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APPROXIMATION OF UNBOUNDED FUNCTIONS BY POSITIVE LINEAR OPERATORS IN LOCALLY-GLOBAL WEIGHTED SPACES

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ABSTRACT

The aim of this paper is studying the approximation of unbounded functions by positive linear operators in locally-global weighted-spaces $L_{P,\delta,w}(X)$.

INTRODUCTION

The theory of approximation of functions constitutes a very wide branch of Mathematics, the basis of approximation of a real function was discovered by Weierstrass in 1885 [1], given by a theorem called "Weierstrass Approximation Theorem".

Afterwards, Stone [2], has generalized this theorem to a compact subsets of the real numbers that known as "Stone Weierstrass Theorem".

In 1912 Bernstein introduced another proof of Weierstrass approximation theorem, which was called Bernstein Theorem.

1950, Szasz [3] defined a sequence of linear positive operators on the interval $[0,\infty)$, called classical Sazasz operators defined as :

$S_n(f,x)=\sum_{i=0}^{\infty} M_{n,i}(x) f\left(\frac{1}{n}\right)$ where $M_{n,i}(x)=e^{-nx}\frac{(nx)^i}{i!}, x\in[0,\infty)$.

The purpose of this paper is studying approximation of unbounded functions by linear positive operator, which has a very similar form to the form of Szasz-Mirakyan operators.

Before proceeding to the study of first and second order of approximation by linear positive operator, it is necessary to know some definitions.

Definition: 1 Let $X=[0,a], a>0$, let $L_{P,w}(X)$ be the spaces of all unbounded functions $f, f(x)=0$ for $x>a$ with the following norm

$\|f\|_{P,w}=\left[\int_0^a \left|(fw)_{(x)}\right|^P dx\right]^{\frac{1}{P}}<\infty$ where $(fw)\in C[0,a], w(x)$ be the positive weighted function and

$$\|f\|_{P,\delta,w}=\left[\int_0^a \left(Sup\{\left|(fw)_{(u)}\right|: u\in N(x,\delta)\}\right)^P dx\right]^{\frac{1}{P}}$$

Be the locally-global of f where $N(x,\delta)=\{y\in X: |x-y|\le\delta\}, \delta\in R^+$ and we denote

$$L_{P,\delta,w}(X)=\{f: \|f\|_{P,\delta,w}<\infty\}$$

Definition: 2 [4] For $n,k,r\in|N|, x\in X=[0,\infty)$

Let us recall some of the common notations:

1) $(n)_k=n(n+1)(n+2)\dots(n+k-1)$ with $(n)_0=1$
2) $e_r(x)=x^r$

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Definition: 3 [4] For $f \in L_{P,w}(X)$ let $\Delta_h^1 f(x) = f(x+h) - f(x)$ and $\Delta_h^2 f(x) = \Delta_h^1 f(x+h) - \Delta_h^1 f(x)$ be the first and second difference with step h at point x.

Definition: 4 Let $f \in L_{P,w}(X), X = [0,\infty)$ then $\omega_k\ (f;\delta)_{P,w} = Sup_{0<P\le\delta}\left\{\left\|\Delta_h^k(fw,x)\right\|_{P,w} : |h| \le \delta, x, x+kh \in X, \delta > 0\right\}, k \in \{1,2\}$ be the usual first and second moduli of smoothness of function *f*. and $\omega_k\ (f;\delta)_{P,\delta,w} = Sup_{0<P\le\delta}\left\{\left\|\Delta_h^k(fw,x)\right\|_{P,\delta,w}\right\}, k \in \{1,2\}$ be the locally usual first and second moduli of smoothness of *f*.

Definition: 5 [4] For $f \in L_{P,w}(X)$, for $n \in N$ then $L_n(f,x) = 2^{-nx}\sum_{k=0}^{\infty}\frac{(nx)_k}{2^k k!} f\left(\frac{k}{n}\right), x \in X = [0,\infty)$ be the linear $+ve$ operator which has a very similar form to Szasz-Mirakyan operators.

AUXILIARY RESULTS

Here we prove some results, which will be useful to prove the main results.

Lemma: 1 For $f, g \in L_{P,w}$, then

a) $L_n(f+g,x) = L_n(f,x) + L_n(g,x)$
b) $L_n(\alpha f,x) = \alpha L_n(f,x), \alpha$ is a constant.

