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ON THE BEST APPROXIMATION OF UNBOUNDED FUNCTIONS BY LINEAR POSITIVE OPERATOR

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ABSTRACT

The purpose of this paper is to study the approximation of unbounded functions by linear positive operator $c_n(f;x)$ in weighted spaces and finding the degree of best approximation of these functions.

INTRODUCTION

Many researchers studied theory of approximation of functions. Many properties and theorems about Jackson Polynomials were introduced by Zygmund [1]. E. S. Bhaya [2] obtained some results about the convergence of periodic functions in the space $L_P(0 < P \leq 1)$. Eman Hassan Muhammed Al-Asady [3] study on the best approximation of function in the spaces $L_P(\mu)$. Hammod A. A. [4] introduced the estimation of any function by the k-functional and he found estimation for positive linear operator.

Definition: 1 [5] Let $C_B[0,\infty) = \{f|f:[0,\infty) \to R\}$ such that $\|f\| = Sup_{X\in[0,\infty)}|f(x)|$and we consider a new linear positive operator

$C_n: C_B[0,\infty) \to C_B[0,\infty)$ Such that

$$C_n(f;x) = \frac{1}{(1+x)^n} f(0) + \sum_{k=1}^{n-1}\binom{n}{k}\frac{x^k}{(1+x)^n}\cdot\frac{1}{\beta(k,n-k+2)}\int_0^\infty f(t)\frac{t^{k-1}}{(1+t)^{n+2}}dt + \left(\frac{x}{1+x}\right)^n f(n),\ \ x \geq 0$$

Where β is beta function.

Definition: 2 [7] (Beta Function) $\int_0^1 t^{a-1}(1-t)^{b-1}\,dt$is called Beta function of $a > 0, b > 0$. It is also written as $\beta(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1}\,dt$

Remark: 1 [7] Properties of Beta function:

i. $\beta(a,b) = \beta(b,a)$, a and$b$ are interchanged.

ii. $\beta(a,b) = \int_0^\infty \frac{t^{a-1}}{(1+t)^{a+b}}dt$

iii. $\beta(a,b) = \frac{(a-1)!}{b(b+1)\ldots\ldots\ldots(b+a-1)}$ if $a \neq b$, a, b are positive integers

Remark: 2

$$\frac{1}{\beta_{(k,n-k+2)}}\int_0^\infty \frac{t^k}{(1+t)^{n+2}}dt = \frac{k}{n-k+1},\ t > 0, 1 \leq k \leq n-1$$

Proof:

$$\frac{1}{\beta_{(k,n-k+2)}}\int_0^\infty \frac{t^k}{(1+t)^{n+2}}dt = \frac{1}{\beta_{(k,n-k+2)}}\int_0^\infty \frac{t^{(k+1)-1}}{(1+t)^{(k+1)+n-k+1}}dt$$

$$= \frac{\beta(k+1,n-k+1)}{\beta(k,n-k+2)}$$ Using remark (1), (iii) we get

$$= \frac{k}{n-k+1}$$

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Definition: 3 Let $L_{P,\alpha} = \{f|f:[0,a] \to R; a > 0\}$

$$\|f\|_{P,\alpha} = \left(\int_0^a |f(x)w(x)|^P \, dx\right)^{\frac{1}{P}} < \infty, \ \ 0 < P < \infty$$

Where $w(x) = e^{-\alpha x}, \alpha > 0$ is weighted positive function, let

$$C_n(f;x) = \frac{1}{(1+x)^n}(f.w)(0) + \sum_{k=1}^{n-1}\binom{n}{k}\frac{x^k}{(1+x)^n}\frac{1}{\beta(k,n-k+2)}\int_0^\infty (f.w)(t)\frac{t^{k-1}}{(1+t)^{n+2}}dt + \left(\frac{x}{1+x}\right)^n (f.w)(n)$$

Definition: 4 [6] The difference of a function *f* of order *k* with step *h* at point x is defined by:

$$\Delta_h^k f(x) = \sum_{m=0}^{k}(-1)^{m+k}\binom{k}{m} f(x+mh)$$

where $\binom{k}{m} = \frac{k!}{m!(k-m)!}$, $\Delta_h f(x) = \Delta_h^1 f(x)$

Definition: 5 [6] The modulus of smoothness of order *k* of function *f*,$\delta > 0$ is defined by:

$$\omega_k(f;\delta) = Sup\{|\Delta_h^k f(x)|: |h| \le \delta, x, x+kh \in [a,b]\}$$

