International Journal of Mathematical Archive-5(4), 2014, 207-210
Available online through www.ijma.info ISSN 2229 – 5046

VALUE DISTRIBUTION OF SMALL FUNCTIONS IN THE UNIT DISK

Renukadevi S. Dyavanal*

Department of Mathematics, Karnatak University, Dharwad - 580 003, India.

(Received on: 10-04-14; Revised & Accepted on: 25-04-14)

ABSTRACT

In this paper, we explore properties of value distribution of differential polynomials of certain class of functions in the disk.

Keywords: *Nevanlinna theory, Unit disk, Small functions, etc.*

Subject Classification: *30D35.*

1. INTRODUCTION AND MAIN RESULTS

If f is a meromorphic function in the complex plane. R. Nevanlinna noted that its characteristic function $T(r,f)$ could be used to categorize f according to its rate of growth as $|z| = r \to \infty$. Later H. Milloux showed for a transcendental meromorphic function in the plane that for each positive integer k,

$$m\left(r, \frac{f^{(k)}}{f}\right) = o\big(T(r,f)\big) \text{ as } r \to \infty \tag{1.1}$$

possibly outside a set of finite measure. If f is a meromorphic function in the unit disk $D = \{z: |z|<1\}$, analogous results to (1.1) exit when

$$\limsup_{n \to \infty} \frac{T(r,f)}{-\log\ (1-r)} = +\infty \tag{1.2}$$

Definition: 1.1 (Class *F*) Class F in defined as

$$F = \{f \in D : f \textit{ is meromorphic and } \limsup_{r \to 1} \frac{T(r,f)}{-\log\ (1-r)} = \alpha < +\infty\}$$

Definition: 1.2 (Index of *f*) For functions f in class F, we say that the index of f denoted by $\alpha(f)$ and given by $\limsup_{r \to 1} \frac{T(r,f)}{-\log\ (1-r)} = \alpha < +\infty$

Definition: 1.3 (Subclass *P* of *F*) Subclass $P \ of \ F$ is defined as

$$P = \left\{ f \in F : m\left(r\,,\, \frac{f'}{f}\right) = o\big(T(r,f)\big) \ \ as \ \ r \to 1 \ \ and \ \ \lim_{r \to 1} T(r,f) = \infty \right\}$$

Definition: 1.4(Closure properties of *F*) If $f \in P$ and $c \neq 0$, then (i) $cf \in P$ (ii) $1/_f \in P$ (iii) $f^n \in P$ (iv) g be a meromorphic functions not identically 0 such that $T(r,g) = o\big(T(r,f)\big) as \ r \to 1 \ and \ m\left(r, \frac{g'}{g}\right) = o\big(T(r,f)\big) \ as \ r \to 1$, then $f\ g \in P$.

Remark: 1.5 The following theorem will show that there is a difference between the disk case and the plane case. In the plane case for transcendental functions, we are guaranteed not only that

$$m\left(r, \frac{f'}{f}\right) = o\big(T(r,f)\big) \textit{ but also that } m\left(r, \frac{f^{(k)}}{f}\right) = o\big(T(r,f)\big)$$

by the theory of Milloux. However, we are not guaranteed this for functions of slow growth in the disk as the following theorem shows.

Corresponding author: Renukadevi S. Dyavanal*
Department of Mathematics, Karnatak University, Dharwad - 580 003, India.
E-mail: renukadyavanal@gmail.com

Theorem: 1.6 There exists an analytic function $f \in P$ such that

$$m\left(r, \frac{f''}{f}\right) \neq o\big(T(r,f)\big) \ as \ r \to 1$$

Theorem: 1.7(First Fundamental Theorem of Nevanlinna) Let f be a meromorphic function in D. Then, for any $a \in C$

$$T(r,f) = m\left(r, \frac{1}{f-a}\right) + N\left(r, \frac{1}{f-a}\right) + O(1) \quad as \quad r \to 1$$

Theorem: 1.8 (Reformulation of the Second Fundamental Theorem): Let f be a nonconstant meromorphic function in F. Then $q \geq 3$ distinct values $a_{1,}\ a_2, \dots, a_q \ \in C \cup \{\infty\}$, we have

$$(q-2)\,T(r,f) \leq \sum_{i-1}^{q} \bar{N}(r,a_i) + log\frac{1}{1-r} + \ O\left(loglog\frac{1}{1-r}\right) \quad as \quad r \to 1$$