Proof: Since $L_n(f,x) = 2^{-nx}\sum_{k=0}^{\infty}\frac{(nx)_k}{2^k k!} f\left(\frac{k}{n}\right)$ then

a) $L_n(f+g,x) = 2^{-nx}\sum_{k=0}^{\infty}\frac{(nx)_k}{2^k k!}(f+g)\left(\frac{k}{n}\right)$

$$= 2^{-nx}\sum_{k=0}^{\infty}\frac{(nx)_k}{2^k k!}\left[f\left(\frac{k}{n}\right) + g\left(\frac{k}{n}\right)\right]$$

$$= 2^{-nx}\sum_{k=0}^{\infty}\frac{(nx)_k}{2^k k!} f\left(\frac{k}{n}\right) + 2^{-nx}\sum_{k=0}^{\infty}\frac{(nx)_k}{2^k k!} g\left(\frac{k}{n}\right)$$

$$= L_n(f,x) + L_n(g,x)$$

b) $L_n(\alpha f,x) = 2^{-nx}\sum_{k=0}^{\infty}\frac{(nx)_k}{2^k k!}(\alpha f)\left(\frac{k}{n}\right)$

$$= \alpha 2^{-nx}\sum_{k=0}^{\infty}\frac{(nx)_k}{2^k k!} f\left(\frac{k}{n}\right) = \alpha L_n(f,x)$$

Hence, L_n be a linear operator.

Remark: 1 [6] For $f \in L_P(X) = \left\{f : \|f\|_P = \left[\int_X |f(x)|^P dx\right]^{\frac{1}{P}} < \infty\right\}$ then

$$\|f\|_P \le \|f\|_{P,\delta} = \left[\int_X \left(Sup_{0<P\le\delta}\{|f(u)| : u \in N(x,\delta)\}\right)^P dx\right]^{\frac{1}{P}}$$

where $N(x,\delta) = \{y \in X : |x-y| \le \delta\}$

Lemma: 2 For $x \in X = [0,a], a > 0$ then

1) $L_n(e_r,x) = e_r \qquad r = \{0,1\}$
2) $L_n(e_2,x) = x^2 + \frac{2x}{n}$

Proof:

1) Since $(nx)_k = nx(nx+1)\ldots(nx+k-1) = nx(nx+1)_{k-1}$

and $L_n(f,x) = 2^{-nx}\sum_{k=0}^{\infty}\frac{(nx)_k}{2^k k!} f\left(\frac{k}{n}\right)$ then

For $r = 0$ we get

$$L_n(e_0,x) = L_n(1,x) = 2^{-nx}\sum_{k=0}^{\infty}\frac{(nx)_k}{2^k k!} = \sum_{k=0}^{\infty} 2^{-nx-k}\frac{(nx)_k}{k!} = 1$$

For r =1 we get

$$L_n(e_1,x) = 2^{-nx}\sum_{k=0}^{\infty}\frac{(nx)_k}{2^k k!}\frac{k}{n} = 2^{-nx-1}\sum_{k=1}^{\infty}\frac{nx(nx+1)_{k-1}}{2^{k-1}k(k-1)!}\frac{k}{n}$$

$$= x2^{-nx-1}\sum_{k=1}^{\infty}\frac{(nx+1)_{k-1}}{2^{k-1}(k-1)!}$$

Let $k-1=j$ then

$$L_n(e_1,x) = x2^{-nx-1}\sum_{j=0}^{\infty}\frac{(nx+1)_j}{2^j j!} = x.1 = x$$

2) $L_n(e_2,x) = 2^{-nx}\sum_{k=0}^{\infty}\frac{(nx)_k}{2^k k!}\frac{k^2}{n^2}$

$$2^{-nx-1}\sum_{k=1}^{\infty}\frac{(nx)(nx+1)_{k-1}}{2^{k-1}k(k-1)!}\frac{k^2}{n^2} = x2^{-nx-1}\sum_{k=1}^{\infty}\frac{(nx+1)_{k-1}}{2^{k-1}(k-1)!}\frac{k}{n}$$