Definition: 6 Let $f \in L_{P,\alpha}$ be an unbounded function and $f.w(x)$ a bounded one, the modulus of smoothness of order *k* of the function $f, \delta > 0$ is defined by:

$$\omega_k(f;\delta)_\alpha = Sup\{|\Delta_h^k f(x)w(x)|: |h| \le \delta, x, x+kh \in [a,b]\}$$

Lemma: 1 [6] Let $f \in C[a,b]$ then

$$\omega_k(f;\delta) \le 2\omega_{k-1}(f;\delta)$$

Lemma: 2 Let $f \in L_{P,\alpha}$, $f.w$ is a bounded function, $\delta > 0$, then

$$\omega_k(f;\delta)_\alpha \le 2\omega_{k-1}(f;\delta)_\alpha$$

Proof: Since $f.w$ is a bounded, using lemma (1), we get

$$\omega_k(f;\delta)_\alpha \le 2\omega_{k-1}(f;\delta)_\alpha$$

Lemma: 3 [6] Let $f \in C[a,b], \delta > 0$, then

$$\omega_k(f;\delta) \le 2\omega_{k-1}(f';\delta)$$

Lemma: 4 Let $f \in L_{P,\alpha}(x), x = [0,a], a > 0, , \delta > 0$, then

$$\omega_k(f;\delta)_\alpha \le \delta\omega_{k-1}(f';\delta)_\alpha$$

Proof: Since $f.w$ is a bounded function, then using lemma (3), we get

$$\omega_k(f;\delta)_\alpha \le \delta\omega_{k-1}(f';\delta)_\alpha$$

Lemma: 5 [6] Let $f \in C[a,b], \delta > 0$, then

$$\omega_k(f;\lambda\delta) \le (\lambda+1)^k\omega_k(f;\delta), \lambda > 0$$

Lemma: 6 Let $f \in L_{P,\alpha}, \delta > 0$, then

$$\omega_k(f;\lambda\delta)_\alpha \le (\lambda+1)^k\omega_k(f;\delta)_\alpha, \lambda > 0$$

Proof: Since $f.w$ is a bounded, and using lemma (5), we get

$$\omega_k(f;\delta)_\alpha \le (\lambda+1)^k\omega_{k-1}(f;\delta)_\alpha$$

Lemma: 7 Let $f \in L_{P,\alpha}[0,a], a > 0$,if $f_0(x) = 1$, $f_1(x) = x$and $f_2(x) = x^2$, then

1) $C_n\ (f_0; x) = C_n\ (1; x) = 1$
2) $C_n\ (f_1; x) = C_n\ (x; x) = x - x\left(\frac{x}{1+x}\right)^n$
3) $C_n\ (f_2; x) = C_n\ (x^2; x) = x^2 + \frac{11x(1+x)^2}{n+2}, n \geq 2$

Proof:

1) $C_n\ (1; x) = \frac{1}{(1+x)^n} f.w(0) + \sum_{k=1}^{n-1} \binom{n}{k} \frac{x^k}{(1+x)^n} \frac{1}{\beta(k,n-k+2)} \int_0^\infty f.w(t) \frac{t^{k-1}}{(1+t)^{n+2}} dt + \left(\frac{x}{1+x}\right)^n f.w(n)$

$$= \frac{1}{(1+x)^n} + \sum_{k=1}^{n-1} \binom{n}{k} \frac{x^k}{(1+x)^n} \frac{\beta(k,n-k+2)}{\beta(k,n-k+2)} + \left(\frac{x}{1+x}\right)^n$$

$$= \frac{1}{(1+x)^n} + \sum_{k=1}^{n-1} \binom{n}{k} \frac{x^k}{(1+x)^n} + \frac{x^n}{(1+x)^n}$$

$$= \sum_{k=0}^{n} \binom{n}{k} \frac{x^k}{(1+x)^n}$$

$$= \frac{1}{(1+x)^n} \sum_{k=0}^{n} \binom{n}{k} x^k$$

$$= \frac{(1+x)^n}{(1+x)^n} = 1$$

2) $C_n\ (x; x) = \frac{1}{(1+x)^n} f.w(0) + \sum_{k=1}^{n-1} \binom{n}{k} \frac{x^k}{(1+x)^n} \frac{1}{\beta(k,n-k+2)} \int_0^\infty f.w(t) \frac{t^{k-1}}{(1+t)^{n+2}} dt + \left(\frac{x}{1+x}\right)^n f.w(n)$

