International Journal of Mathematical Archive-5(2), 2014, 281-292
Available online through www.ijma.info ISSN 2229 – 5046

VERTEX- EDGE DOMINATION POLYNOMIALS OF GRAPHS

[1]A. Vijayan* and [2]T. Nagarajan

[1]Associate Professor, Department of Mathematics,
Nesamony Memorial Christian College, Marthandam, Tamil Nadu, India.

[2]Assistant Professor, Department of Mathematics, Sivanthi Aditanar College,
Pillayarpuram, Nagercoil, Tamil Nadu, India.

(Received on: 10-02-14; Revised & Accepted on: 25-02-14)

ABSTRACT

In this paper, we introduce the concept of vertex-edge domination polynomial for any Graph. The vertex-edge domination polynomial of a graph G of order n is the polynomia $D_{ve}(G, x) = \sum_{i=\gamma_{ve}(G)}^{|V(G)|} d_{ve}(G, i)x^i$, *where* $d_{ve}(G, i)$ *is the number of vertex-edge dominating sets of G of size i, and* $\gamma_{ve}(G)$ *is the vertex-edge domination number of G. We obtain some properties of* $D_{ve}(G, x)$ *and its co-efficients. Also, we find the vertex-edge domination polynomial for the complete Graph* K_n, $G \circ K_1$ *and* $G \circ K_2$.

Keywords: *Vertex-edge dominating sets, vertex-edge domination number, vertex-edge domination polynomial.*

1. INTRODUCTION

Let G = (V, E) be a Graph. For any vertex $v \in V$, the open neighbourhood of v is the set $N(v) = \{u \in V \mid uv \in E\}$ and the closed neighbourhood of v is the set $N[v] = N(v) \cup \{v\}$. For a set $S \subseteq V$, the open neighbourhood of S is $N(S) = \bigcup_{v\in S} N(v)$ and the closed neighbourhood of S is $N[S] = N(S) \cup S$. A set S of vertices in a Graph G is said to be a dominating set if every vertex $u \in V$ is either an element of S or is adjacent to an element of S. The minimum cardinality of a dominating set of G is said to be domination number and is denoted by $\gamma(G)$.

A set S of vertices in a Graph G is said to be a vertex-edge dominating set, if for every edge $e \in E(G)$, there exists a vertex $v \in S$ such that v dominates e. In other words, for a Graph G = (V, E), a vertex $u \in V(G)$ vertex-edge dominates an edge $vw \in E(G)$ if (i) u = v or u = w (u is incident to vw), or (ii) uv or uw is an edge in G (u is incident to an edge is adjacent to vw).

The minimum cardinality of a vertex-edge dominating set of G is called vertex-edge domination number of G, and is denoted by $\gamma_{ve}(G)$.

The join of two Graphs G_1 and G_2, denoted by $G_1 \vee G_2$ is a graph with the vertex set $V = V_1 \cup V_2$ and edge set $E_1 \cup E_2 \cup \{uv | u \in V_1 \text{ and } v \in V_2\}$. The corona of two graphs G_1 and G_2 is the graph $G = G_1 \circ G_2$ formed from one copy of G_1 and $|V(G_1)|$ copies of G_2, where the i^{th} vertex of G_1 is adjacent to every vertex in the i^{th} copy of G_2. A graph is an empty graph if it contains no edges.

2. INTRODUCTION TO VERTEX-EDGE DOMINATION POLYNOMIAL

In this section, we are going to state the definition of vertex-edge domination polynomial and derive some properties.

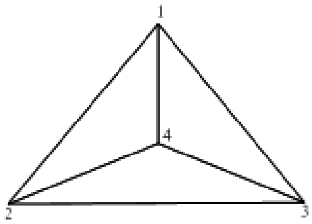
Definition: 2. 1 Let $D_{ve}(G, i)$ be the family of vertex-edge dominating sets of a graph G with cardinality i and let $d_{ve}(G, i) = |D_{ve}(G, i)|$. Then the vertex-edge domination polynomial, $D_{ve}(G, x)$ of G is defined as

Corresponding author: [1]A. Vijayan*
[1]Associate Professor, Department of Mathematics,
Nesamony Memorial Christian College, Marthandam, Tamil Nadu, India.
E-mail: naacnmccm@gmail.com

$$D_{ve}(G, x) = \sum_{i=\gamma_{ve}(G)}^{|V(G)|} d_{ve}(G, i)\, x^i ,$$

where $\gamma_{ve}(G)$ is the vertex-edge domination number of G.

Example: 2.2 Consider K_4



Vertex-edge dominating sets of cardinality 1 are {1}, {2}, {3}, {4}.

∴ K_4 has 4 vertex-edge dominating sets of cardinality 1 and $\gamma_{ve}(K_4)=1$
Vertex-edge dominating sets of cardinality 2 are {1, 2}, {1, 3}, {1, 4}, {2, 3}, {2, 4}, {3, 4}.

∴ K_4 has 6 vertex-edge dominating sets of cardinality 2
Vertex-edge dominating sets of cardinality 3 are {1, 2, 3}, {1, 2, 4}, {1, 3, 4}, {2, 3, 4}.