Let $k-1=j$ then

$$L_n(e_2,x) = x2^{-nx-1}\sum_{k=1}^{\infty}\frac{(nx+1)_{k-1}}{2^{k-1}(k-1)!}\frac{k}{n} = x2^{-nx-1}\sum_{j=0}^{\infty}\frac{(nx+1)_j}{2^j j!}\frac{(j+1)}{n}$$

$$= x2^{-nx-1}\sum_{j=0}^{\infty}\frac{(nx+1)_j}{2^j j!}\frac{j}{n} + \frac{x}{n}2^{-nx-1}\sum_{j=0}^{\infty}\frac{(nx+1)_j}{2^j j!} \qquad (k=j+1)$$

$$= x2^{-nx-2}\sum_{k=0}^{\infty}\frac{(nx+1)(nx+2)_k}{2^k k!}\frac{x}{n} = x^2+\frac{x}{n}+\frac{x}{n} = x^2+\frac{2x}{n}$$

Then

$$L_n(e_2,x) = x^2+\frac{2x}{n}$$

Lemma: 3 [5] For $f\in C[a,b]$ then

$$\omega_k\ (f,\lambda\delta)_P \le (\lambda+1)^k\omega_k\ (f,\delta)_P\delta \ge 0, \lambda > 0$$

Lemma: 4 For $f\in L_{P,w}(X), X=[0,a], a>0, \lambda>0$ then

$$\omega_k\ (f,\lambda\delta)_{P,\delta,w} \le (\lambda+1)^k\omega_k\ (f,\delta)_{P,\delta,w} \quad k=\{1,2\}$$

Proof: To prove this lemma we must prove that

$\omega_k\ (f,n\delta)_{P,\delta,w} \le n^k\omega_k\ (f,\delta)_{P,\delta,w}$ where *n* is a natural number let's use the identity that

$\Delta_{nh}^k f(x) \le \sum_{i_1=0}^{n-1}\sum_{i_2=0}^{n-1}\dots\sum_{i_k}\Delta_h^k f(x+i_1h+i_2h+\cdots i_kh)$ by [5]

We can prove this identity by induction on *k*

For k=1 we have

$$\Delta_{nh}f(x) = f(x+nh)-f(x)$$

$$= \sum_{i=0}^{n-1}[f(x+ih+h)-f(x+ih)]$$

$$= \sum_{i=0}^{n-1}\Delta_h^1 f(x+ih)$$

Let's suppose that the identity is valid for a given natural number *k* then

$$\Delta_{nh}^{k+1}f(x) = \Delta_{nh}^k[f(x+nh)-f(x)] = \Delta_{nh}^k[\Delta_{nh}^1(f(x)]$$

$$= \sum_{i_1=0}^{n-1}\sum_{i_2=0}^{n-1}\dots\sum_{i_k=0}^{n-1}\Delta_h^k\left[\Delta_{nh}f(x+i_1h+i_2h+\cdots i_kh)\right]$$

$$= \sum_{i_1=0}^{n-1}\sum_{i_2=0}^{n-1}\dots\sum_{i_k=0}^{n-1}\Delta_h^k\left[\sum_{i_{k+1}=0}^{n-1}\Delta_h\ f(x+i_1h+i_2h+\cdots i_kh+i_{k+1}h)\right]$$

$$= \sum_{i_1=0}^{n-1}\sum_{i_2=0}^{n-1}\dots\sum_{i_k=0}^{n-1}\sum_{i_{k+1}=0}^{n-1}\Delta_h^{k+1}f(x+i_1h+i_2h+\cdots i_kh+i_{k+1}h)$$

Then

$$\omega_k\,(f,n\delta)_{P,\delta,w} = \underset{0\le P<\delta}{Sup}\left\{\left\|\Delta_{nh}^k f(x)\right\|_{P,\delta,w} : |nh| \le \delta\right\}$$

$$= Sup\quad\left[\int_0^a\left(Sup\left\{\left|\Delta_{nh}^k(fw)_{(u)}\right|^P : u\in N(x,\delta)\right\}\right)^P dx\right]^{\frac{1}{P}}$$

$$\le Sup\quad\left[\int_0^a\left|\sum_{i_1=0}^{n-1}\dots\sum_{i_k=0}^{n-1}\Delta_h^k\,(fw)(u+i_1h+i_2h+\cdots i_kh) : u\in N(x,\delta)\right|^P dx\right]^{\frac{1}{P}}$$

$$\le \sum_{i_1=0}^{n-1}\sum_{i_2=0}^{n-1}\dots\sum_{i_k=0}^{n-1} Sup\quad\left[\int_0^a\left|\Delta_h^k(fw)(u+i_1h+i_2h+\cdots i_kh) : u\in N(x,\delta)\right|^P dx\right]^{\frac{1}{P}}$$

$$\le n^k\omega_k\,(f,\delta)_{P,\delta,w}$$

Then

$$\omega_k\,(f,n\delta)_{P,\delta,w} \le n^k\omega_k\,(f,\delta)_{P,\delta,w}$$

Now since $[\lambda]\le\lambda,\ \ \lambda>0$ then $\lambda\le[\lambda]+1$ then by proposition monotonic, we have

$$\omega_k\,(f,\lambda\delta)_{P,\delta,w} \le \omega_k\,(f,([\lambda]+1)\delta)_{P,\delta,w}$$

