\right)$, for $j = 1,2, \dots \dots , n$

Proof: Let β_n with its factors such that $\beta_n = \prod_{j=1}^{n} k_j$, provided $\theta_n = n + \frac{a_{n-1}}{a_n}$, (3)

Then by theorem (3), there exist linear factors of $\mathbf{P}(\mathbf{x})$ such that $g^{k_j}(x) = \left(x + \left(k_j - 1\right)\right)$, for $j = 1, . . , n$, (4).

Hence $g^{k_j}(x) = \left(x + \left(k_j - 1\right)\right)$ of orders k_j, for $j = 1,2, . . n$ are the linear factors of $P(x)$ with order β_n.

Therefore, the factorization of $g^{\beta_n}(x)$ with order β_n is given by;

$P(x) = a_n \prod_{j=1}^{n} g^{k_j}(x)$, (5), where $g^{k_j}(x) = \left(x - r_j\right)$, $j = 1,2, . . , n$

To prove uniqueness: let's assume that there exists pair of linear factors of

$$g^{k_j}(x) = \left(x + \left(k_j - 1\right)\right) \text{ and } g^{q_j}(x) = \left(x + \left(q_j - 1\right)\right), \text{ for } j = 1,2,3, \dots \dots n - 1, n.$$

Polynomial factors of none zero orders k_j and q_j, for $j = 1,2, . . n - 1, n$ of $P(x)$ are;

$$a_n \prod_{j=1}^{n} g^{k_j}(x) = a_n \prod_{j=1}^{n} g^{q_j}(x), (6)$$

Provided that the finite sums of those possible pair of factors k_j and q_j, for $j = 1,2, . . , n$ of β_n given by $\theta_n = \text{Min} \sum_{j=1}^{n} k_j = n + \frac{a_{n-1}}{a_n}$ and $\theta_n = \text{Min} \sum_{j=1}^{n} q_j = n + \frac{a_{n-1}}{a_n}$, (7). Then from (7); $\sum_{j=1}^{n} (k_j - q_j) = 0$, (8).

Since k_j and q_j, for $j = 1,2, \dots n$ are none zero and (8) satisfy, if only if $k_j = q_j$, for $j = 1,2, . . , n$.

Thus, linear factors $g^{k_j}(x)$ and $g^{q_j}(x)$ are monic and have the same orders; $g^{k_j}(x) = g^{q_j}(x)$, for all $j = 1,2, , \dots n$, (9).

Therefore, $P(x)$ written uniquely as the factorization of its linear factors of unique orders k_j;

$$P(x) = g^{\beta_n}(x) = a_n \prod_{j=1}^{n} g^{j}(x), (10), \text{ for } g^{k_j}(x) = \left(x - r_j\right), \text{ for } j = 1,2,3, . . , n$$

Corollary: **2** Let $p(x)$ be a reducible monic polynomial of degree n with order β_n and let $f(x)$ be a factor of $P(x)$. Let k_j be the possible factors of $\beta_n = \prod_{j=1}^{n} k_j$, for $\theta_n = n + \frac{a_{n-1}}{a_n}$, then there exists an index point $z \epsilon I$, for the set $I = \{1,2,3, \dots n\}$ such that k_z is order of $f(x)$.

Proof: Directly from uniqueness of polynomials factorization

VANISHING ORDERS OF POLYNOMIALS

The above all theorems hold for nonzero order $\beta_n \neq 0$ of polynomial. Now, we will see construction of the condition in which some factors of β_n vanish provided that $\theta_n = \min \sum_{j=2}^{n} k_j = \frac{a_{n-1}}{a_n} + n$, remains constant. By fixing the number I zero's factors of an order we can use the following criterions.

Lemma: 5 Let P(x) be monic reducible n^{th} degree polynomial with β_n reducible order $\beta_n = 0$, then the next possible order β_{n-1} is defined by $\beta_{n-1} = \prod_{j=2}^{n} k_j = n + \sum_{j=1}^{n-1} \frac{(n-j)a_{n-j}}{a_n}$, provided that $\theta_n = \min \sum_{j=2}^{n} k_j = \frac{a_{n-1}}{a_n} + n$. (In higher term β_{n-1} also defined by the order of the derivatives of P(x) or **P'(x)**)

Proof: suppose $\beta_n = \prod_{j=1}^{n} k_j = 0$, then at least one factor of β_n is zero, let it be $k_1 = 0$. Assume that $\beta_{n-1} \neq 0$ with at most (n-1) none zero possible factors $k_2, k_3, k_4, k_5, \dots k_{n-1}, k_n$ and provided $\theta_n = \min \sum_{j=2}^{n} k_j = \frac{a_{n-1}}{a_n} + n$ are the same for all condition, because $k_1 = 0$. Then by induction on degree n, let it is true for n=2, because simply from θ_2, $k_2 = \frac{a_1}{a_2} + 2$. Let n =3 with order β_3 are defined by;

$$k_1 + k_2 + k_3 = \frac{a_2}{a_3} + 3 = \theta_3, \text{ and } k_1\, k_2 k_3 = \frac{a_2}{a_3} + \frac{a_1}{a_3} + \frac{a_0}{a_3} + 1 = \beta_3, (2)$$

