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BARGMANN ANALYTIC REPRESENTATION FOR TWO-MODE SYSTEMS

M. Tabuni*
Department of Mathematics, University of Tripoli, Libya.

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ABSTRACT

The paper contains an investigation of zeros Of Bargmann analytic representation. A brief introduction to Harmonic oscillator formalism is given. The Bargmann analytic representation has been studied. The zeros of Bargmann analytic function are considered. The Q or Husimi functions are introduced. The The Bargmann functions and the Husimi functions have the same zeros. The Bargmann functions $f(z)$ have exactly q zeros. The evolution time of the zeros μ_n are discussed. Various examples have been given.

1. INTRODUCTION

This Paper is devoted to study the zeros of Bargmann analytic representation in the complex plane. The Bargmann function is very important kind of analytic functions [1, 2, 3] in the complex plane [4, 5, 6]. The zeros of Bargmann functions and the zeros of the Q or Husimi function which are identical, have been used to consider of various models [7, 8, 9, 10, 11, 12, 13]. The analytic Bargmann functions $f(z)$ have exactly q zeros which subjected to the constraint.(33). The growth of an entire function $f(z)$ is described by the order ρ and type σ [14, 15, 16, 17]. The entire function $f(z)$ is polynomial of order q and has q zeros. The q zeros of the analytic functions $f(z)$ depends on the distribution of the coefficients $f_0, f_1, ..., f_n$. If the coefficients $f_0, f_1, ..., f_n$ are real then the zeros μ_n are real or appear as complex conjugate pairs and draw symmetric graph with respect to the z_r axis.

2. HARMONIC OSCILLATOR IN ONE-MODE SYSTEMS

Let H_q be the Hilbert space with number eigenstates $|n\rangle$. We consider a harmonic oscillator corresponding the Hamiltonian:

$$H = \frac{1}{2}(x^2 + p^2); \tag{1}$$

where $x;\ p;$ the position and momentum operators with $[x, p] = i1$.

Let $a, a^\dagger$ be the creation and annihilation operators:

$$a = \frac{x+ip}{\sqrt{2}}; \quad a^\dagger = \frac{x-ip}{\sqrt{2}}; \tag{2}$$

where

$$aa^\dagger |n\rangle = n|n\rangle. \tag{3}$$

These two operators obey the canonical commutation relation

$$[a, a^\dagger] = 1; \tag{4}$$

and act on the number state as follows:

$$a^\dagger |n\rangle = (n+1)^{1/2} |n+1\rangle;$$
$$a|n\rangle = n^{1/2} |n-1\rangle; \tag{5}$$

*Corresponding author: M. Tabuni**
Department of Mathematics, University of Tripoli, Libya.

The displacement operators are defined as

$$D(z) = exp(za^{\dagger} - z^{*}a); \quad z = (x + ip)/\sqrt{2}. \tag{6}$$

We consider the coherent states

$$|z\rangle = exp\left(-\frac{1}{2}|z|^{2}\right)\sum_{n=0}^{\infty}(n!)^{-1/2} z^{n} |n\rangle. \tag{7}$$

The coherent states are defined as the eigenstate of the annihilation operator a

$$a|z\rangle = z|z\rangle; \tag{8}$$

and the position representation of the coherent state is a Gaussian function

$$f_{z}(x) = \pi^{-1/4} exp(-\frac{x^{2}}{2} + \sqrt{2}zx - zz_{R}); \quad z = z_{R} + iz_{I}. \tag{9}$$

The inner product of two coherent states $|z_1\rangle$ and $|z_2\rangle$ is

$$\langle z_1 | z_2\rangle = exp\left(-\frac{1}{2}|z_1|^{2} - \frac{1}{2}|z_2|^{2} + z_1 z_2^{*}\right). \tag{10}$$

3. HARMONIC OSCILLATOR IN TWO-MODE SYSTEMS

Let $H_q = H_1 \otimes H_2$ be the two-mode Hilbert space. We consider the two-mode orthonormal basis

$$|n,m\rangle = \left[\frac{H_n(x)e^{\frac{-1}{2}x^2}}{\sqrt{\pi}\,2^{n} n!}\right]\left[\frac{H_m(x)e^{\frac{-1}{2}x^2}}{\sqrt{\pi}\,2^{m} m!}\right] \tag{11}$$

where H_n is the Hermite polynomials.