Let $[\lambda]+1=n$ then

$$\omega_k\,(f,\lambda\delta)_{P,\delta,w} \le \omega_k\,(f,([\lambda]+1)\delta)_{P,\delta,w}$$

$$= \omega_k\,(f,n\delta)_{P,\delta,w}$$

$$\le n^k\,\omega_k\,(f,\delta)_{P,\delta,w}$$

$$= ([\lambda]+1)^k\,\omega_k\,(f,\delta)_{P,\delta,w}$$

$$\le (\lambda+1)^k\,\omega_k\,(f,\delta)_{P,\delta,w}$$

Hence

$$\omega_k\,(f,\lambda\delta)_{P,\delta,w} \le (\lambda+1)^k\omega_k\,(f,\delta)_{P,\delta,w}$$

Lemma: 5 [5] For $f\in C[a,b]$ then

$$\omega_k\,(f,\delta)_P \le \delta\omega_{k-1}(f^{'},\delta)_P,\qquad k=1,2,\dots,\quad \delta\ge 0$$

Lemma: 6 For $f\in L_{P,w}(X).\,X=[0,a],a>0,\delta\ge 0$ then

$$\omega_2\,(f,\delta)_{P,\delta,w} \le \delta\omega_1\,(f',\delta)_{P,\delta,w}\ \text{ where } f^{'}=(fw)'$$

Proof:

$$\omega_k\,(f,\delta)_{P,\delta,w} = \underset{0<P\le\delta}{Sup}\left\|\Delta_h^k(fw,x)\right\|_{P,\delta,w}$$

$$\le Sup\quad\left[\int_0^a\left|\Delta_h^k(fw)_{(u)} : u\in N(x,\delta)\right|^P dx\right]^{\frac{1}{P}}$$

$$= Sup\quad\left[\int_0^a\left|\Delta_h^{k-1}(\Delta_h(fw)_{(u)}) : u\in N(x,\delta)\right|^P dx\right]^{\frac{1}{P}}$$

$$\le Sup\quad\left[\int_0^a\left|\Delta_h^{k-1}\left[(fw)_{(u+h)}-(fw)_{(u)}\right] : u\in N(x,\delta)\right|^P dx\right]^{\frac{1}{P}}$$

$$\le Sup\quad\left[\int_0^a\left|\Delta_h^{k-1}\int_0^h(fw)'_{(u+t)}dt : u\in N(x,\delta)\right|^P dx\right]^{\frac{1}{P}}$$

$$\le Sup\quad\left[\int_0^a\left|\Delta_h^{k-1}(fw)'_{(u)} : u\in N(x,\delta)\int_0^h dt\right|^P dx\right]^{\frac{1}{P}}$$

$$\le Sup\quad\left\|\Delta_h^{k-1}(fw)'\right\|_{P,\delta,w}.\,h \le \delta\omega_{k-1}(f',\delta)_{P,\delta,w}$$

Then
$\omega_k\ (f,\delta)_{P,\delta,w} \leq \delta\omega_{k-1}(f',\delta)_{P,\delta,w}$ where $f^{'} = (fw)'$

Hence, for $k = 2$, we get

$$\omega_2\ (f,\delta)_{P,\delta,w} \leq \delta\omega_1\ (f',\delta)_{P,\delta,w}$$

Lemma: 7 For L_n be a linear positive operator in the space $L_{P,w}(X)$. $X = [0,a], a > 0$ such that

1) $L_n(1,x) = 1$
2) $L_n(t,x) = x + \alpha(x)$
3) $L_n(t^2,x) = x^2 + \beta(x)$ then for every $f \in L_{P,w}(X)$

$$\|L_n(f,.) - f\|_{P,\delta,w} \leq 3\omega_2\ \left(f,\sqrt{\beta(x) - 2x\alpha(x)}\right)_{P,\delta,w}$$

Proof: For $f \in C[a,b]$ then

$\|L_n(f,.) - f\|_P \leq 3\omega\ \left(f,\sqrt{\beta(x) - 2x\alpha(x)}\right)_P$ by Korevkin theorem [5]