$$k_1.k_2 + k_1.k_3 + k_2\, k_3 = \frac{2a_2}{a_3} + \frac{a_1}{a_3} + 3, (3)$$

Where k_1, k_2 and k_3 are linear factors of order β_3 in terms of coefficients a_3, a_2, a_1 and a_0. In similar fashion, we can construct the relationship between all factors k_1, $k_2, k_3, \dots k_{n-1}, k_n$ of β_n with coefficients $a_n, a_{n-1}, \dots\dots a_1$ and a_0. By our supposition $k_1 = 0$, we missing one variable so we should determine the remaining (n-1) none zero factors of β_n. For n=3, since $\beta_3 = 0$, let it be $k_1 = 0$, then we obtain that;

$$k_2 + k_3 = \frac{a_2}{a_3} + 3 = \theta_3, \text{ and } k_2\, k_3 = \frac{2a_2}{a_3} + \frac{a_1}{a_3} + 3, (4)$$

For n=4, $\beta_4 = 0$, let $k_1 = 0$ such that $k_2 + k_3 + k_4 = \frac{a_3}{a_4} + 4 = \theta_4$ and $k_2 k_3 k_4 = \frac{3a_3}{a_4} + \frac{2a_2}{a_4} + \frac{a_1}{a_4} + 4$, (5). Therefore, by induction on n the remaining factors $k_2, k_3, \dots k_{n-1}, k_n$ of β_{n-1} in terms of coefficients $a_n, a_{n-1}, \dots\dots a_1$ and a_0 are given by;

$$\beta_{n-1} = \prod_{j=2}^{n} k_j = n + \sum_{j=1}^{n-1} \frac{(n-j)a_{n-j}}{a_n}, \text{ provided } \theta_n = \min \sum_{j=2}^{n} k_j = \frac{a_{n-1}}{a_n} + n, (6)$$

Therefore, from (6) we simply conclude an order β_{n-1} is the finite sum of the coefficients of the derivative of P(x). In other word, β_{n-1} defined by order of **P'(x)** provided $\theta_n = \min \sum_{j=2}^{n} k_j = \frac{a_{n-1}}{a_n} + n$.

Definition: 7 Let P(x) polynomial of degree greater than or equal to 2;

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} \dots\dots\dots + a_2 x^2 + a_1 x + a_0, (1).$$

Where $a_n \neq 0, a_{n-1}, \dots, a_1$ and a_0 are real coefficients. Let $b_{1,i}$ vertex points and $(x - b_{1,i})^2$ quadratic factors of P(x), for $i = 1,2,., n-1$. Then by division algorithm, there exist constant remainder values $P(b_{1,i})$ and degenerate quotient factors $G_i(x)$ of P(x), for $i = 1,2,.., n-1$ such that;

$$P(x) = G_i(x)\left(x - b_{1,i}\right)^2 + P\left(b_{1,j}\right), \text{for } i, j = 1,2,.., n-1, \ (2)$$

Where $G_i(x) = a_n \prod_{j=2}^{n-1}\left(x + b_{j,i}\right)$, for $i = 1,2,.., n-1$ and $j = 2,3,.., n-1$ are degenerate factors of degree (n-2) with $-b_{n-1,i}, -b_{n-2,i}, \dots, -b_{2,i}$, for $i = 1,2, \dots, n-1$ degenerate point and $P(b_{1,i})$ for $i = 1,2,.. n-1$ are vertex values at the vertex points $b_{1,i}$, for $i = 1,2,.. n-1$.

Lemma: 6 Let's consider a monic polynomial P(x) of degree n. Then there exist unique degenerate quotient factors $G_i(x)$ and unique reminder vertex values $K_i = P(b_{1,i})$ for all $i = 1,2,.. n-1$ of P(x).

$$P(x) = G_i(x)\left(x - b_{1,i}\right)^2 + K_i, \text{for } i = 1,2,.., n-1, (1)$$

where $Q(x) = \left(x - b_{1,i}\right)^2$ quadratic factors of P(x) and $P\left(b_{1,i}\right) = K_i$, for $i = 1,2,.., n-1$ vertex values of P(x).

Proof: suppose there exist degenerate quotient factors $G_{1,m}(x)$ and $G_{2,m}(x)$ and constant reminder values $K_{1,m} = P\left(b_{1,m}\right)$ and $K_{2,m} = P(b_{1,m})$, for a particular index i=m, and vertex point $b_{1,m}$ such that;

$$P(x) = G_{1,m}(x)\left(x - b_{1,m}\right)^2 + K_{1,m} \text{ and } P(x) = G_{2,m}(x)\left(x - b_{1,m}\right)^2 + K_{2,m}, (2)$$

With $\deg K_{1,m} < deg Q(x)$ and $\deg K_{2,m} < deg Q(x) = \left(x - b_{1,i}\right)^2$. from (2), we have

$$K_{1,m} = P(x) - G_{1,m}(x)\left(x - b_{1,m}\right)^2 \text{and } K_{2,m} = P(x) - G_{2,m}(x)\left(x - b_{1,m}\right)^2, (3).$$

Then $K_{1,m} - K_{2,m} = \left(x - b_{1,m}\right)^2\left[G_{2,m}(x) - G_{1,m}(x)\right]$, (4).