∴ K_4 has 4 vertex-edge dominating sets of cardinality 3.
Vertex-edge dominating sets of cardinality 4 is {1, 2, 3, 4}

∴ K_4 has 1 vertex-edge dominating set of cardinality 4.

∴ The vertex-edge domination polynomial is

$$D_{ve}(K_4, x) = \sum_{i=\gamma_{ve}(K_4)}^{|K_4|} d_{ve}(K_4, i)\, x^i$$

$$= \sum_{i=1}^{4} d_{ve}(K_4, i)\, x^i$$

$$= d_{ve}(K_4, 1)\, x^1 + d_{ve}(K_4, 2)\, x^2 + d_{ve}(K_4, 3)\, x^3 + d_{ve}(K_4, 4)\, x^4$$

$$= 4\, x^1 + 6x^2 + 4\, x^3 + 1.\, x^4$$

$$= x^4 + 4\, x^3 + 6\, x^2 + 4\, x + 1 - 1$$

$$= (1 + x)^4 - 1$$

Theorem: 2.2 If G is a Graph without isolated vertices, consisting of two components G_1 and G_2, then $D_{ve}(G, x) = D_{ve}(G_1, x).\, D_{ve}(G_2, x)$.

Proof: Let G_1 and G_2 be the components of a Graph G without isolated vertices. Let the vertex-edge domination number of G_1 and G_2 be $\gamma_{ve}(G_1)$ and $\gamma_{ve}(G_2)$. For any $k \geq \gamma_{ve}(G)$, the vertex–edge dominating set of k vertices in G arises by choosing a vertex-edge dominating set of j vertices of G_1 and a vertex-edge dominating set of k-j vertices in G_2.

The number of vertex-edge dominating sets in $G_1 \cup G_2$ is equal to the coefficient of x^k in $D_{ve}(G_1, x).\, D_{ve}(G_2, x)$. The number of vertex-edge dominating sets of G is the co-efficient of x^k in $D_{ve}(G, x)$.

Hence the co-efficient of x^k in $D_{ve}(G, x)$ and $D_{ve}(G_1, x).\, D_{ve}(G_2, x)$ are equal.
∴ $D_{ve}(G, x) = D_{ve}(G_1, x).\, D_{ve}(G_2, x)$.

Theorem: 2.3 If G is a Graph without isolated vertices consists of m components $G_1, G_2, \ldots, G_m$. Then $D_{ve}(G, x) = D_{ve}(G_1, x).\, D_{ve}(G_2, x) \ldots D_{ve}(G_m, x)$.

Proof: The proof of the theorem follows from theorem 2.2.

Example: 2.4 Consider the graph given in figure 1.

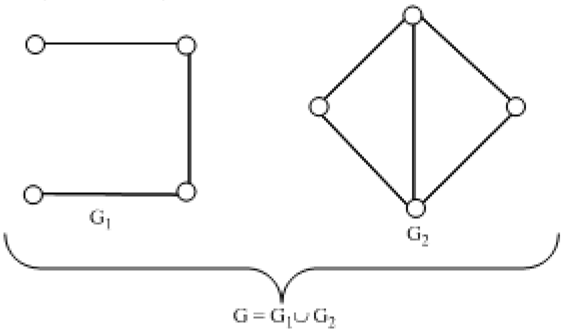


Figure 1

$\gamma_{ve}(G) = \gamma_{ve}(G_1)+\gamma_{ve}(G_2)=1+1=2$ a vertex-edge dominating sets of k=2 vertices in G arises by choosing a vertex-edge dominating set of j = 1 in G_1 ($j\in\{1, 2, 3, 4\}$) and a vertex edge dominating set k – j= 2 –1 = 1 vertex in G_2

$$D_{ve}(G_1, x) = \sum_{i=\gamma_{ve}(G_1)}^{|V(G_1)|} d_{ve}(G_1, i)x^i$$

$$= \sum_{i=1}^{4} d_{ve}(G_1, i)x^i$$

$$= d_{ve}(G_1, 1)\, x^1 + d_{ve}(G_1, 2)\, x^2 + d_{ve}(G_1, 3)\, x^3 + d_{ve}(G_1, 4)\, x^4$$

$$= 2x + 6x^2 + 4x^3 + x^4$$

$$= x^4 + 4x^3 + 6x^2 + 2x$$

$$D_{ve}(G_2, x) = \sum_{i=\gamma_{ve}(G_2)}^{|V(G_2)|} d_{ve}(G_2, i)x^i$$

$$= \sum_{i=1}^{4} d_{ve}(G_2, i)x^{i\ \mathrm{i}}$$

$$= d_{ve}(G_2, 1)\, x^1 + d_{ve}(G_2, 2)\, x^2 + d_{ve}(G_2, 3)\, x^3 + d_{ve}(G_2, 4)\, x^4$$

$$= 4x^1 + 6x^2 + 4x^3 + x^4$$

$$= x^4 + 4x^3 + 6x^2 + 4x$$

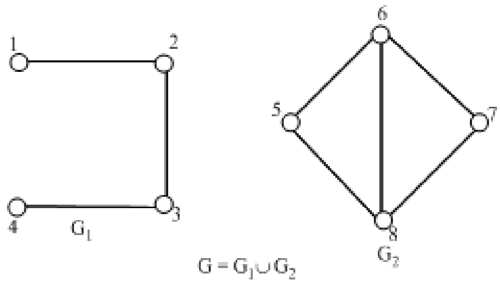
$$D_{ve}(G_1, x) \,.\, D_{ve}(G_2, x) = (x^4 + 4x^3 + 6x^2 + 2x) \times (x^4 + 4x^3 + 6x^2 + 4x)$$