And since $(fw) \in C[0,a]$ by definition1

$$\|L_n(fw,.) - fw\|_P \leq 3\omega\ \left(fw,\sqrt{\beta(x) - 2x\alpha(x)}\right)_P$$

Then by remark 1, we get

$$\|L_n(fw,.) - fw\|_{P,\delta} \leq 3\omega\ \left(fw,\sqrt{\beta(x) - 2x\alpha(x)}\right)_{P,\delta}$$

Thus

$$\|L_n(f,.) - f\|_{P,\delta,w} \leq 3\omega\ \left(f,\sqrt{\beta(x) - 2x\alpha(x)}\right)_{P,\delta,w}$$

$$\leq 3\omega_2\ \left(f,\sqrt{\beta(x) - 2x\alpha(x)}\right)_{P,\delta,w}$$

Hence

$$\|L_n(f,.) - f\|_{P,\delta,w} \leq 3\omega_2\ \left(f,\sqrt{\beta(x) - 2x\alpha(x)}\right)_{P,\delta,w}$$

Remark: 2 [4] From the linear operator $L_n(f,x) = 2^{-nx}\sum_{k=0}^{\infty}\frac{(nx)_k}{2^k k!}f\left(\frac{k}{n}\right)$

Let's introduce a new linear positive operator as

$$K_n(f,x) = n\sum_{k=0}^{\infty} M_{n,k}(x)\int_{k/n}^{(k+1)/n} f(t)\ dt$$

Where $f{:}\,[0,\infty) \to R$ and $M_{n,k}(x) = 2^{-nx}\frac{(nx)_k}{2^k k!},\ k \geq 0$

Remark: 3 In order to obtain an approximation process in spaces of an unbounded functions, let's introduce the new linear positive operator as

$$K_n(f,x) = n\sum_{k=0}^{\infty} M_{n,k}(x)\int_{k/n}^{(k+1)/n}(fw)(t)\ dt$$

Where $M_{n,k}(x) = 2^{-nx}\frac{(nx)_k}{2^k k!}$ and w is the positive weighted function, $k \geq 0$

Lemma: 8 For $f \in L_{P,w}(X), X = [0,a], a > 0$ then

1) $K_n(e_0,x) = 1$
2) $K_n(e_1,x) = x + \frac{1}{n}$
3) $K_n(e_2,x) = x^2 + \frac{3x}{n} + \frac{1}{3n^2}$

Proof: $K_n(f,x) = n\sum_{k=0}^{\infty} 2^{-nx}\frac{(nx)_k}{2^k k!}\int_{k/n}^{(k+1)/n}(fw)(t)\ dt$

Then

$$K_n(e_0,x) = K_n(1,x) = n\sum_{k=0}^{\infty} 2^{-nx}\frac{(nx)_k}{2^k k!}\int_{k/n}^{(k+1)/n} dt$$

$$= n\sum_{k=0}^{\infty} 2^{-nx}\frac{(nx)_k}{2^k k!}\left[\frac{k+1}{n}-\frac{k}{n}\right] = n\sum_{k=0}^{\infty} 2^{-nx}\frac{(nx)_k}{2^k k!}\left[\frac{1}{n}\right]$$

$$= \sum_{k=0}^{\infty} 2^{-nx}\frac{(nx)_k}{2^k k!} = 1$$

$$K_n(e_1,x) = K_n(t,x) = n\sum_{k=0}^{\infty} 2^{-nx}\frac{(nx)_k}{2^k k!}\int_{k/n}^{(k+1)/n} t\,dt$$

$$= n\sum_{k=0}^{\infty} 2^{-nx}\frac{(nx)_k}{2^k k!}\left[\frac{\frac{(k+1)^2}{n^2}}{2}-\frac{\frac{k^2}{n^2}}{2}\right]$$

$$= n\sum_{k=0}^{\infty} 2^{-nx}\frac{(nx)_k}{2^k k!}\left[\frac{(k+1)^2}{2n^2}-\frac{k^2}{2n^2}\right]$$

$$= n\sum_{k=0}^{\infty} 2^{-nx}\frac{(nx)_k}{2^k k!}\left[\frac{2k+1}{2n^2}\right]$$

$$= n\sum_{k=0}^{\infty} 2^{-nx}\frac{(nx)_k}{2^k k!}\frac{k}{n^2} + n\sum_{k=0}^{\infty} 2^{-nx}\frac{(nx)_k}{2^k k!}\frac{1}{2n^2}$$