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Therefore, from (4) $Q(x) = (x - b_{1,i})^2$ factor to $K_{1,m} - K_{2,m}$ and by division theorem, as degree $(K_{1,m} - K_{2,m}) \leq \text{Max}\{\deg K_{1,m},\ \deg K_{2,m}\} < degQ(x)$, (5), it follows that $K_{1,m} - K_{2,m} = 0$ as a result $K_{1,m} = K_{2,m}$ and from (4); $(x - b_{1,m})^2[G_{2,m}(x) - G_{1,m}(x)] = 0, \forall x \in \mathbb{R}$ and it suggests that $\forall x \in \mathbb{R} | b_{1,m}, (x - b_{1,m})^2 \neq 0$.

so, $G_{2,m}(x) - G_{1,m}(x) = 0, \forall x \in \mathbb{R}$, it implies $G_{2,m}(x) = G_{1,m}(x), \forall x \in \mathbb{R}$. Hence, we can uniquely define the vertices differentiation polynomials in (1).

Definition: 8 Vertices differentiation polynomial: a function of P(x) that is uniquely represents a particular vertex point with the corresponding vertex value of polynomial such that for any degree n can be defined as;

$$P(x) = G_i(x)(x - b_{1,i})^2 + P(b_{1,j}), \text{for } i, j = 1,2,..,n-1, (1)$$

Thus (1) are called Vertices differentiation polynomials of P(x). Where $G_i(x) = a_n \prod_{j=2}^{n-1}(x + b_{j,i})$, for $i = 1,2,..,n-1$ and $j = 2,..,n-1$ are degenerate polynomials of degree (n-2) , $K_i = P(b_{1,i})$ for $i = 1,2,..n-1$ are vertices values of P(x) at the vertex points $b_{1,i}$,for $i = 1,2,..n-1$, and $-b_{n-1,i}, -b_{n-2,i}, \dots, -b_{2,i}$, for $i = 1,2,\dots,n-1$ are degenerate points.

Definition: 9 The name degenerate points refer to the zero's $-b_{n-1,i}, -b_{n-2,i}, \dots, -b_{2,i}$ of degenerate polynomials $G_i(x)$ and vertex points $b_{1,j}$, for $j = 1,2,..,n-1$ satisfy;

$$P(-b_{z,i},) = P(b_{1,j}),\ \text{for } i, j = 1,2,..,n-1 \text{ and } z = 2,3,\dots,n-1, (1)$$

where $P(b_{1,i})$ vertex values of P(x) at the vertex points $b_{1,i}$,for $i = 1,2,..n-1$ and $-b_{n-1,i}, -b_{n-2,i}, \dots, -b_{2,i}$, for $i = 1,2,\dots,n-1$ are degenerate points.

Lemma: 7 Let a monic polynomial P(x) of degree n≥2 and $a_n \neq 0$. Let $b_{1,i}$ be vertex points; $(x - b_{1,i})^2$ quadratic factors and $G_i(x) = a_n \prod_{j=2}^{n-1}(x + b_{j,i})$ degenerate factors of P(x), for $i = 1,2,..,n-1$. Then by division algorithm, there exist polynomials equations of degree (n-2) with variables $b_{j,i}, i = 1,2,\dots.n-1$ and$j = 2,3\dots,n-2$ such that;

$$C_{n-2}{b_{k,i}}^{n-2} + C_{n-3}{b_{k,i}}^{n-3} + \cdots \dots + C_2{b_{k,i}}^2 + C_1 b_{k,i} + C_0 = 0, (1)$$
$$\text{For } i = 1,2,\dots\dots,n-1 \text{ and } k = 2,3,\dots\dots,n-1$$

Where coefficients C_j, for $j = 0,1,2,\dots,n-2$ described in terms of the vertex points $b_{1,k}$, for $k = 1,2,3,\dots.,n-1$ and coefficients a_j, for j=0,1,2,….n of P(x) such that for each k, the coefficients C_j given by;

$$C_j = \sum_{i=j+1}^{n-1}(-1)^{j+1}\,(i-j)a_{i+1}{b_{1,k}}^{i-(j+1)}, (2)$$

Proof: By induction on n, generalized (2). Let's consider P(x) of degree n=3, such that $P(x) = a_3x^3 + a_2x^2 + a_1x + a_0$, (3). Where $a_3 \neq 0$, a_2 a_1, and a_0 are real coefficients of (1).From (3), we have two vertices differentiation polynomial such that;

$$P(x) = G_i(x)(x - b_{1,i})^2 + P(b_{1,i}), \text{for } i = 1, 2, (4)$$

Let $G_i(x) = a_3(x + b_{2,i})$ degenerate polynomials, for i=1,2, then $b_{1,1}, b_{1,2}$ and $b_{2,1}, b_{2,2}$ satisfy;

$$b_{2,i} = \frac{a_2}{a_3} + 2b_{1,i}, \text{for } i = 1,2, (5)$$
$${b_{1,i}}^2 - 2b_{1,i}b_{2,i} = \frac{a_1}{a_3}, \text{for } i = 1,2, (6)$$