$$= \sum_{k=1}^{n-1} \binom{n}{k} \frac{x^k}{(1+x)^n} \frac{1}{\beta(k,n-k+2)} \int_0^\infty \frac{t^k}{(1+t)^{n+2}} dt + \left(\frac{x}{1+x}\right)^n n$$

$$= \sum_{k=1}^{n-1} \binom{n}{k} \frac{x^k}{(1+x)^n} \frac{k}{n-k+1} + n\left(\frac{x}{1+x}\right)^n$$

$$= \sum_{k=1}^{n-1} \binom{n}{k-1} \frac{x^k}{(1+x)^n} + n\left(\frac{x}{1+x}\right)^n$$

$$= \sum_{j=0}^{n-2} \binom{n}{j} \frac{x^{j+1}}{(1+x)^n} + n\left(\frac{x}{1+x}\right)^n$$

$$= \frac{x}{(1+x)^n} \sum_{j=0}^{n-2} \binom{n}{j} x^j$$

$$= \frac{x}{(1+x)^n} [(1+x)^n - nx^{n-1} - x^n] + n\left(\frac{x}{1+x}\right)^n$$

$$= \frac{x(1+x)^n}{(1+x)^n} - \frac{nx^n}{(1+x)^n} - \frac{x^n x}{(1+x)^n} + n\left(\frac{x}{1+x}\right)^n$$

$$= x - n\left(\frac{x}{1+x}\right)^n - x\left(\frac{x}{1+x}\right)^n + n\left(\frac{x}{1+x}\right)^n$$

$$= x - x\left(\frac{x}{1+x}\right)^n$$

3) $C_n\ (x^2; x) = \frac{1}{(1+x)^n} f.w(0) + \sum_{k=1}^{n-1} \binom{n}{k} \frac{x^k}{(1+x)^n} \frac{1}{\beta(k,n-k+2)} \int_0^\infty f.w(t) \frac{t^{k-1}}{(1+t)^{n+2}} dt + \left(\frac{x}{1+x}\right)^n f.w(n)$

$$= \sum_{k=1}^{n-1} \binom{n}{k} \frac{x^k}{(1+x)^n} \frac{1}{\beta(k,n-k+2)} \int_0^\infty t^2 \frac{t^{k-1}}{(1+t)^{n+2}} dt + n^2\left(\frac{x}{1+x}\right)^n$$

$$= \sum_{k=1}^{n-1} \binom{n}{k} \frac{x^k}{(1+x)^n} \frac{1}{\beta(k,n-k+2)} \int_0^\infty \frac{t^{(k+2)-1}}{(1+t)^{(k+2)+n-k}} dt + n^2\left(\frac{x}{1+x}\right)^n$$

$$= \sum_{k=1}^{n-1} \binom{n}{k} \frac{x^k}{(1+x)^n} \frac{\beta(k+2,n-k)}{\beta(k,n-k+2)} + n^2\left(\frac{x}{1+x}\right)^n$$

$$= \sum_{k=1}^{n-1} \binom{n}{k} \frac{x^k}{(1+x)^n} \frac{k(k+1)}{(n-k)(n-k+1)} + n^2 \left(\frac{x}{1+x}\right)^n$$

$$= \sum_{k=1}^{n-1} \binom{n}{k-1} \left(\frac{n+1}{n-k} - 1\right) \frac{x^k}{(1+x)^n} + n^2 \left(\frac{x}{1+x}\right)^n$$

$$= \sum_{k=1}^{n-1} \binom{n}{k-1} \frac{n+1}{n-k} \frac{x^k}{(1+x)^n} - \sum_{k=1}^{n-1} \binom{n}{k-1} \frac{x^k}{(1+x)^n} + n^2 \left(\frac{x}{1+x}\right)^n$$

$$= \sum_{k=1}^{n-1} \binom{n+1}{k-1} \frac{n-k+2}{n-k} \frac{x^k}{(1+x)^n} - \sum_{j=0}^{n-2} \binom{n}{j} \frac{x^{j+1}}{(1+x)^n} + n^2 \left(\frac{x}{1+x}\right)^n$$