$$= n\sum_{k=1}^{\infty} 2^{-nx}\frac{nx(nx+1)_{k-1}}{2\,2^{k-1}k(k-1)!}\frac{k}{n^2} + \sum_{k=0}^{\infty} 2^{-nx-1}\frac{(nx)_k}{2^k k!}\frac{1}{n}$$

$$= x\sum_{k=1}^{\infty} 2^{-nx-1}\frac{(nx+1)_{k-1}}{2^{k-1}(k-1)!} + \frac{1}{n}\sum_{k=0}^{\infty} 2^{-nx-1}\frac{(nx)_k}{2^k k!}\ \text{let } k-1=j$$

$$= x\sum_{j=0}^{\infty} 2^{-nx-1}\frac{(nx+1)_j}{2^j j!} + \frac{1}{n}.1 = x.1 + \frac{1}{n} = x + \frac{1}{n}$$

$$K_n(e_2,x) = K_n(t^2,x) = n\sum_{k=0}^{\infty} 2^{-nx}\frac{(nx)_k}{2^k k!}\int_{k/n}^{(k+1)/n} t^2 dt$$

$$= n\sum_{k=0}^{\infty} 2^{-nx}\frac{(nx)_k}{2^k k!}\left[\frac{\frac{(k+1)^3}{n^3}}{3}-\frac{\frac{k^3}{n^3}}{3}\right] = n\sum_{k=0}^{\infty} 2^{-nx}\frac{(nx)_k}{2^k k!}\left[\frac{(k+1)^3}{3n^3}-\frac{k^3}{3n^3}\right]$$

$$= n\sum_{k=0}^{\infty} 2^{-nx}\frac{(nx)_k}{2^k k!}\left[\frac{3k^2+3k+1}{3n^3}\right] = n\sum_{k=0}^{\infty} 2^{-nx}\frac{(nx)_k}{2^k k!}\left[\frac{k^2}{n^3}+\frac{k}{n^3}+\frac{1}{3n^3}\right]$$

$$= n\sum_{k=0}^{\infty} 2^{-nx}\frac{(nx)_k}{2^k k!}\frac{k^2}{n^3} + n\sum_{k=0}^{\infty} 2^{-nx}\frac{(nx)_k}{2^k k!}\frac{k}{n^3} + n\sum_{k=0}^{\infty} 2^{-nx}\frac{(nx)_k}{2^k k!}\frac{1}{3n^3}$$

$$= \sum_{k=0}^{\infty} 2^{-nx}\frac{(nx)_k}{2^k k!}\frac{k^2}{n^2} + \sum_{k=0}^{\infty} 2^{-nx}\frac{(nx)_k}{2^k k!}\frac{k}{n^2} + n\sum_{k=0}^{\infty} 2^{-nx}\frac{(nx)_k}{2^k k!}\frac{1}{3n^2}$$

$$= 2^{-nx-1}\sum_{k=1}^{\infty}\frac{(nx)(nx+1)_{k-1}}{2^{k-1}k(k-1)!}\frac{k^2}{n^2} + 2^{-nx-1}\sum_{k=1}^{\infty}\frac{(nx)(nx+1)_{k-1}}{2^{k-1}k(k-1)!}\frac{k}{n^2} + \frac{1}{3n^2}\sum_{k=0}^{\infty} 2^{-nx}\frac{(nx)_k}{2^k k!}$$

$$= x2^{-nx-1}\sum_{k=1}^{\infty}\frac{(nx+1)_{k-1}}{2^{k-1}(k-1)!}\frac{k}{n} + x2^{-nx-1}\sum_{k=1}^{\infty}\frac{(nx+1)_{k-1}}{2^{k-1}(k-1)!}\frac{1}{n} + \frac{1}{3n^2}1$$

Let $k-1=j$ then

$$K_n(e_2,x) = x2^{-nx-1}\sum_{j=0}^{\infty}\frac{(nx+1)_j}{2^j j!}\frac{j+1}{n} + x2^{-nx-1}\sum_{j=0}^{\infty}\frac{(nx+1)_j}{2^j j!}\frac{1}{n} + \frac{1}{3n^2}$$

$$= x2^{-nx-1}\sum_{j=0}^{\infty}\frac{(nx+1)_j}{2^j j!}\frac{j}{n} + \frac{x}{n}2^{-nx-1}\sum_{j=0}^{\infty}\frac{(nx+1)_j}{2^j j!} + \frac{x}{n}2^{-nx-1}\sum_{j=0}^{\infty}\frac{(nx+1)_j}{2^j j!} + \frac{1}{3n^2}$$

$$= x^2 + \frac{x}{n} + \frac{x}{n} + \frac{x}{n} + \frac{1}{3n^2} = x^2 + \frac{3x}{n} + \frac{1}{3n^2}$$