From (5) and (6), there exist polynomial equation of degree (n-2) = 1 containing the points $b_{j,i}$, for i=1,2, j=2;

$$a_3b_{2,i} - [a_2 + 2a_3b_{1,i}] = 0, \text{for } i = 1,2, (7). \text{ Where } C_1 = a_3 \text{ and } C_0 = -[a_2 + 2a_3b_{1,i}]$$

Equations (7) for $i = 1,2$ and $j = 2$ are called polynomial equations of degree (n-2) = 1 and the points $b_{2,i}$, for $i = 1,2$ are called degenerate points of P(x).

Next, let n=4; $P(x) = a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$, (8). Where $a_4 \neq 0$, a_3 , a_2 a_1, and a_0 are real constant coefficients. Here, we have two vertices differentiation polynomials;

$$P(x) = G_i(x)(x - b_{1,i})^2 + P(b_{1,i}), \text{for } i = 1, 2,3, (9)$$

Let $G_i(x) = a_4(x + b_{3,i})(x + b_{2,i})$ be degenerate polynomials, for i=1,2,3, then $b_{1,i}$ and $b_{j,i}$, for i=1,2,3 , j=2,3 satisfy;

$$b_{3,i} + b_{2,i} = \frac{a_3}{a_4} + 2b_{1,i}, \text{for } i = 1,2,3, (10)$$
$${b_{1,i}}^2 - 2b_{1,i}(b_{3,i} + b_{2,i}) + b_{2,i}b_{3,i} = \frac{a_2}{a_4}, \text{for } i = 1,2,3, (11)$$
$${b_{1,i}}^2(b_{3,i} + b_{2,i}) - 2b_{1,i}b_{2,i}b_{3,i} = \frac{a_1}{a_4}, \text{for } i = 1,2,3, (12)$$

By using (10) - (12), there exist polynomial equations of degree (n-2) = 2 containing the points $b_{j,i}$, for i=1,2….,3, j=2,3,3;
$-a_4 b_{j,i}{}^2 + [2b_{1,i} + a_3]b_{j,i} - [3a_4 b_{1,i}{}^2 + 2a_3 b_{1,i} + a_2] = 0$, (13)

Then, for $i = 1,2,3$ (13) are called two polynomial equation of degree 2
$C_2 = -a_4$, $C_1 = 2b_{1,i} + a_3$ and $C_0 = -[3a_4 b_{1,i}{}^2 + 2a_3 b_{1,i} + a_2]$,

Finally, let n = 5; $P(x) = a_5x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$, (14).Where $a_5 \neq 0, a_4, a_3$, a_2 a_1, and a_0 are real constant coefficients. From (14), we have four vertices differentiation polynomials;
$P(x) = G_i(x)(x - b_{1,i})^2 + P(b_{1,i})$, for $i = 1,2,..,4$ (15)

Let $G_i(x) = a_5(x + b_{4,i})(x + b_{3,i})(x + b_{2,i})$ be degenerate polynomials, for i=1, 2, 3, 4, then there exist polynomial equations of degree (n-2) = 3 containing the points $b_{j,i}$, for i=1,2….,4, j=2,3,4;
$a_5 b_{j,i}{}^3 - [2a_5 + a_4]b_{j,i}{}^2 + [3a_5 b_{1,i}{}^2 + 2a_4 b_{1,i} + a_3]b_{j,i} - [4a_5 b_{1,i}{}^3 + 3a_4 b_{1,i}{}^2 + 2a_3 b_{1,i} + a_2] = 0$, (16)

From (16), we can reduce in to the 3rd degree polynomial equations;
$C_3 b_{ji}{}^3 + C_2 b_{j,i}{}^2 + C_1 b_{j,i} + C_0 = 0$, for $j = 2,3,4$ and $i = 1,2,..4$, (17)

Where the coefficients C_j with $b_{1,i}$, for $i = 1,2,3,4$ given by; $C_3 = a_5$,$C_2 = -[2a_5 + a_4]$,
$C_1 = 3a_5 b_{1,i}{}^2 + 2a_4 b_{1,i} + a_3$ and $C_0 = -[4a_5 b_{1,i}{}^3 + 3a_4 b_{1,i}{}^2 + 2a_3 b_{1,i} + a_2]$

Therefore, by induction on n, there exist polynomial equations of degree $(n - 2)$ such that;
$C_{n-2} b_{j,i}{}^{n-2} + C_{n-3} b_{j,i}{}^{n-3} + C_{n-4} b_{j,i}{}^{n-4} + \cdots C_1 b_{j,i} + C_0 = 0$, (18)