In the year 1986, Shea and Sons [4] explores varies results for admissible functions in class F be refined further by restricting the functions to class P. We now state a theorem from Shea and Sons that can be refined

Theorem: 1.9 Let f be a meromorphic function in D which is in class F and for which

$$N\left(r, \frac{1}{f}\right) + N(r,f) = o\big(T(r,f)\big) \ as \ r \to 1$$

Let n be a positive integer and for $k = 0,1,2 \dots n$, let a_k be a meromorphic function in D for which

$T(r, a_k) = o\big(T(r,f)\big) \ as \ r \to 1$. If ψ is defined in D by

$$\psi = \sum_{k=0}^{n} a_k f^{(k)}$$

and ψ is nonconstant, then ψ assumes every complex number except possibly zero infinitely often provided the index of f is $\alpha > 1 + \frac{n(n+1)}{2}$.

Theorem: 1.10 Let f be a meromorphic function in D which is in class P and for which

$$N\left(r, \frac{1}{f}\right) + N(r,f) = o\big(T(r,f)\big) \ as \ r \to 1$$

Let n be a positive integer and for $k = 0,1,2, \dots, n$, let a_k be a meromorphic function in D for which $T(r, a_k) = o\big(T(r,f)\big) \ as \ r \to 1$. Also, define E to be the set defined by

$E = \left\{k : \ m\left(r, \frac{f^{(k)}}{f}\right) = o\big(T(r,f)\big) \ as \ r \to 1\right\}$ and $\psi = \sum_{k=0}^{n} f^{(k)} \ in \ D.$

And ψ is nonconstant, then ψ assumes every complex number except possibly zero infinitely often provided the index of f is $\alpha > 1 + \frac{n(n+1)}{2}$ - $\sum E$, where $\sum E$ is the sum of the values of E.

In this paper, we extend Theorem 1.9 and Theorem 1.10 to general homogeneous differential polynomials and prove the following theorem.

Theorem: 1.11 Let f be a meromorphic function in D which is in class P and for which

$$N\left(r, \frac{1}{f}\right) + N(r,f) = o\big(T(r,f)\big) \ as \ r \to 1 \tag{1.3}$$

If non constant function ψ is defined in D as

$$\psi = \sum_{j=0}^{m} a_j f^{n_{0,j}} (f')^{n_{1,j}} (f'')^{n_{2,j}} \dots \big(f^{(s)}\big)^{n_{s,j}} \tag{1.4}$$

Where $n = \sum_{i=0}^{s} n_{i,j}$ ($\forall \ j = 0,1,2, \dots, m$) and let a_k be a meromorphic function in D for which

$$T(r, a_k) = o\big(T(r,f)\big) \ as \ r \to 1 \tag{1.5}$$

and $E = \{k : \ m\left(r, \frac{f^{(k)}}{f}\right) = o\big(T(r,f)\big)\} \ as \ r \to 1\}$ for $k = 0,1,2, \dots, s.$

Then ψ assumes every complex number except possibly zero infinitely often provided the index of f is given by $n\alpha > 1 + \sum_{j=0}^{m} \sum_{k \neq E} \ k \ n_{k,j}$.

2. PROOF OF THEOREM: 1.1

Since Class F is closed under differentiation, addition and multiplication and ψ is in class F. Therefore we can apply the reformulation of the second fundamental theorem for class F to ψ

$$T(r,\psi) \leq \bar{N}\left(r,\frac{1}{\psi}\right)+\bar{N}\left(r,\frac{1}{\psi-c}\right)+\bar{N}(r,\psi)+\log\left(\frac{1}{1-r}\right)+O\left(\log\log\left(\frac{1}{1-r}\right)\right) \quad as \quad r\to 1. \tag{2.1}$$

From (1.4) and (1.5), we have

$$\bar{N}(r,\psi) \leq \bar{N}(r,f)+\sum_{j=0}^{m}\bar{N}(r,a_j) \leq \bar{N}(r,f)+o(T(r,f)) \quad as\, r\to 1. \tag{2.2}$$