Hence $K_n(e_2,x) = x^2 + \frac{3x}{n} + \frac{1}{3n^2}$

MAIN RESULTS

Theorem: 1 For $f \in L_{P,w}(X), X = [0,a]$ with $f(x) = 0$ for $x > a > 0$ then

$$\|L_n(f,.) - f\|_{P,\delta,w} \leq 3\left(1+\sqrt{2a}\right)^2 \omega_2\left(f,\frac{1}{\sqrt{n}}\right)_{P,\delta,w}$$

Proof: From lemma 2 we have

1) $L_n(e_r,x)=e_r, \qquad r=\{0,1\}$
2) $L_n(e_2,x)=x^2+\frac{2x}{n}$

Then

$L_n(1,x)=1$

$L_n(t,x)=x+\alpha(x)$ where $\alpha(x)=0$

$L_n(t^2,x)=x^2+\beta(x)$ where $\beta(x)=\frac{2x}{n}$

And since L_n is a linear positive operator by lemma 1, then from lemma 7, we get

$$\begin{aligned}\|L_n(f,.)-f\|_{P,\delta,w} &\le 3\omega_2\left(f,\sqrt{\beta(x)-2x\alpha(x)}\right)_{P,\delta,w}\\ &=3\omega_2\left(f,\sqrt{\frac{2x}{n}-0}\right)_{P,\delta,w}\\ &=3\omega_2\left(f,\frac{\sqrt{2x}}{\sqrt{n}}\right)_{P,\delta,w}\\ &\le 3\omega_2\left(f,\frac{\sqrt{2a}}{\sqrt{n}}\right)_{P,\delta,w}\end{aligned}$$

Thus

$$\|L_n(f,.)-f\|_{P,\delta,w}\le 3\omega_2\left(f,\frac{\sqrt{2a}}{\sqrt{n}}\right)_{P,\delta,w}$$

And from lemma 4 we have

$$\|L_n(f,.)-f\|_{P,\delta,w}\le 3\omega_2\left(f,\sqrt{2a}\frac{1}{\sqrt{n}}\right)_{P,\delta,w}\le 3\left(\sqrt{2a}+1\right)^2\omega_2\left(f,\frac{1}{\sqrt{n}}\right)_{P,\delta,w}$$

Thus, $\|L_n(f,.)-f\|_{P,\delta,w}\le 3\left(\sqrt{2a}+1\right)^2\omega_2\left(f,\frac{1}{\sqrt{n}}\right)_{P,\delta,w}$

Theorem: 2 For $f\in L_{P,w}(X), X=[0,a]$ with $f(x)=0$ for $x>a>0$ then

$$\|L_n(f,.)-f\|_{P,\delta,w}\le\frac{3\left(\sqrt{2a}+1\right)^2}{\sqrt{n}}\omega\left(f',\frac{1}{\sqrt{n}}\right)_{P,\delta,w}\ \text{where} f'=(fw)'$$

Proof: From theorem 1, we get

$$\|L_n(f,.)-f\|_{P,\delta,w}\le 3\left(\sqrt{2a}+1\right)^2\omega_2\left(f,\frac{1}{\sqrt{n}}\right)_{P,\delta,w}$$

Then by using lemma 6, we get

$$\begin{aligned}\|L_n(f,.)-f\|_{P,\delta,w}&\le 3\left(\sqrt{2a}+1\right)^2\omega_2\left(f,\frac{1}{\sqrt{n}}\right)_{P,\delta,w}\\ &\le 3\left(\sqrt{2a}+1\right)^2\frac{1}{\sqrt{n}}\omega\left(f',\frac{1}{\sqrt{n}}\right)_{P,\delta,w}\end{aligned}$$

Thus

$$\|L_n(f,.)-f\|_{P,\delta,w}\le\frac{3\left(\sqrt{2a}+1\right)^2}{\sqrt{n}}\omega\left(f',\frac{1}{\sqrt{n}}\right)_{P,\delta,w}$$