Where coefficients C_j, for $j = 2,3, \dots n - 1$ and $i = 1,2,..n - 1$ can be generalize for $n \geq 3$;
$C_j = \sum_{i=j+1}^{n-1} (-1)^{j+1}(i - j)a_{i+1}\, b_{1,k}{}^{i-(j+1)}$, for $j = 0,1,2, \dots n - 2$, $i = 1,2, \dots, n - 1$, (19)

Lemma: 8 Let $G_i(x)$ be degenerate polynomials for $i = 1,2, \dots, n - 1$ defined by;
$G_i(x) = d_{n-2}x^{n-2} + d_{n-3}x^{n-3} + \cdots \dots + d_2x^2 + d_1x + d_0$, for $i = 1,2, \dots, n - 1$, (1)

Let $-b_{m,j}$, for $m = 2,3, \dots n - 2$ and $j = 1,2, \dots n - 2$ are degenerate points/roots of $G_i(x)$, only if $b_{m,j}$, for $m = 2,3, \dots n - 2$ and $j = 1,2, \dots n - 2$ roots of the polynomials equation;
$C_{n-2}x^{n-2} + C_{n-3}x^{n-3} + \cdots \dots + C_2x^2 + C_1x + C_0 = 0$, (2)

Where the coefficients C_j, for $j = 0,1,2, \dots n - 2$ given by;
$C_j = \sum_{i=j+1}^{n-1} (-1)^{j+1}(i - j)a_{i+1}\, b_{1,k}{}^{i-(j+1)}$, for $j = 0,1,2, \dots n - 2$, $i = 1,2, \dots, n - 1$, (3)

Then the coefficients d_j, for $j = 0,1,2, \dots n - 2$ of $G_i(x)$ in terms of coefficients C_j of (2) given;
$d_j = \sum_{i=j+1}^{n-1} (-1)^{2j+1}(i - j)a_{i+1}\, b_{1,k}{}^{i-(j+1)}$, for $j = 0,1,2, \dots n - 2$, $i = 1,2, \dots, n - 1$, (4)

Proof: let $G_i(x) = a_n \prod_{m=2}^{n-1}(x + b_{m,i})$ be degenerate polynomials for $i = 1,2, \dots, n - 1$, then by lemma (), there exists polynomial equations of degree (n-2) with roots $b_{m,j}$, for $m = 2,3, \dots n - 2$ and $j = 0,1,2, \dots n - 2$ at a particular vertex point $b_{1,k} = 1$;
$c_{n-2}\, b_{m,k}{}^{n-2} + c_{n-3} b_{m,k}{}^{n-3} + c_{n-4} b_{m,k}{}^{n-4} + \cdots + c_1 b_{m,k} + c_0 = 0$, (5)

$$C_j = \sum_{i=j+1}^{n-1} (-1)^{j+1}(i - j)a_{i+1}, \text{ for } j = 0,1,2, \dots n - 2, (6)$$

For arbitrary x, then there exists a polynomial h (x) of degree (n-2) such that;
$h(x) = c_{n-2}x^{n-2} + c_{n-3}x^{n-3} + c_{n-4}x^{n-4} + \cdots + c_1x + c_0$, (7)
Where $C_j = \sum_{i=j+1}^{n-1} (-1)^{j+1}(i - j)a_{i+1}$, for $j = 0,1,2, \dots n - 2$

From (5), at most (n-2) points of $b_{m,k}$ are roots of (5). But, we need to have negative roots $x = -b_{m,k}$, for $m = 0,1,2, \dots n - 2$, of (5), to have the coefficients of $G_i(x) = a_n \prod_{m=2}^{n-1}(x + b_{m,i})$ so we should convert all coefficients to the same sign. Because, for each even index $j = 0,2,4,6 \dots$, we have negative C_j and for each odd index $j = 1,3,5,7 \dots$, we have positive coefficients. Assume n an even degree, so by using odd power of $x = -b_{m,k}$, $m = 0,1,2, \dots n - 2$, then we obtain the same sign, i.e;

$c_{n-2}\left(-b_{m,k}\right)^{n-2}+c_{n-3}\left(-b_{m,k}\right)^{n-3}+c_{n-4}\left(-b_{m,k}\right)^{n-4}+\cdots+c_1\left(-b_{m,k}\right)+c_0=0,$ (8)

$c_{n-2}\ {b_{m,k}}^{n-2}-c_{n-3}{b_{m,k}}^{n-3}+c_{n-4}{b_{m,k}}^{n-4}-c_{n-5}{b_{m,k}}^{n-5}+\cdots+-c_1b_{m,k}+c_0=0,$ (9)

Then, we have $d_{n-2}\ {b_{m,k}}^{n-2}+d_{n-3}{b_{m,k}}^{n-3}+d_{n-4}{b_{m,k}}^{n-4}+\cdots+d_1b_{m,k}+d_0=0$, (10).