Let $a_2, a_2^{\dagger}$ be the creation and annihilation operators in two-mode systems:

$$a_2 = 1 \otimes a, a_2^{\dagger} = 1 \otimes a^{\dagger}, [a_2, a_2^{\dagger}] = 1. \tag{12}$$

The displacement operators are defined as

$$D_2(z) = exp(za_2^{\dagger} - z^{*}a_2), \quad z = (x + ip)/\sqrt{2}. \tag{13}$$

We consider the coherent states

$$|z_1, z_2\rangle = exp\left(-\frac{1}{2}(|z_1|^{2} + |z_2|^{2})\right)\sum_{n=0}^{\infty}\frac{z_1^{n} z_2^{m}}{\sqrt{n!m!}}|n,m\rangle. \tag{14}$$

The state $|f\rangle$ can be analyzed in above basis 11 as follows

$$|f\rangle = \sum_{n,m} f_{nm}|n,m\rangle, \quad \sum_{n,m}|f_{nm}|^{2} = 1$$
$$|f\rangle^{*} = \sum_{n,m} f_{nm}^{*}|n,m\rangle$$
$$\langle f|^{*} = \sum_{n,m} f_{nm}\langle n,m|. \tag{15}$$

4. BARGMANN ANALYTIC REPRESENTATION OF ONE VARIABLE

We cosider an arbitrary $|f\rangle$ state

$$|f\rangle = \sum_{n=0}^{\infty} f_n |n\rangle; \quad \sum_{n=0}^{\infty}|f_n|^{2} = 1. \tag{16}$$

In The Bargmann representation [2, 4, 5, 6] , the state $|f\rangle$ is represented by

$$f(z) = exp(\frac{|z|^2}{2})\langle z^* \mid f\rangle = \sum_{n=0}^{\infty} \frac{f_n z^n}{\sqrt{n!}}; \tag{17}$$

which is an entire function (i.e. analytic function in the complex plane $\mathbb{C}$) defined on a torus, satisfying the quasi-periodic condition [7]

$$f\left[z+1/\sqrt{2}\right] = \exp(q\pi(\frac{1}{2}+z\sqrt{2}))f(z);$$
$$f\left[z+i/\sqrt{2}\right] = \exp(q\pi(\frac{1}{2}-iz\sqrt{2}))f(z). \tag{18}$$

The inner product of the two states[2] is given by

$$\langle f \mid g\rangle = \frac{1}{\pi}\int_{\mathbb{C}}[f(z)]^* g(z)exp(-|z|^2)\frac{d^2z}{\pi}$$
$$= \sum_n f_n^* g_n \quad , d^2z = dz_R dz_I. \tag{19}$$

Therefore we can represent the creation and annihilation operators by the two variable analytic functions in the Bargmann analytic [18] representation (see [1])as following

$$a \to \mu^* \exp(z\mu^*); \quad a^\dagger \to z\exp(z\mu^*) \tag{20}$$

The Bargmann analytic representation of the creation and annihilation operator is

$$a \to \partial_z; \quad a^\dagger \to z. \tag{21}$$

5. BARGMANN ANALYTIC REPRESENTATION OF TWO VARIABLE

The Bargmann analytic representation of the state $|f\rangle$ is given by

$$f(z_1,z_2) = \exp(\frac{|z_1|^2+|z_2|^2}{2})\langle f^* \mid z_1,z_2\rangle$$
$$= \sum_{n,m} f_{nm}\frac{z_1^n z_2^m}{\sqrt{n!m!}}; \tag{22}$$

which is an entire function.