Also, since the index of f is greater than 0, we have $O\left(\log\log\left(\frac{1}{1-r}\right)\right)=o(T(r,f))\ as\ r\to 1$. Therefore, using (2.1) and (2.2) and the first fundamental theorem, we get $as\ r\to 1$

$$\begin{aligned} T(r,\psi) &= m\left(r,\frac{1}{\psi}\right)+N\left(r,\frac{1}{\psi}\right)+O(1) \\ &\leq \bar{N}\left(r,\frac{1}{\psi}\right)+\bar{N}\left(r,\frac{1}{\psi-c}\right)+\bar{N}(r,\psi)+\log\left(\frac{1}{1-r}\right)+O\left(\log\log\left(\frac{1}{1-r}\right)\right) \\ &\leq \bar{N}\left(r,\frac{1}{\psi}\right)+\bar{N}\left(r,\frac{1}{\psi-c}\right)+\bar{N}(r,f)+\log\left(\frac{1}{1-r}\right)+o(T(r,f)) \end{aligned}$$

Now, solving for $m\left(r,\frac{1}{\psi}\right)$ in the above calculation, we have the following inequality

$$\begin{aligned} m\left(r,\frac{1}{\psi}\right) &\leq \bar{N}\left(r,\frac{1}{\psi}\right)-N\left(r,\frac{1}{\psi}\right)+\bar{N}\left(r,\frac{1}{\psi-c}\right)+\bar{N}(r,f)+\log\left(\frac{1}{1-r}\right)+o(T(r,f))\ as\, r\to 1. \\ &\leq \bar{N}\left(r,\frac{1}{\psi-c}\right)+\bar{N}(r,f)+\log\left(\frac{1}{1-r}\right)+o(T(r,f))as\, r\to 1. \end{aligned} \tag{2.3}$$

By the first fundamental theorem, properties of the proximity function and (2.3) give us the following

$$\begin{aligned} nT(r,f) &= m\left(r,\frac{1}{f^n}\right)+N\left(r,\frac{1}{f^n}\right)+O(1) \leq m\left(r,\frac{1}{\psi}\frac{\psi}{f^n}\right)+N\left(r,\frac{1}{f^n}\right)+O(1) \\ &\leq m\left(r,\frac{1}{\psi}\right)+m\left(r,\frac{\psi}{f^n}\right)+N\left(r,\frac{1}{f^n}\right)+O(1) \\ &\leq \bar{N}\left(r,\frac{1}{\psi-c}\right)+\bar{N}(r,f)+\log\left(\frac{1}{1-r}\right)+m\left(r,\frac{\psi}{f^n}\right)+N\left(r,\frac{1}{f^n}\right)+o(T(r,f))as\, r\to 1. \end{aligned} \tag{2.4}$$

Noticing the fact that $\bar{N}(r,f) \leq N(r,f)$ and using (1.3)

$$N(r,f)+N\left(r,\frac{1}{f^n}\right)=N(r,f)+nN\left(r,\frac{1}{f}\right)=o(T(r,f)) \quad as \quad r\to 1 \tag{2.5}$$

From (2.4) and (2.5), we have

$$n\,T(r,f) \leq \bar{N}\left(r,\frac{1}{\psi-c}\right)+m\left(r,\frac{\psi}{f^n}\right)+\log\left(\frac{1}{1-r}\right)+o(T(r,f)) \quad as \quad r\to 1 \tag{2.6}$$