Where $f'=(fw)'$

Theorem: 3 For $f\in L_{P,w}(X), X=[0,a], a>0$ then one has $\lim_{n\to\infty}L_n(f,x)=f(x)$ uniformly on $[0,a]$

Proof: From theorem 1, we get

$$\|L_n(f,.)-f\|_{P,\delta,w}\le 3\left(\sqrt{2a}+1\right)^2\omega_2\left(f,\frac{1}{\sqrt{n}}\right)_{P,\delta,w}\ \text{then}$$

$$\lim_{n\to\infty} \|L_n(f,.) - f\| \le 3\left(\sqrt{2a}+1\right)^2 \lim_{n\to\infty} \omega_2 \left(f, \frac{1}{\sqrt{n}}\right)_{P,\delta,w}$$
$$= 3\left(\sqrt{2a}+1\right) \ \omega_2\ (f,0)$$
$$= 3\left(\sqrt{2a}+1\right).(0) = 0$$

Thus

$$\lim_{n\to\infty} \|L_n(f,.) - f\| = 0$$

Hence, $L_n(f,x) \xrightarrow{c.u.} f(x)$on $[0,a]$

Theorem: 4 For $f \in L_{P,w}(X), X = [0,a]$with $f(x) = 0$ for $x > a > 0$ then

$$\|K_n(f,.) - f\|_{P,\delta,w} \le \left(1 + \frac{\sqrt{2a+1}}{\sqrt{3}}\right)^2 \omega_2\ \left(f, \frac{1}{\sqrt{n}}\right)_{P,\delta,w}$$

Proof: From lemma 8, we get

1) $K_n(e_0,x) = 1$
2) $K_n(e_1,x) = x + \frac{1}{n}$
3) $K_n(e_2,x) = x^2 + \frac{3x}{n} + \frac{1}{3n^2}$

Then

$K_n(1,x) = 1, K_n(t,x) = x + \alpha(x)$ where $\alpha(x) = \frac{1}{n}$

$K_n(t^2,x) = x^2 + \beta(x)$ where $\beta(x) = \frac{3x}{n} + \frac{1}{3n^2}$

Thus from lemma 7, we get

$$\|K_n(f,.) - f\|_{P,\delta,w} \le 3\omega_2\ \left(f, \sqrt{\beta(x) - 2x\alpha(x)}\right)_{P,\delta,w}$$
$$= 3\omega_2\ \left(f, \sqrt{\frac{3x}{n} + \frac{1}{3n^2} - \frac{2x}{n}}\right)_{P,\delta,w} = 3\omega_2\ \left(f, \sqrt{\frac{3nx+1}{3n^2}}\right)_{P,\delta,w}$$
$$\le 3\omega\ \left(f, \sqrt{\frac{3an+1}{3n^2}}\right)_{P,\delta,w}$$

Then for $n \to \infty$ we have

$$\|K_n(f,.) - f\|_{P,\delta,w} \le 3\omega_2 \left(f, \sqrt{\frac{3an+1}{3n^2}}\right)_{P,\delta,w} \le 3\omega_2\ \left(f, \frac{\sqrt{3a+1}}{\sqrt{3}\sqrt{n}}\right)_{P,\delta,w}$$

Thus

$$\|K_n(f,.) - f\|_{P,\delta,w} \le 3\omega_2\ \left(f, \frac{\sqrt{3a+1}}{\sqrt{3}\sqrt{n}}\right)_{P,\delta,w}$$

Then from lemma 4, we have

$$\|K_n(f,.) - f\|_{P,\delta,w} \le 3\omega_2\ \left(f, \frac{\sqrt{3a+1}}{\sqrt{3}\sqrt{n}}\right)_{P,\delta,w} \le 3\left(1 + \frac{\sqrt{3a+1}}{\sqrt{3}}\right)^2 \omega_2\ \left(f, \frac{1}{\sqrt{n}}\right)_{P,\delta,w}$$

Hence,

$$\|K_n(f,.) - f\|_{P,\delta,w} \le 3\left(1 + \frac{\sqrt{3a+1}}{\sqrt{3}}\right)^2 \omega_2\ \left(f, \frac{1}{\sqrt{n}}\right)_{P,\delta,w}$$

CONCLUSION

We are using linear positive operators $L_n(f,x)$ and $K_n(f,x)$, for $f \in L_{P,w}(X)$ and obtaining the degree of best approximation of this function in weighted space in terms of weighted modulus of function f.

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