Coefficient of x^2 in $D_{ve}(G_1, x) \,.\, D_{ve}(G_2, x)$ is

= 8

k = 2

$j_1 = 1$, $k - j_1 = 2 - 1 = 1$



The vertex-edge dominating set of cardinality 2 of

$G = \{\{2,5\},\{2,6\},\{2,7\},\{2,8\},\{3,5\},\{3,6\},\{3,7\},\{3,8\}\}$

$d_{ve}(G, 2) = 8$

$\therefore$ coefficient of x^2 in $D_{ve}(G, x)$ is 8

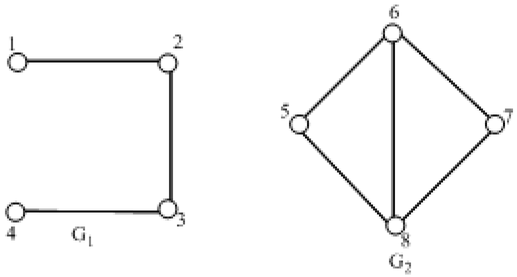
$\therefore$ coefficient of x^2 in $D_{ve}(G, x)$ is same as coefficient of x^2 in

$D_{ve}(G_1, x)\ .\ D_{ve}(G_2, x)$

$\therefore$ $D_{ve}(G, x) = D_{ve}(G_1, x).D_{ve}(G_2, x)$

$k = 3$

$j_1 = 1,\ k - j_1 = 3 - 1 = 2$



$G = G_1 \cup G_2$

$d_{ve}(G_1, 1) = \{\{2\}, \{3\}\} = 2$

The coefficient of x^1 in $D_{ve}(G_1, x)$ is 2

$d_{ve}(G_2, 2) = \{\{5, 6\},\{5, 7\},\{5, 8\},\{6, 7\},\{6, 8\},\{7, 8\}\} = 6$

$\therefore$ The coefficient of x^2 in $D_{ve}(G_2, x)$ is 6

$\therefore$ The coefficient of x^3 in $D_{ve}(G_1, x).\ D_{ve}(G_2, x)$ is 2 x 6 = 12 $G = G_1 \cup G_2$

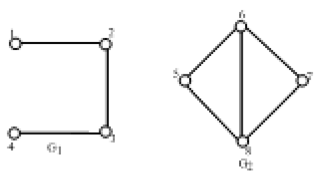
$d_{ve}(G, 3) = \{\{2, 5, 6\}, \{2, 5, 7\}, \{2, 5, 8\}, \{2, 6, 7\}, \{2, 6, 8\}, \{2, 7, 8\}, \{3, 5, 6\}, \{3, 5, 7\},\{3, 5, 8\}, \{3, 6, 7\}, \{3, 6, 8\}, \{3, 7, 8\}\}$
$= 12$

$\therefore$ The coefficient of x^3 in $D_{ve}(G, x)$ is 12

$\therefore$ $D_{ve}(G, x) = D_{ve}(G_1, x)\ D_{ve}(G_2, x)$

$k = 3$

$j_1 = 2,\ k - j_1 = 3 - 2 = 1$



$G = G_1 \cup G_2$

$d_{ve}(G_1, 2) = \{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{3, 4\}\}$

$= 6$

$\therefore$ The coefficient of x^2 in $D_{ve}(G_1, x)$ is 6

$d_{ve}(G_2, 1) = \{\{5\}, \{6\}, \{7\}, \{8\}\} = 4$

$\therefore$ The coefficient of x^1 in $D_{ve}(G_2, x)$ is 4

Coefficient of x^3 in $D_{ve}(G_1, x)\ .\ D_{ve}(G_2, x) = 6 \text{ x } 4 = 24$

$G = G_1 \cup G_2$

$$\begin{aligned} d_{ve}(G, 3) = {} & \{\{1, 2, 5\}, \{1, 2, 6\}, \{1, 2, 7\}, \{1, 2, 8\}, \{1, 3, 5\}, \{1, 3, 6\}, \{1, 3, 7\}, \\ & \{1, 3, 8\}, \{1, 4, 5\}, \{1, 4, 6\}, \{1, 4, 7\}, \{1, 4, 8\}, \{2, 3, 5\}, \{2, 3, 6\}, \\ & \{2, 3, 7\}, \{2, 3, 8\}, \{2, 4, 5\}, \{2, 4, 6\}, \{2, 4, 7\}, \{2, 4, 8\}, \{3, 4, 5\}, \\ & \{3, 4, 6\}, \{3, 4, 7\}, \{3, 4, 8\}\}. \\ = {} & 24 \end{aligned}$$