Where $d_j=(-1)^jC_j=(-1)^j\sum_{i=j+1}^{n-1}(-1)^{j+1}(i-j)a_{i+1}$, for $j=0,1,\ldots n-2$, (11)

Therefore, $d_j=\sum_{i=j+1}^{n-1}(-1)^{2j+1}(i-j)a_{i+1}$, for $j=0,1,2,\ldots n-2$ the coefficients of degenerate polynomials $G_i(x)$, for $i=1,2,\ldots n-1$

Definition: 10 A monic polynomial P(x), for $n\geq 2$; $a_n\neq 0, a_{n-1},\ldots,a_1$ and a_0 are real coefficients.
$P(x)=a_nx^n+a_{n-1}x^{n-1}+a_{n-2}x^{n-2}\ldots\ldots\ldots..+a_2x^2+a_1x+a_0$, (1). For each vertex points $b_{1,k}$ of P(x), for $k=1,2,,\ldots,n-1$, there exist unique degenerate polynomials $G_i(x)$, for i=1,2,…..,n-1 of degree (n-2) such that;

$$G_i(x)=d_{n-2}x^{n-2}+d_{n-3}x^{n-3}+\cdots\ldots.+d_2x^2+d_1x+d_0, \text{for } i=1,2,\ldots..,n-1, (2)$$

Then coefficients d_j, for $j=0,1,2,\ldots,n-2$ of (2) described in terms of the vertex points $b_{1,k}$, for $k=1,2,3,.,n-1$ and coefficients of P(x), for particular index k the coefficients d_j given by;

$$d_j=\sum_{i=j+1}^{n-1}(-1)^{2j+1}(i-j)a_{i+1}{b_{1,k}}^{i-(j+1)}, (3)$$

Lemma: 9 let P(x) be a monic reducible n^{th} degree polynomial and suppose two consecutive orders of P(x) vanish $\beta_n=\beta_{n-1}=0$, if and only if $b_{1,m}$ =1 is the double root and vertex point of P(x) so that the vertex value at this vertex point vanishes. Where β_{n-1} reduced order and has n-1 possible factors of the same order β_n.

Proof: Since $\beta_n=\beta_{n-1}=0$, implies that for $k_1=k_2=0$ the point $b_{1,m}=1$ is the root of P(x) and its derivative P'(x), so $b_{1,m}=1$ is the double root and the vertex point of P(x).

Similarly,for $k_1=k_2=0$ and a double root of P(x) at $b_{1,m}=1$, so the vertex value of P(x) could be $P\left(b_{1,m}\right)=0$ at $b_{1,m}=1$, (1). Then $P(x)=G_m(x)(x-1)^2$, for $k_1=k_2=0$. Where the degenerate polynomial $G_m(x)=a_n\prod_{k=2}^{n-1}\left(x+b_{k,m}\right)$, (2). Conversely, if $b_{1,m}$=1 is a double root of P(x), then the sum of coefficients of P(x) and its derivative P'(x) vanish such that $\beta_n=\beta_{n-1}=0$.

Theorem: 5 let a monic polynomial P(x) of degree n has order $\beta_n=\beta_{n-1}=0$. Then there exists a degenerate polynomial G(x) of degree (n-2) with non zero order β_{n-2} such that;

$$G_i(x)=d_{n-2}x^{n-2}+d_{n-3}x^{n-3}+d_{n-4}x^{n-4}+\cdots+d_1x+d_0, \text{for } i=1,2,\ldots n-1, (1)$$

$$\beta_{n-2}=\sum_{j=0}^{n-2}d_j=\prod_{j=3}^{n}k_j, (2)$$

where k_j, for $j=3,4,\ldots n$ the possible factors of β_{n-2}, where $k_1=0$ and $k_2=0$ dropped. Then the coefficients d_j, for $j=0,1,2,\ldots n-2$ of $G_i(x)$ are given by;

$$d_j=\sum_{i=j+1}^{n-1}(-1)^{2j+1}(i-j)a_{i+1}, \text{for } j=0,1,2,\ldots n-2, (3)$$

Proof: Suppose $\beta_n=\beta_{n-1}=0$ of P(x), then by **lemma** (9), P(x) have a double root and vertex points at $k_1=k_2=0$, so that the vertex value at this point vanishes. Thus $P(x)=G_k(x)(x-1)^2$, (4). Similarly, since $k_1=k_2=0$, we can remove two factors of an order. So, there exists a degenerate polynomial $G_k(x)$ of degree (n-2) such that;

$$G_k(x)=d_{n-2}x^{n-2}+d_{n-3}x^{n-3}+d_{n-4}x^{n-4}+\cdots+d_1x+d_0, \quad (5)$$

Then by using the above lemma (8); we can fix the coefficients d_j, for $j=0,1,2,\ldots n-2$ of $G_k(x)$ given by;

$$d_j=\sum_{i=j+1}^{n-1}(-1)^{2j+1}(i-j)a_{i+1}, \text{for } j=0,1,2,\ldots n-2, (6)$$