$$= \sum_{k=1}^{n-1} \binom{n+1}{k-1} \left[\frac{n-k}{n-k} + \frac{2}{n-k}\right] \frac{x^k}{(1+x)^n} - \sum_{j=0}^{n-2} \binom{n}{j} \frac{x^{j+1}}{(1+x)^n} + n^2 \left(\frac{x}{1+x}\right)^n$$

$$= \sum_{k=1}^{n-1} \binom{n+1}{k-1} \frac{x^k}{(1+x)^n} + 2\sum_{k=1}^{n-1} \binom{n+1}{k-1} \frac{1}{n-k} \frac{x^k}{(1+x)^n} - x + n\left(\frac{x}{1+x}\right)^n + x\left(\frac{x}{1+x}\right)^n + n^2 \left(\frac{x}{1+x}\right)^n$$

$$= \sum_{j=0}^{n-2} \binom{n+1}{j} \frac{x^{j+1}}{(1+x)^n} + 2\sum_{j=0}^{n-2} \binom{n+1}{j} \frac{1}{n-j-1} \frac{x^{j+1}}{(1+x)^n} - x + n\left(\frac{x}{1+x}\right)^n + x\left(\frac{x}{1+x}\right)^n + n^2 \left(\frac{x}{1+x}\right)^n$$

$$= \frac{x}{(1+x)^n}\left[(1+x)^{n+1} - \frac{n(n+1)}{2}x^{n-1} - (n+1)x^n - x^{n+1}\right]$$
$$+2\sum_{j=0}^{n-2} \binom{n+1}{j} \frac{1}{n-j-1} \cdot \frac{x^{j+1}}{(1+x)^n} - x + n\left(\frac{x}{1+x}\right)^n + x\left(\frac{x}{1+x}\right)^n + n^2 \left(\frac{x}{1+x}\right)^n$$

$$= x^2 + \frac{n(n+1)}{2}\left(\frac{x}{1+x}\right)^n - nx\left(\frac{x}{1+x}\right)^n - \frac{x^{n+2}}{(1+x)^n} + I$$

Put $I = 2\sum_{j=0}^{n-2} \binom{n+1}{j} \frac{1}{n-j-1} \frac{x^{j+1}}{(1+x)^n}$

Since $\frac{4}{n-j+2} \geq \frac{1}{n-j-1}$ $\qquad n-2 \geq j \geq 0, 2 \leq n$

$$I \leq 8\sum_{j=0}^{n-2} \binom{n+1}{j} \frac{1}{n-j+2} \frac{x^{j+1}}{(1+x)^n}$$

$$< 8x\sum_{j=0}^{n+1} \binom{n+1}{j} \frac{1}{n-j+2} \frac{x^j}{(1+x)^j}\left(1 - \frac{x}{1+x}\right)^{n-j}$$

Let $\frac{1}{n+2-v}$ be a random variable with v has binomial distribution with parameters $n = n+1$ and $p = \frac{x}{1+x}$

$$I < 8x(1+x)\sum_{j=0}^{n+1} \binom{n+1}{j} \frac{1}{n-j+2} \frac{x^j}{(1+x)^j}\left(1 - \frac{x}{1+x}\right)^{n+1-j}$$

$$= 8x(1+x)\mathrm{E}\left[\frac{1}{n+2-v}\right]$$

Using the result of Chao and Strawderman [8], the random variable $n+1-v$ has parameters $n = n+1$ and $q = 1-p = \frac{1}{1+x}$, we get

$$\mathrm{E}\left[\frac{1}{n+2-v}\right] = \mathrm{E}\left[\frac{1}{1+(n+2-v)}\right] = \frac{1-P^{n+2}}{(n+2)q}$$

$$< \frac{1}{(n+2)q}$$

$$= \frac{1+x}{n+2}$$

Thus, $I \leq \frac{8x(1+x)^2}{n+2}$

$$C_n\ (x^2; x) \leq x^2 + \frac{n(n+1)}{2}\left(\frac{x}{1+x}\right)^n - nx\left(\frac{x}{1+x}\right)^n - \frac{x^{n+2}}{(1+x)^n} + \frac{8x(1+x)^2}{n+2}$$
$$\leq x^2 + \frac{n(n+1)}{2}\left(\frac{x}{1+x}\right)^n + 8\frac{x(1+x)^2}{n+2}$$