$\therefore$ The coefficient of x^3 in $D_{ve}(G, x)$ is 24

$\therefore \quad D_{ve}(G, x) = D_{ve}(G_1, x) \,.\, D_{ve}(G_2, x)$

Theorem: 2.5 If G_1 and G_2 are Graphs of order n_1 and n_2 respectively, then

$$D_{ve}(G_1 \vee G_2, x) = [((1+x)^{n1} - 1)((1+x)^{n2} - 1)] + D_{ve}(G_1, x) + D_{ve}(G_2, x)$$

Proof: From the definition of $G_1 \vee G_2$, if D_1 is any vertex-edge domination set of G_1, then D_1 is a vertex-edge domination set of $G_1 \vee G_2$. Similarly, if D_2 is any vertex-edge domination set of G_2, then D_2 is a vertex-edge domination set of $G_1 \vee G_2$.

Also, the sets consist of any one vertex of G_1 and any one vertex of G_2, forms the vertex-edge Dominating sets of $G_1 \vee G_2$ of cardinality two. There are $\binom{n_1}{1}\binom{n_2}{1}$ such sets. Similarly, the number of vertex-edge dominating sets of cardinality three other than the first two cases is $\binom{n_1}{1}\binom{n_2}{2} + \binom{n_2}{1}\binom{n_1}{2}$.

Proceeding like this, we obtain the other vertex-edge dominating sets of cardinality $n_1 + n_2$.

Therefore,
$$\begin{aligned} D_{ve}(G_1 \vee G_2, x) = {} & D_{ve}(G_1, x) + D_{ve}(G_2, x) + \binom{n_1}{1}\binom{n_2}{1}x^2 + \left[\binom{n_1}{1}\binom{n_2}{2} + \binom{n_1}{2}\binom{n_2}{1}\right]x^3 \\ & + \left[\binom{n_1}{1}\binom{n_2}{3} + \binom{n_1}{2}\binom{n_2}{2} + \binom{n_1}{3}\binom{n_2}{1}\right]x^4 \\ & + \ldots + \left[\binom{n_1}{1}\binom{n_2}{n_1+n_2-1} + \ldots + \binom{n_2}{n_1+n_2-1}\binom{n_2}{1}\right]x^{n1+n2} \\ & D_{ve}(G_1, x) + D_{ve}(G_2, x) + \left[\binom{n_1}{1}x + \binom{n_2}{2}x^2 + \ldots + \binom{n_1}{n_1}x^{n_1}\right] \\ & \times\left[\binom{n_2}{1}x + \binom{n_2}{2}x^2 + \ldots + \binom{n_2}{n_2}x^{n_2}\right] \end{aligned}$$

$$\begin{aligned} = {} & D_{ve}(G_1, x) + D_{ve}(G_2, x) \\ & + \left[\binom{n_1}{0} + \binom{n_1}{1}x + \binom{n_1}{2}x^2 + \ldots + \binom{n_1}{n_1}x^{n_1} - \binom{n_1}{0}\right] \\ & \times\left[\binom{n_2}{0} + \binom{n_2}{1}x + \binom{n_2}{2}x^2 + \ldots + \binom{n_2}{n_2}x^{n_2} - \binom{n_2}{0}\right] \end{aligned}$$

$\therefore D_{ve}(G_{1,} \vee G_2, x) = D_{ve}(G_1, x) + D_{ve}(G_2, x) + [(1+x)^{n1} - 1][(1+x)^{n2} - 1]$

$\therefore D_{ve}(G_1 \vee G_2, x) = [(1+x)^{n1} - 1][(1+x)^{n2} - 1] + D_{ve}(G_1, x) + D_{ve}(G_2, x)$

3. CO-EFFICIENT OF VERTEX-EDGE DOMINATION POLYNOMIAL

Theorem: 3.1 Let G be a graph with $|V(G)| = n$. Then
- (i) If G is connected, then $d_{ve}(G, n) = 1$ and $d_{ve}(G, n-1) = n$.
- (ii) $d_{ve}(G, i) = 0$ iff $i < \gamma_{ve}(G)$ or $i > n$.

(iii) $D_{ve}(G, x)$ has no constant term.
(iv) $D_{ve}(G, x)$ is a strictly increasing function in $[0, \infty)$.
(v) Let G be a Graph and H be any induced subgraph of G. Then, $\deg(D_{ve}(G, x)) \geq \deg(D_{ve}(H, x))$
(vi) Zero is a root of $D_{ve}(G, x)$ with multiplicity $\gamma_{ve}(G)$.

Proof:
(i) Since G has n vertices, there is only one way to choose all these vertices and it dominates all the vertices and edges. Therefore, $d_{ve}(G, n) = 1$. If we delete one vertex v, the remaining $n - 1$ vertices dominate all the vertices and edges of G. (This is done in n ways). Therefore, $d_{ve}(G, n - 1) = n$.
(ii) Since $D_{ve}(G, i) = \phi$ if $i < \gamma_{ve}(G)$ or $D_{ve}(G, n + k) = \phi$, $k = 1, 2, \ldots$.
Therefore, we have $d_{ve}(G, i)$□ $= 0$ if $i < \gamma_{ve}(G)$ or $i > n$
Conversely, if $i < \gamma_{ve}(G)$ or $i > n$, $d_{ve}(G, i) = 0$ .Hence the result.