Now, assume a none zero next order $\beta_{n-2}\neq 0$, then we can define β_{n-2} with coefficients of $G_k(x)$;

$$\beta_{n-2}=\sum_{j=0}^{n-2}d_j=\prod_{j=3}^{n}k_j, (7)$$

Where k_j, for $j=3,4,\ldots n$ possible factors of β_{n-2}, where $k_1=0$ and $k_2=0$ dropped and criterion of reducibility provided the same

$$\theta_n=\frac{na_n}{a_n}+\frac{a_{n-1}}{a_n}=\sum_{j=0}^{n-2}k_j, (8)$$

Corollary: 3 Let I be the number of factors $k_j = 0$, for $j = 1,2, \dots i$ of β_n in which the preceding order's vanish until $\beta_{n-i+1} = 0$, so that the next proceeding order β_{n-i} with the remaining (n-i) number of factors should be determined by;

$$\beta_{n-i} = \prod_{j=i+1}^{n} k_j = \frac{1}{i!}\sum_{j=0}^{n-i} \frac{(n-j)!a_{n-j}}{((n-(i+j))!a_n}, (1)$$

EXTENDED THEOREMS AND CONCEPTS OF ALGEBRA

1. EXTENDED REMAINDER THEOREM

Theorem: 6 Let β_n an order of a polynomial f(x) of degree n greater than or equal to one, and let c be any real number. If f(x) is divided by the linear polynomial $(x - c)$ of order α, then the remainder is a polynomial f(c) of order f(c), if and only if, an order β_n is divided by the linear order α and the remainder is a constant polynomial of order f $(-\alpha + 1)$ such that;

$$f(x)=(x-c)q(x)+f(c) \leftrightarrow \beta_n = \alpha q(1) + f(-\alpha + 1), (1)$$

Where $q(1)$ an order of q(x), and α order of linear polynomial $(x+ \alpha - 1)$

Proof: $(\rightarrow)$ Suppose f(x) is divided by $(x - c)$, the remainder always constant such that; f (x)=(x-c)q(x)+k,(1) where k constant and since, for x=c such that k=f(c).
$\leftrightarrow$ f (x)=(x-c)q(x)+f(c),(2)

Moreover, let α be real order of linear polynomial $(x - c)$ that divides β_n, then by division of numbers, there exist real numbers ω and k such that $\beta_n = \alpha\omega + k$, (3) and since β_n an order of f(x), so $\beta_n = f(1)$, (4). Then from (3) and (4), we can compute equality as;
$\beta_n = \alpha\omega + k = f(1) = (1-c)q(1)+f(c)$,(5)

Since α be an order of $(x - c)$ such that $c = -\alpha + 1$ and then from (5); we have; $\omega = q(1)$ and $k = f(c)$, (6). Therefore, $\beta_n = \alpha q(1) + f(-\alpha + 1)$, (7)

Where q(1) an order of quotient q(x), $f(-\alpha + 1)$ an order of a remainder polynomial $f(-\alpha + 1)$ and α order of a linear polynomial (x-c).

Conversely, let β_n an order of f(x) such that $\beta_n = \alpha q(1) + f(-\alpha + 1)$,(8) where $q(1)$ an order of q(x) and α an order of linear polynomial $(x+ \alpha - 1)$, if and only if, α an order of $(x-c)$, for a real number c such that $\alpha= (1-c)$, (9). Then, from (8) and (9) we have;

$\beta_n = (1 - c)q(1) + f(c)$, (10). But, β_n an order of f(x) such that $\beta_n = f(1)$, (11).

So, using (10), (11), and variable x, so we have $f(x) = (x - c)q(x) + f(c)$. Hence the results

2. EXTENDED FACTOR THEOREM

Theorem: 7 Let β_n an order of a polynomial P(x) of degree n greater than or equal to one, and let c be any real number, then;

I. x – c is a factor of P(x), if and only if **f(c)=0.**
II. ω is a linear factor of β_n, if and only if f (1-ω)=0. Where ω order of x-c.

Proof (I): Suppose x – c is a factor of P(x), if and only if f(c) = 0. By using polynomials division theorem with factor (x-c) to expresses f(x) as; f (x) = d(x)q(x)+r(x).Where d(x)=(x-c) and using remainder theorem r(x)=f(c)=k, that give us f (x) = (x-c)q(x)+f(c). Where q(x) is a polynomial of degree n less than the degree of f(x) such that f(c) = 0. In particular, let P(x) be a polynomial of degree n greater than or equal to one and let c be any real number, then x – c is a factor of P(x), if and only if f(c)=0.

Proof (II): Suppose f (1-ω)=0, if and only if an order ω linear factor of β_n. Where ω order of x-c.

Suppose f (1-ω)=0 then there exists a of linear factor $(x + \omega - 1)$such that

$f(x) = (x + \omega - 1)g(x)$, for $g(x) \in \mathbb{Z}[x]$, (1)

Let β_n be an order of f(x) such that $\beta_n = f(1)$, (2)

$\leftrightarrow f(1) = (1 + \omega - 1)g(1)$, (3)

$\leftrightarrow \beta_n = \omega g(1)$, (4)

Where ω order of linear factor and g (1) order of g(x). Therefore, ω is a linear factor of β_n.

Conversely, suppose an order ω a linear factor of β_n and $\beta_n = \sum_{j=1}^{n} \frac{a_j}{a_n} = f(1)$ order of f(x).