Since

$\binom{n+2}{3}x^{n-1} \le (1+x)^{n+2}, x \ge 0$

$\frac{(n+2)!}{3!(n-1)!}x^{n-1} \le (1+x)^{n+2}$

$\frac{(n+2)(n+1)n(n-1)!}{6(n-1)!}x^{n-1} \le (1+x)^{n+2}$

$\frac{n(n+2)(n+2)}{6}x^{n-1} \le (1+x)^{n+2}$

$\frac{6x}{n(n+1)(n+2)x^n} \ge \frac{1}{(1+x)^{n+2}}$

$\frac{6x}{n(n+1)(n+2)} \ge \frac{x^n}{(1+x)^n(1+x)^2}$

$\frac{6x}{n(n+1)(n+2)} \ge \left(\frac{x}{1+x}\right)^n \frac{1}{(1+x)^2}$

$\frac{6x}{n+2} \ge \left(\frac{x}{1+x}\right)^n \frac{n(n+1)}{(1+x)^2}$

$\frac{3x}{n+2} \ge \left(\frac{x}{1+x}\right)^n \frac{n(n+1)}{2(1+x)^2}$

$\frac{3x(1+x)^2}{n+2} \ge \left(\frac{x}{1+x}\right)^n \frac{n(n+1)}{2}$, we get

$$C_n\ (x^2;x) \le x^2 + \frac{3x(1+x)^2}{n+2} + \frac{8x(1+x)^2}{n+2}$$

$$\le x^2 + \frac{11x(1+x)^2}{n+2}$$

Here under we listed some theorems that we need to prove our reslts.

Theorem: 1 (Korevkin's Theorem) [6] Let L be a linear positive operator in the space $C[a,b]$i. e, $L: C[a,b] \to C[a,b]$ and

1) $L(1;x) = 1$
2) $L(t;x) = x + \alpha(x)$
3) $L(t^2;x) = x^2 + \beta(x)$, then for every function $f \in C[a,b]$,
   we have $\|L(f;.) - f\|_{C[a,b]} \le 3\omega(f;\sqrt{d}),\ d = \|\beta(x) - 2x\alpha(x)\|_{C[a,b]}$

Theorem: 2 Let L_n be a linear positive operator, $f \in L_{P,\alpha}[0,\alpha], a > 0$, $L_n: L_{P,\alpha}[0,\alpha] \to L_{P,\alpha}[0,\alpha]$and

1) $L_n(1,x) = 1$
2) $L_n(t,x) = x + \alpha(x)$
3) $L_n(t^2,x) = x^2 + \beta(x)$, then $\|L_n(f;.) - f(.)\|_{P,\alpha} \le 3\omega(f;\sqrt{d})_\alpha$, $d = \|\beta(x) - 2x\alpha(x)\|_{P,\alpha}$

Proof: Let $w(x) = e^{-\alpha x}$, since f is an unbounded function, thus $f(x).w(x)$ is a bounded one, using theorem (1), we get

$$\|L_n(f;.) - f(.)\|_{P,\alpha} \le 3\omega\left(f;\sqrt{\|\beta(x) - 2x\alpha(x)\|_{P,\alpha}}\right)_\alpha$$

Lemma: 8 Let $f \in L_{P,\alpha}$ be an unbounded function, $x \in [0,a], a > 0, n \ge 2$, then $C_n(f;x)$ is a linear positive operator.

Proof: Since $x \in [0,a],\ a > 0 \to 0 \le x \le a \to x \ge 0$, then $C_n(f;x)$ is a positive operator, also, since $C_n: L_{P,\alpha}[0,a] \to L_{P,\alpha}[0,a]$ take $f \le g$, then

$$\frac{1}{(1+x)^n}f.w(0) + \sum_{k=1}^{n-1}\binom{n}{k}\frac{x^k}{(1+x)^n}\frac{1}{\beta(k,n-k+2)}\int_0^\infty f(t)\frac{t^{k-1}}{(1+t)^{n+2}}dt + \left(\frac{x}{1+x}\right)^n f(n)$$
$$\le \frac{1}{(1+x)^n}g.w(0) + \sum_{k=1}^{n-1}\binom{n}{k}\frac{x^k}{(1+x)^n}\frac{1}{\beta(k,n-k+2)}\int_0^\infty g.w(t)\frac{t^{k-1}}{(1+t)^{n+2}}dt + \left(\frac{x}{1+x}\right)^n g(n),$$