The inner product of the two states $|f\rangle$ and $|g\rangle$ see [19] is given by

$$\langle f \mid g\rangle = \int [f(z_1,z_2)]^* g(z_1,z_2)\exp(-(|z_1|^2+|z_2|^2))\frac{d^2z_1}{\pi}\frac{d^2z_2}{\pi}. \tag{23}$$

We can represent the creation and annihilation operators [19] as following

$$a_1 \to \partial_{z_1}, \quad a_2 \to z_2$$
$$a_1^\dagger \to \partial_{z_1}, \quad a_2^\dagger \to \partial_{z_2}. \tag{24}$$

5.1 The growth of Bargmann analytic functions of one variable

The growth of an entire function $f(z)$ is described by the order ρ and type σ [14, 15, 16, 17, 18]:

$$\rho = \lim_{R\to\infty} sup\frac{lnlnM(R)}{lnR}, \sigma = \lim_{R\to\infty} sup\frac{lnM(R)}{R^\rho}, \tag{25}$$

where $M(R)$ is the maximum value of $f(z)$ on $|z| = R$. The space $H(\rho,\sigma)$ is a subspace of $H(\rho',\sigma')$ if $\rho < \rho'$ or if $\rho = \rho'; \sigma < \sigma'$.

We can now derive the Bargmann analytic representation of some quantum states as examples.

- The number state $|n\rangle$ is represented as

$$f(z) = \frac{z^n}{\sqrt{n!}}. \tag{26}$$

It is of order 0.

- The coherent state $|\alpha\rangle$ is represented as

$$f(z) = exp(\alpha z - \frac{1}{2}|\alpha|^2). \tag{27}$$

It is of order $\rho = 1$ and type $|\alpha|$.

5.2 The growth of Bargmann analytic functions of two variables

The growth of an entire function $f(z_1, z_2)$ is described by the order ρ and type σ [14, 15, 16, 17, 18].

let $M(R)$ be the maximum value of $f(z_1, z_2)$ on the shere $\sqrt{|z_1|^2 + |z_2|^2} = R$. As $|z_1|^2 + |z_2|^2 \to \infty$ the function grows as following

$$|f(z_1, z_2)| \approx \exp[\sigma(|z_1|^2 + |z_2|^2)^{\rho/2}$$

see [19].

We can now derive the Bargmann analytic representation of some quantum states as examples.

- The number state $|n, m\rangle$ is represented as

$$f(z_1, z_2) = \frac{z_1^n z_2^m}{\sqrt{n!m!}}. \tag{28}$$

It is of order 0.

- The coherent state $|\omega_1, \omega_2\rangle$ is represented as

$$f(z_1, z_2) = \exp(\frac{-(|\omega_1|^2 + |\omega_2|^2)}{2})e^{\omega_1 z_1} e^{\omega_2 z_2}. \tag{29}$$

6. ZEROS OF BARGMANN ANALYTIC FUNCTIONS OF ONE VARIABLE

We denote as μ_n the zeros of $f(z)$, i.e. $f(\mu_n) = 0$. Let ℓ be the boundary of the fundamental domain of analyticity, $S = [0, 1/\sqrt{2}] \times [0, 1/\sqrt{2}]$. We consider the integrals

$$I = \oint_\ell \frac{dz}{2\pi i} \frac{f(z)'}{f(z)}; \quad J = \oint_\ell \frac{dz}{2\pi i} \frac{f(z)'}{f(z)} z. \tag{30}$$

I is equal to the number of zeros of this function (with the multiplicities taken into account), inside the contour ℓ. J is equal to the sum of these zeros. Using the quasi-periodicity of Eq. (18) we prove that the integral I, for a contour along the boundary ℓ, is equal to q. Therefore the analytic functions $f(z)$ have exactly q zeros [7, 8].