(iii) Since $\gamma_{ve}(G) \geq 1$, the vertex-edge domination polynomial has no term of degree 0. Therefore, it has no constant term. The proof of (iv) follows from the definition of vertex-edge domination polynomial.
(v) We have $\deg(D_{ve}(H, x))$= Number of vertices in H, Also, $\deg(D_{ve}(G, x))$ = Number of vertices in G since Number of vertices in H $\leq$ Number of vertices in G,
$\deg(D_{ve}(H, x)) \leq \deg(D_{ve}(G, x))$

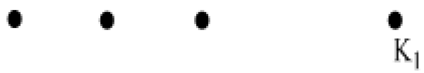
4. Vertex-edge Domination Polynomial of $G \circ K_1$

Lemma: 4.1 Let G be an empty graph of order n. Then, $\gamma_{ve}(G \circ K_1) = n$

Proof: Since G has n vertices, $G \circ K_1$ has 2n vertices. Let $V(G)=\{u_1, u_2, \ldots u_n\}$. Clearly $\{u, u_2, \ldots, u_n\}$ is the minimal vertex-edge dominating set of $G \circ K_1$.

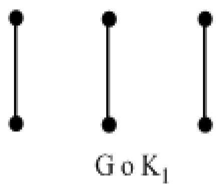
Therefore, $\gamma_{ve}(G \circ K_1) = n$

Example: 4.2



G

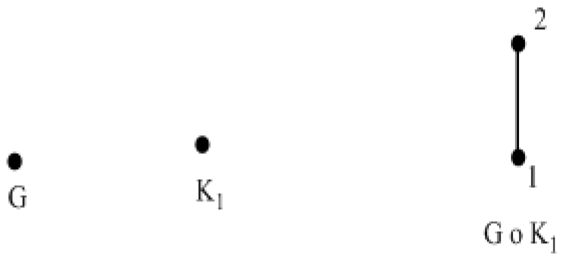
(Empty graph with 3 vertices)



There are three vertices required to cover all the vertices and edges of $G \circ K_1$.

Therefore, Minimum cardinality =3
$\gamma_{ve}(G \circ K_1) = 3$.

Example: 4.3 Let G be a complete Graph with n vertices. First, let n = 1.



$d_{ve}(G \circ K_1, 1) = 2$

$d_{ve}(G \circ K_1, 2) = 1$

$$\therefore\ D_{ve}(G \circ K_1, x) = \sum_{i=\gamma_{ve}(GoK_1)}^{|V(GoK_1)|} d_{ve}(GoK_1, i)x^i$$

$$= \sum_{i=1}^{2} d_{ve}(GoK_1, i)x^i$$

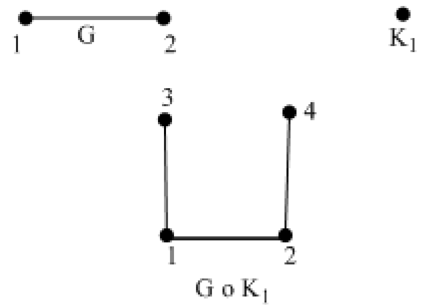
$$= d_{ve}(G \circ K_1, 1)x^1 + d_{ve}(G \circ K_1, 2)x^2$$

$$= 2x + x^2$$

$$= (1 + x)^2 - 1$$

$$= (1 + x)^2 - \binom{1}{0}$$

Let n = 2,



$d_{ve}(G \circ K_1, 1) = 2$

$d_{ve}(G \circ K_1, 2) = 6$

$d_{ve}(G \circ K_1, 3) = 4$

$d_{ve}(G \circ K_1, 4) = 1$

$$\therefore\ D_{ve}(G \circ K_1, x) = \sum_{i=\gamma_{ve}(GoK_1)}^{|V(GoK_1)|} d_{ve}(GoK_1, i)x^i$$

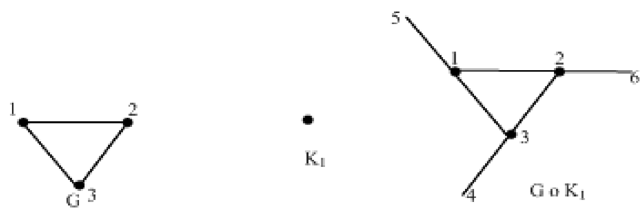
$$= \sum_{i=1}^{4} d_{ve}(GoK_1, i)x^i$$

$$= 2x + 6x^2 + 4x^3 + x^4$$

$$= (1 + x)^4 - (2x + 1)$$

$$= (1 + x)^4 - \left(\binom{2}{1}x + \binom{2}{0}\right)$$

n = 3,



$d_{ve}(G \circ K_1, 1) = 3$

$d_{ve}(G \text{ o } K_1, 2) = 12$

$d_{ve}(G \text{ o } K_1, 3) = 20$

$d_{ve}(G \text{ o } K_1, 4) = 15$

$d_{ve}(G \text{ o } K_1, 5) = 6$

$d_{ve}(G \text{ o } K_1, 6) = 1$

$$\therefore D_{ve}(G \text{ o } K_1, x) = \sum_{i=\gamma_{ve}(GoK_1)}^{|V(GoK_1)|} d_{ve}(GoK_1, i)x^i$$