we get $C_n(f;x) \le C_n(g;x)$,

Thus
$C_n(f;.)$is monotone

Now let

$$C_n((\alpha_1 f+\beta_1 g),x) = \frac{1}{(1+x)^n}(\alpha_1 f+\beta_1 g).w(0)$$
$$\sum_{k=1}^{n-1}\binom{n}{k}\frac{x^k}{(1+x)^n}\frac{1}{\beta(k,n-k+2)}\int_0^\infty(\alpha_1 f+\beta_1 g).w(t)\frac{t^{k-1}}{(1+t)^{n+2}}dt+\left(\frac{x}{1+x}\right)^n(\alpha_1 f+\beta_1 g)(n)$$

$$= \frac{1}{(1+x)^n}(\alpha_1 fw)(0)+\frac{1}{(1+x)^n}(\beta_1 gw)(0)+\sum_{k=1}^{n-1}\binom{n}{k}\frac{x^k}{(1+x)^n}\frac{1}{\beta(k,n-k+2)}\int_0^\infty \alpha_1 fw(t)\frac{t^{k-1}}{(1+t)^{n+2}}dt$$
$$+\sum_{k=1}^{n-1}\binom{n}{k}\frac{x^k}{(1+x)^n}\frac{1}{\beta(k,n-k+2)}\int_0^\infty \beta_1 gw(t)\frac{t^{k-1}}{(1+t)^{n+2}}dt$$
$$+\left(\frac{x}{1+x}\right)^n\alpha_1 f.w(n)+\left(\frac{x}{1+x}\right)^n\beta_1 g.w(n)$$

$= \alpha_1 C_n(f;x)+\beta_1 C_n(g;x)$, $C_n$is a linear operator

MAIN RESULTS

Theorem: 3 Let $f\in L_{P,\alpha}, x\in[0,a], a>0, \delta>0, n\ge 2$, then

$$\|C_n\ (f;.)-f(.)\|_{P,\alpha}\le\left(1,\frac{1}{\delta}\sqrt{\frac{13a(1+a^2)}{n+2}}\right)\omega(f;\delta)_\alpha$$

Proof: Since $C_n(f;.)$ is a linear positive operator, and lemma (7) holds, using theorem (2), we get

$$\alpha(x) = -x\left(\frac{x}{1+x}\right)^n$$

$$\beta(x)=\frac{11x(1+x)^2}{n+2}$$

$$\sqrt{\beta(x)-2x\alpha(x)}=\sqrt{\frac{11x(1+x)^2}{n+2}+2x^2\left(\frac{x}{1+x}\right)^n}$$

Since $x\in[0,a]$, $a>0\to 0\le x\le a$, then

$$\sqrt{\beta(x)-2x\alpha(x)}\le\sqrt{\beta(a)-2a\alpha(a)}$$

$$\le\sqrt{\frac{11a(1+a)^2}{n+2}+2a^2\left(\frac{a}{1+a}\right)^n}$$

$$\le\sqrt{\frac{13a(1+a)^2}{n+2}}$$

Then

$$\|C_n\ (f;.)-f(.)\|_{P,\alpha}\le 3\omega\left(f;\ \sqrt{\frac{13a(1+a)^2}{n+2}}\right)_\alpha$$

$$\le 3\omega\left(f;\ \frac{\delta}{\delta}\sqrt{\frac{13a(1+a)^2}{n+2}}\right)_\alpha$$

From lemma (6), we have

$\omega(f;\lambda\delta)_\alpha\le(\lambda+1)\omega(f;\delta)_\alpha$, we get

$$\|C_n\ (f;.)-f(.)\|_{P,\alpha}\le 3\left(1+\ \frac{1}{\delta}\sqrt{\frac{13a(1+a)^2}{n+2}}\right)\omega(f;\delta)_\alpha$$

Corollary: Let $f \in L_{P,\alpha}, x \in [0,a], a > 0$, then $C_n(f;.) \xrightarrow{u.c} f$ Uniform converence as $n \to \infty$

Proof: Since $\alpha(x) = -x\left(\frac{x}{1+x}\right)^n$

$$\lim_{n\to\infty}\|\alpha_n\ (.)\|_{P,\alpha} = \lim_{n\to\infty}\left\|x\left(\frac{x}{1+x}\right)^n\right\|_{P,\alpha}$$

$$= \lim_{n\to\infty}\left(\int_0^a\left|x\left(\frac{x}{1+x}\right)^n e^{-nx}\right|^P dt\right)^{\frac{1}{P}}$$

$$\leq \lim_{n\to\infty} \text{Sup}_x\left|xe^{-nx}\left(\frac{x}{1+x}\right)^n\right|$$

$$\leq \lim_{n\to\infty} \text{Sup}_x\left|\frac{x}{e^{nx}}\left(\frac{x}{1+x}\right)^n\right|$$