Since ω is a linear factor of β_n then there exists $u \in \mathbb{Z}$ such that $\beta_n = \omega u$, (5). For, every linear factor ω of β_n, there exists $r = 1 - \omega$ such that $f(1-\omega) = 0$.

3. EXTENDED MULTIPLICITY THEOREM

Theorem: 8 Let's consider a monic polynomial P(x) of degree n with $a_n \neq 0$ and suppose the vertex value $P(b_{1,u}) = 0$, of P(x) at fixed index u, then there exists $x = b_{1,k}$, for a particular index k be the root of P(x) with multiple, $m \geq 2$.

Proof: ($\rightarrow$) Suppose P(x) has at most (n-1) vertex points, vertex values and vertices differentiation polynomials. Then P(x) can be written as;
$P(x) = G_i(x)\left(x - b_{1,i}\right)^2 + P(b_{1,i})$, (1)

where $b_{1,i}$, for $i = 1, 2, \ldots \ldots n-1$ vertex points; $P(b_{1,i})$, for $i = 1, 2, \ldots \ldots n-1$ the vertex values and $-b_{n-1,i}, -b_{n-2,i}, \ldots\ldots\ldots$-,$b_{3,i}$ and $-b_{2,i}$ for $i = 1, 2, \ldots \ldots n-1$ degenerate points.

Here let's assume that for a particular index i=m, such that from (1);

$P(x) = a_n\left(x + b_{n-1,m}\right)\left(x + b_{n-2,m}\right) \ldots\ldots\left(x + b_{2,m}\right)\left(x - b_{1,m}\right)^2 + P(b_{1,m})$, (2) and since for a particular index i =m, $P(b_{1,m}) = 0$, (3). From (2) and (3), for some fixed index m;

$P(x) = a_n\left(x + b_{n-1,m}\right)\left(x + b_{n-2,m}\right) \ldots\ldots\left(x + b_{2,m}\right)\left(x - b_{1,m}\right)^v = 0$, for $v \geq 2$ (4). Thus, there exist a factor $\left(x - b_{1,m}\right)^v$ of P(x) with multiplicity $v \geq 2$, if and only if ω^v a factor of P(1) with multiplicity $v \geq 2$. Where $\omega = 1 - b_{1,m}$ is a factor of an order $\beta_n = P(1)$. Therefore, $x_{n-1} = -b_{n-1,m}$, $x_{n-2} = -b_{n-2,m}$, …, $x_3 = -b_{3,m}$ and $x_2 = -b_{2,m}$ are simple roots and then $x_1 = -\omega + 1$ multiple roots of P(x) with multiplicity $v \geq 2$.

In particular, if $(x - c)^k$ is a factor of P(x), but $(x - c)^{k+1}$ is not, then $(x - c)$ is said to be a zero of P(x) with multiplicity k, if and only if ω^k is a factor of an order β_n of P (x), but ω^{k+1} is not, then $c = -\omega + 1$ is said to be a zero of P(x) with multiplicity k.

4. EXTENDED THEOREM OF RATIONAL ROOT TEST

Theorem: 9 If the rational number $\frac{p}{q}$, in its lowest terms, is a zero of;
$P(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} \ldots\ldots\ldots\ldots + a_2x^2 + a_1x + a_0$, for $a_n \neq 0$, (1). With integer coefficients. Then $\frac{q-p}{q}$ must be a factor of $\beta_n = \sum_{j=1}^{n} \frac{a_j}{a_n}$, if and only if $\frac{p}{q}$ must be a factor of $\frac{a_o}{a_n}$.

Proof: Let's consider a monic polynomial of degree n given by;

$P(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_2x^2 + a_1x + a_0$, (1)

Let P(x) of order β_n and $\beta_n = P(1) = \sum_{j=1}^{n} \frac{a_j}{a_n}$ is reducible over $\mathbb{Q}$, if and only if the constant term $\frac{a_o}{a_n} = P(0)$ of P(x) is reducible over $\mathbb{Q}$. Since P(x) reducible, then β_n is reducible. So β_n can be written as $\beta_n = P(1) = \prod_{j=1}^{n} k_j$, (2). Where k_j , for j=1, 2,….n-1,n are at most n factors of β_n.

Then, let $r_j = -k_j + 1$, for j=1, 2,.. ,n the corresponding n possible roots of P(x)such that $P(x) = \prod_{j=1}^{n}(x + k_j - 1)$,(3). Since P(x) reducible, from (3); $P(0) = \frac{a_o}{a_n} = \prod_{j=1}^{n}\left(k_j - 1\right)$, (4). Hence, $P(0)$ is reducible to its possible factors $k_j - 1$, for $j = 1,2, \ldots . n$. Conversely, suppose that by factor theorem $P(0) = \frac{a_o}{a_n}$ is reducible to its possible roots r_j, for $j = 1,2, \ldots . n$, then there exist possible factors $k_j = -r_j + 1$, for $j = 1,2, \ldots . n$ such that
$\beta_n = \prod_{j=1}^{n} k_j = P(1)$, (4).

Therefore, $\beta_n = P(1)$ is reducible over $\mathbb{Q}$.

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