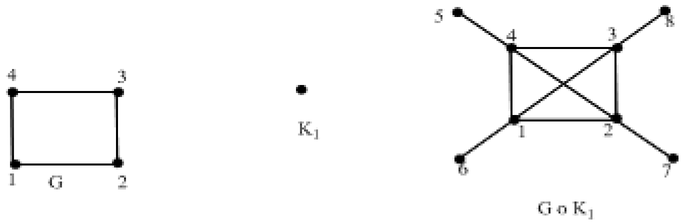
$$= \sum_{i=1}^{6} d_{ve}(GoK_1, i)x^i$$

$$= 3x + 12x^2 + 20x^3 + 15x^4 + 6x^5 + x^6$$

$$= (1 + x)^6 - (1 + 3x + 3x^2)$$

$$= (1 + x)^6 - \left(\binom{3}{2}x^2 + \binom{3}{1}x + \binom{3}{0} \right)$$

n = 4,



$d_{ve}(G \text{ o } K_1, 1) = 4$

$d_{ve}(G \text{ o } K_1, 2) = 22$

$d_{ve}(G \text{ o } K_1, 3) = 52$

$d_{ve}(G \text{ o } K_1, 4) = 70$

$d_{ve}(G \text{ o } K_1, 5) = 56$

$d_{ve}(G \text{ o } K_1, 6) = 28$

$d_{ve}(G \text{ o } K_1, 7) = 8$

$d_{ve}(G \text{ o } K_1, 8) = 1$

$$\therefore D_{ve}(G \text{ o } K_1, x) = \sum_{i=\gamma_{ve}(GoK_1)}^{|V(GoK_1)|} d_{ve}(GoK_1, i)x^i$$

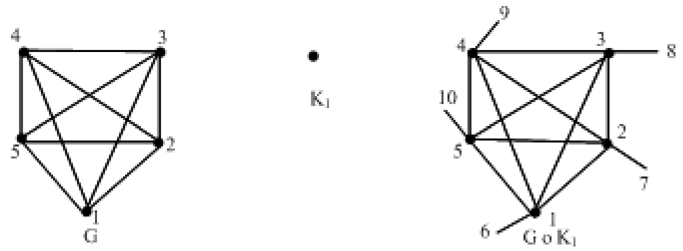
$$= \sum_{i=1}^{8} d_{ve}(GoK_1, i)x^i$$

$$= 4x + 22x^2 + 52x^3 + 70x^4 + 56x^5 + 28x^6 + 8x^7 + x^8$$

$$= (1 + x)^8 - (4x^3 + 6x^2 + 4x + 1)$$

$$= (1 + x)^8 - \left(\binom{4}{3}x^3 + \binom{4}{2}x^2 + \binom{4}{1}x + \binom{4}{0} \right)$$

n = 5,



$d_{ve}(G \text{ o } K_1, 1) = 5$

$d_{ve}(G \text{ o } K_1, 2) = 35$

$d_{ve}(G \text{ o } K_1, 3) = 110$

$d_{ve}(G \text{ o } K_1, 4) = 205$

$d_{ve}(G \text{ o } K_1, 5) = 252$

$d_{ve}(G \text{ o } K_1, 6) = 210$

$d_{ve}(G \text{ o } K_1, 7) = 120$

$d_{ve}(G \text{ o } K_1, 8) = 45$

$d_{ve}(G \text{ o } K_1, 9) = 10$

$d_{ve}(G \text{ o } K_1, 10) = 1$

$$\therefore D_{ve}(G \text{ o } K_1, x) = \sum_{i=\gamma_{ve}(GoK_1)}^{|V(GoK_1)|} d_{ve}(GoK_1, i)x^i$$

$$= 5x + 35x^2 + 110x^3 + 205x^4 + 252x^5 + 210x^6 + 120x^7 + 45x^8 + 10x^9 + x^{10}$$

$$= (1 + x)^{10} - (5x^4 + 10x^3 + 10x^2 + 5x + 1)$$

$$= (1 + x)^{10} - \left(\binom{5}{4}x^4 + \binom{5}{3}x^3 + \binom{5}{2}x^2 + \binom{5}{1}x + \binom{5}{0} \right)$$

Now, we generalize the vertex-edge domination polynomial of G o K_1, where G is the complete graph with n vertices as

$$D_{ve}(G \text{ o } K_1, x) = (1 + x)^{2n} - \left(\binom{n}{0} + \binom{n}{1}x + \ldots + \binom{n}{n-1}x^{n-1} \right)$$

$$= (1 + x)^{2n} - \sum_{k=0}^{n-1} \binom{n}{k}x^k.$$

Theorem: 4.4 G is a complete Graph of order n. We have

$$d_{ve}(G \text{ o } K_1, r) = \begin{cases} \binom{2n}{r} - \binom{n}{r}, & r < n \\ \binom{2n}{r}, & r \geq n \end{cases}$$

Hence, $D_{ve}(G \text{ o } K_1, x) = (1 + x)^{2n} - \sum_{r=0}^{n-1} \binom{n}{r}x^r.$

Proof: Since G has n vertices, G o K_1 has 2n vertices. One vertex of G o K_1 is enough to cover all the vertices and edges of G o K_1. Therefore the minimum cardinality is one.