Then $\|\alpha_n\ (.)\|_{P,\alpha} \to 0 \quad as \quad n \to \infty$

Also

$$\lim_{n\to\infty}\|\beta_n\ (.)\|_{P,\alpha} = \lim_{n\to\infty}\left\|\frac{11x(1+x)^2}{n+2}\right\|_{P,\alpha}$$

$$= \lim_{n\to\infty}\left(\int_0^a\left|\frac{11x(1+x)^2}{n+2}e^{-nx}\right|^P dt\right)^{\frac{1}{P}}$$

$$\leq \lim_{n\to\infty} \text{Sup}_x\left|\frac{11x(1+x)^2}{n+2}e^{-nx}\right|$$

$$\leq \lim_{n\to\infty} \text{Sup}_x\left|\frac{11x}{e^{nx}}.\frac{(1+x)^2}{n+2}\right|$$

$$\leq \text{Sup}_x\left|\lim_{n\to\infty}\frac{11x}{e^{nx}}.\lim_{n\to\infty}\frac{(1+x)^2}{n+2}\right| = 0$$

Then $\|\beta_n\ (.)\|_{P,\alpha} \to 0 \quad as \quad n \to \infty$

$\sqrt{\|\beta_n\ (x) - 2x\alpha(x)\|_{P,\alpha}} \to 0$ as $n \to \infty$

$$\lim_{n\to\infty}\|C_n\ (f;.) - f(.)\|_{P,\alpha} \leq 3\omega\left(f, \sqrt{\|\beta_n\ (x) - 2x\alpha(x)\|_{P,\alpha}}\right)_\alpha$$

$$= 3\omega(f,0)$$

$$= 0$$

Thus $\|C_n\ (f;.) - f(.)\|_{P,\alpha} \to 0$ as $n \to \infty$

$C_n\ (f) \xrightarrow{u.c} f$

Theorem: 4 Let $f \in L_{P,\alpha}, [0,a], a > 0,\ (fe^{-\alpha})' \in L_{P,\alpha}$ is differentiable, then

$$\|C_n\ (f;.) - f(.)\|_{P,\alpha} \leq 3C(\delta + A)\omega(f^{'};\delta)_\alpha$$

$A = \sqrt{\frac{13a(1+a)^2}{n+2}}$, C is a constant

Proof:

$$\|C_n\ (f;.) - f(.)\|_{P,\alpha} \leq 3\omega\left(f; \sqrt{\beta(x) - 2x\alpha(x)}\right)_\alpha$$

$$\leq 3\omega_1\left(f; \sqrt{\beta(x) - 2x\alpha(x)}\right)_\alpha$$

$$\leq 3\omega_1\left(f; \sqrt{\frac{13a(1+a)^2}{n+2}}\right)_\alpha$$

$$\leq 3\omega\left(f; \frac{\delta}{\delta}\sqrt{\frac{13a(1+a)^2}{n+2}}\right)_\alpha$$

Using lemma (6), we have

$\omega(f;\lambda\delta)_\alpha \leq (\lambda+1)\omega(f;\delta)_\alpha$, we get

$$\|C_n\ (f;.) - f\|_{P,\alpha} \leq 3\left(1 + \frac{1}{\delta}\sqrt{\frac{13a(1+a)^2}{n+2}}\right)\omega(f;\delta)_\alpha$$

$$\|C_n\ (f;.) - f(.)\|_{P,\alpha} \leq 3C\left(1 + \frac{1}{\delta}\sqrt{\frac{13a(1+a)^2}{n+2}}\right)\omega_2(f;\delta)_\alpha$$

By using lemma (4), we have:

$\omega_2(f;\delta)_\alpha \leq \delta\omega_1(f';\delta)_\alpha$, we get

$$\|C_n\ (f;.) - f(.)\|_{P,\alpha} \leq 3C\delta\left(1 + \frac{1}{\delta}\sqrt{\frac{13a(1+a)^2}{n+2}}\right)\omega(f';\delta)_\alpha$$

$$\leq 3C(\delta + A)\omega(f';\delta)_\alpha$$

$$A = \sqrt{\frac{13a(1+a)^2}{n+2}}$$

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