As $r\to 1$. We now estimate $m\left(r,\frac{\psi}{f^n}\right)$ as follows

$$\begin{aligned} m\left(r,\frac{\psi}{f^n}\right) &= m\left[r,\frac{\sum_{j=0}^{m} a_j f^{n_{0,j}}(f')^{n_{1,j}}(f'')^{n_{2,j}}\ldots(f^{(n)})^{n_{n,j}}}{f^n}\right] \\ &= \sum_{j=0}^{m} m\left(r,\frac{a_j f^{n_{0,j}}(f')^{n_{1,j}}(f'')^{n_{2,j}}\ldots(f^{(n)})^{n_{n,j}}}{f^n}\right)+\log(m+1) \\ &= \sum_{j=0}^{m} m\left(r,\ a_j\left(\frac{f^{n_{0,j}}}{f}\right)\left(\frac{f'}{f}\right)^{n_{1,j}}\left(\frac{f''}{f}\right)^{n_{2,j}}\ldots\ldots\left(\frac{f^{(n)}}{f}\right)^{n_{n,j}}\right)+\log(m+1) \\ &= \sum_{j=0}^{m}\left\{m(r,a_j)+\sum_{k=1}^{n} m\left(r,\left(\frac{f^{(k)}}{f}\right)^{n_{k,j}}\right)\right\}+\log(m+1) \\ &= \sum_{j=0}^{m}\sum_{k=1}^{n} m\left(r,\left(\frac{f^{(k)}}{f}\right)^{n_{k,j}}\right)+o(T(r,f))+\log(m+1) \\ &= \sum_{j=0}^{m}\left\{\sum_{k\in E} m\left(r,\left(\frac{f^{(k)}}{f}\right)^{n_{k,j}}\right)+\sum_{k\notin E} m\left(r,\left(\frac{f^{(k)}}{f}\right)^{n_{k,j}}\right)\right\}+o(T(r,f)). \\ &= \sum_{j=0}^{m}\sum_{k\notin E} m\left(r,\left(\frac{f^{(k)}}{f}\right)^{n_{k,j}}\right)+o(T(r,f)) \\ &= \sum_{j=0}^{m}\sum_{k\notin E} n_{k,j}\ m\left(r,\frac{f^{(k)}}{f}\right)+o(T(r,f)) \end{aligned} \tag{2.7}$$

By using (4.5.1) in Chapter 4 of [1], we have

$$m\left(r,\frac{f^{(k)}}{f}\right) \leq k\log\left(\frac{1}{1-r}\right) + k\big(2+o(1)\big)\log\log\left(\frac{1}{1-r}\right) \quad as \quad r \to 1 \tag{2.8}$$

Hence by (2.7) and (2.8), we get

$$m\left(r,\frac{\psi}{f^n}\right) = \sum_{j=0}^{m}\sum_{k\notin E} k\,n_{k,j} \;\log\left(\frac{1}{1-r}\right) + \; o\big(T(r,f)\big) \quad as \quad r \to 1 \tag{2.9}$$

Then, by (2.6) and (2.9), we can write

$$n\,T(r,f) \;\leq\; \bar{N}\left(r,\frac{1}{\psi-\mathrm{c}}\right) + \left[\sum_{j=0}^{m}\sum_{k\notin E} k\,n_{k,\,j} \;+1\right]\log\left(\frac{1}{1-r}\right) + \; o\big(T(r,f)\big) \;\; as\, r \to 1$$

Since the index α of f is given by $n\alpha > \sum_{j=0}^{m}\sum_{k\notin E} \; k\,n_{k,j} \;+1,$ we have

$$vT(r,f) \;\leq\; \bar{N}\left(r,\frac{1}{\psi-\mathrm{c}}\right) + o\big(T(r,f)\big) \quad as\, r \to 1$$

Where $v > 0$. Since $T(r,f)$ is unbounded, we have proved the claim that ψ assumes every complex number except possibly zero infinitely often.

REFERENCES

[1] P. A. Gunsul, "A Class of Small Functions in the Unit Disk", PhD Thesis, Northern Illinois University, 2009.
[2] W. K. Hayman, "Meromorphic Functions", Oxford University Press, London, 1964.
[3] R. Korhonen and J. Ratty, "Finite order solutions of linear differential equations in the unit disk", J. Math. Anal. Appl. 349(2009), 43-54.
[4] D. Shea and L. R. Sons, "Value distribution theory for meromorphic functions of slow growth in the unit disk", Houston. J. Math. 12(2) (1886), 249-226.
[5] L. R. Sons, "Values for differential polynomials in the disk", Abstracts of the AMS. Vol. 28 (2007), P-640.
[6] L. Yang, "Value Distribution Theory", Springer-Verlag, Beijing, 1993.

Source of support: Nil, Conflict of interest: None Declared