Therefore, $\gamma_{ve}(G \circ K_1) = 1$. If $r < n$, then the $V(G) = \{u_1 \ldots u_n\}$ and $V(G \circ K_1) = \{u_1 \ldots u_n, v_1 \ldots v_n\}$ consists of any r vertices from $\{u_1 \ldots u_n, v_1 \ldots v_n\}$ excluding sets having any r vertices from $\{v_1 \ldots v_n\}$.

Thus, If $r < n$, then the vertex-edge dominating sets of $G \circ K_1$ is $\binom{2n}{r} - \binom{n}{r}$.

$\therefore$ The number of vertex – edge dominating sets is $\binom{2n}{r} - \binom{n}{r}$. Let $r \geq n$, we know that any set of n vertices in $\{u_1, u_2, \ldots, u_n, v_1, v_2 \ldots v_n\}$ is a vertex-edge dominating set of $G \circ K_1$. Then the number of vertex-edge dominating sets of $G \circ K_1$ is $\binom{2n}{r}$.

$$D_{ve}(G \circ K_1, x) = \sum_{i=\gamma_{ve}(GoK_1)}^{|V(GoK_1)|} d_{ve}(GoK_1, r)x^r$$

$$= \sum_{r=1}^{2n} d_{ve}(GoK_1, r)x^r$$

$$= \sum_{r=1}^{n-1} d_{ve}(GoK_1, r)x^r + \sum_{r=n}^{2n} d_{ve}(GoK_1, r)x^r$$

$$= \sum_{r=1}^{n-1}\left[\binom{2n}{r} - \binom{n}{r}\right]x^r + \sum_{r=n}^{2n}\binom{2n}{r}x^r$$

$$= \left[\binom{2n}{1} - \binom{n}{1}\right]x^1 + \left[\binom{2n}{2} - \binom{n}{2}\right]x^2 + \ldots + \left[\binom{2n}{n-1} - \binom{n}{n-1}\right]x^{n-1} + \binom{2n}{n}x^n + \binom{2n}{n+1}x^{n+1} + \ldots + \binom{2n}{2n}x^{2n}$$

$$= \binom{2n}{1}x^1 + \binom{2n}{2}x^2 + \binom{2n}{3}x^3 + \ldots + \binom{2n}{n}x^n + \binom{2n}{n+1}x^{n+1} + \ldots + \binom{2n}{2n}x^{2n} - \left(\binom{n}{1}x^1 + \binom{n}{2}x^2 + \ldots + \binom{n}{n-1}x^{n-1}\right) = \binom{2n}{0} + \binom{2n}{1}x^1 + \binom{2n}{2}x^2 + \binom{2n}{3}x^3 + \ldots + \binom{2n}{n}x^n + \binom{2n}{n+1}x^{n+1} + \ldots + \binom{2n}{2n}x^{2n} - \left(\binom{n}{0} + \binom{n}{1}x^1 + \binom{n}{2}x^2 + \ldots + \binom{n}{n-1}x^{n-1}\right)$$

$$= (1 + x)^{2n} - \sum_{r=0}^{n-1}\binom{n}{r}x^r$$

5. VERTEX-EDGE DOMINATION POLYNOMIAL OF G o K_2

Theorem: 5.1 If G is a complete Graph of order n. we have

$$d_{ve}(GoK_2, r) = \begin{cases} \binom{3n}{r} - \binom{n}{1}\binom{3n-3}{r} + \binom{n}{2}\binom{3n-6}{r} - \ldots + (-1)^k\binom{n}{k}\binom{3n-3k}{r} \\ \text{k is the largest integer satisfying } 3n - 3k \geq r, \text{ if } r \leq 3n - 3 \\ \binom{3n}{r}, \text{ if } r > 3n - 3 \end{cases}$$

Hence, $D_{ve}(G \circ K_2, x) = [(1 + x)^3 - 1]^n$

Proof: Since G has n vertices, $G \circ K_2$ has 3n vertices. The n vertices of G cover all the vertices and edges of $G \circ K_2$. Therefore, the minimum cardinality of vertex-edge dominating sets is n.

$$\therefore \gamma_{ve}(G \circ K_2) = n$$

If $r \le 3n-3$, then, The number of vertex-edge dominating sets of $G \circ K_2$ of cardinality r is

$$\binom{3n}{r} - \binom{n}{1}\binom{3n-3}{r} + \binom{n}{2}\binom{3n-6}{r} - \ldots + (-1)^k \binom{n}{k}\binom{3n-3k}{r}$$

When $r > 3n-3$, The number of vertex-edge dominating sets of $G \circ K_2$ of cardinality r is $\binom{3n}{r}$.

Therefore,

$$D_{ve}(G \circ K_2, x) = \sum_{r=\gamma_{ve}(G \circ K_2)}^{|V(G \circ K_2)|} d_{ve}(G \circ K_2, r)\, x^r$$

$$= \sum_{r=n}^{3n} d_{ve}(G \circ K_2, r)\, x^r$$

$$= \sum_{r=1}^{3n} d_{ve}(G \circ K_2, r)\, x^r \text{ (Since, } d_{ve}(G \circ K_2, r) = 0 \text{ for } r = 1, 2, \ldots, n-1)$$

$$= \sum_{r=1}^{3n-3} d_{ve}(G \circ K_2, r)\, x^r + \sum_{r=3n-2}^{3n} d_{ve}(G \circ K_2, r)\, x^r$$

$$= \sum_{r=1}^{3n-3} \left[\binom{3n}{r} - \binom{n}{1}\binom{3n-3}{r} + \binom{n}{2}\binom{3n-6}{r} - \ldots + (-1)^k \binom{n}{k}\binom{3n-3k}{r}\right] x^r + \sum_{r=3n-2}^{3n} \binom{3n}{r} x^r$$

$$= \sum_{r=1}^{3n-3} \binom{3n}{r} x^r - \binom{n}{1} \sum_{r=1}^{3n-3} \binom{3n-3}{r} x^r + \binom{n}{2} \sum_{r=1}^{3n-3} \binom{3n-6}{r} x^r - \ldots + (-1)^k \binom{n}{k} \sum_{r=1}^{3n-3} \binom{3n-3k}{r} x^r + \sum_{r=3n-2}^{3n} \binom{3n}{r} x^r$$

$$= \binom{3n}{1} x^1 + \binom{3n}{2} x^2 + \ldots + \binom{3n}{3n-3} x^{3n-3}$$
$$- \binom{n}{1}\left[\binom{3n-3}{1} x^1 + \binom{3n-3}{2} x^2 + \ldots + \binom{3n-3}{3n-3} x^{3n-3}\right]$$
$$+ \binom{n}{2}\left[\binom{3n-6}{1} x^1 + \binom{3n-6}{2} x^2 + \ldots\ldots + \binom{3n-6}{3n-6} x^{3n-6}\right]$$
$$- \ldots + (-1)^k \binom{n}{k}\left[\binom{3n-3k}{r} x^1 + \ldots + \binom{3n-3k}{3n-3k} x^{3n-3k}\right]$$
$$+ \ldots + \binom{3n}{3n-2} x^{3n-2} + \binom{3n}{3n-1} x^{3n-1} + \binom{3n}{3n} x^{3n}$$

$$= 1 + \binom{3n}{1} x^1 + \binom{3n}{2} x^2 + \ldots + \binom{3n}{3n-3} x^{3n-3} +$$
$$-1 - \binom{n}{1}\left[1 + \binom{3n-3}{1} x^1 + \binom{3n-3}{2} x^2 + \ldots + \binom{3n-3}{3n-3} x^{3n-3} - 1\right]$$
$$+ \binom{n}{2}\left[1 + \binom{3n-6}{1} x^1 + \binom{3n-6}{2} x^2 + \ldots + \binom{3n-6}{3n-6} x^{3n-6} - 1\right] + \ldots + (-1)^n \binom{n}{n}$$

$$= (1+x)^{3n} - \binom{n}{1}(1+x)^{3(n-1)} + \binom{n}{2}\left((1+x)^{3(n-2)}\right) - \ldots + (-1)^n$$
$$-\left[\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \ldots + (-1)^n\binom{n}{n}\right]$$
$$= [(1+x)^3 - 1]^n - \left[\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \ldots + (-1)^n\binom{n}{n}\right]$$
$$= [(1+x)^3 - 1]^n - 0$$
$$= [(1+x)^3 - 1]^n \quad \left(\because \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \ldots + (-1)^n\binom{n}{n} = 0\right)$$

$\therefore D_{ve}(G \circ K_2, x) = [(1+x)^3 - 1]^n$

REFERENCES

[1] Alikhani.S and Peng Y.H, 2008, Introduction to Domination Polynomial of a Graph. arXiv: 0905 . 2251v1 [math.co] 14 may.

[2] Alikhani.S and Peng Y.H, 2008. Domination sets and Domination polynomial of cycles, Global Journal of Pure and Applied Mathematics Vol.4, no.2.

[3] Bondy J.A, Murthy. U.S.R., Graph theory with applications, Elsevier Science Publishing Co, sixth printing, 1984.

[4] Frucht. R, and Harary. F, On the Corona of two graphs, Aequations Math.4 (1970) 322-324.

[5] Gray Chartand, Ping Zhang, 2005, Introduction to graph theory, Mc Graw Hill, Higher Education.

[6] Haynes T.W, Hedetniemi S.T., Slater P.J., Fundamentals of Domination in Graphs, Marcel Dekker, Newyork, 1998.

Source of support: Nil, Conflict of interest: None Declared