$$\oint_\ell \frac{dz}{2\pi i} \frac{f(z)'}{f(z)} = q. \tag{31}$$

Using the quasi-periodicity of Eq. (??) we also prove that [7, 8]

$$\oint_\ell \frac{dz}{2\pi i} \frac{\partial_z f(z)}{f(z)} z = 2^{-3/2} q(1+i); \tag{32}$$

giving the sum of the zeros μ_n of $f(z)$. Therefore the analytic functions $f(z)$ [7, 8] have exactly q zeros subjected to the constraint

$$\sum_{n=1}^{q} \mu_n = 2^{-3/2} q(1+i). \tag{33}$$

The Husimi function and Bargmann function $f(z)$ are related to each other and it easy to see that there zeros are identical (i.e μ is a zero of $f(z)$ providing ζ is a zero of the Husimi function). The Weierstrass-Hadamard factorization allows the reconstruction of entire functions from their zeros [2, 18]. We suppose that q zeros μ_n of $f(z)$ are given, and that they satisfy the constraint of Eq. (33). The Weierstrass-Hadamard reconstructs the Bargmann functions $f(z)$ as following [2]

$$f(z) = z^m \prod_{n=1}^{q} \exp(Q_p(z)E(\mu_n, d); \tag{34}$$

where

$$E(\mu_n, d) = (1 - \frac{z}{\mu}) \exp\left(\frac{z}{\mu} + \frac{z^2}{\mu^2} + ... + \frac{z^d}{\mu^d} \right); \tag{35}$$

m is the multiplicity of the zero, $Q_p(z)$ is polynomial of degree p and d is a positive number. As an example we consider the function

$$f(z) = \sum_{n=0}^{14} \frac{f_n z^n}{\sqrt{n!}}, \tag{36}$$

The coefficients f_n are given in Table. 1.

i	$f_i(0)$	i	$f_i(0)$	i	$f_i(0)$
0	0.1-0.2i	5	0.3-0.2i	10	0.1-0.1
1	0.3+0.3i	6	0.9-0.03i	11	-0.1+0.2
2	0.3+0.2i	7	0.3+0.01i	12	0.2+0.3i
3	0.01-0.3i	8	0.1+0.01i	13	-0.01-0.1i
4	0.1-0.01i	9	0.1-0.2i	14	-0.01+0.1i

Table - 1: The coefficients f_n of function in Eq.(36)

In Fig.1 we show the distribution of zeros of function $f(z)$ of Eq.(39) which is polynomial of order 14 and has 14 zeros.

The q zeros of the analytic functions $f(z)$ depends on the distribution of the coefficients $f_0, f_1, ..., f_n$. This coefficients subjected to the constraint

$$\sum_{n=0}^{q} f_n^2 = 1, \tag{37}$$

which comes from the normalization.

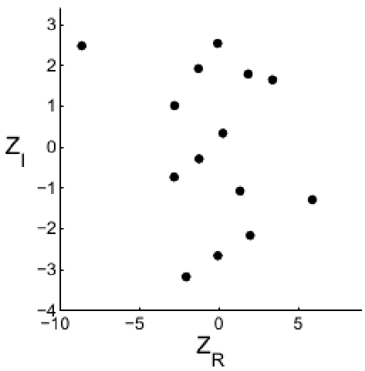

Figure 1: The distributions of zeros of function $f(z)$ of Eq. (39). The $|F(t)\rangle$ at $t = 0$ is described through the coefficients f_n in Eq.1

7. ZEROS OF BARGMANN ANALYTIC FUNCTIONS OF TWO VARIABLE

We denote as (μ_n, ν_n) the zeros of $f(z_1, z_2)$, i.e. $f(\mu_n, \nu_n) = 0$. If (μ_n, ν_n) is a zero of the Bargmann function $f(z_1, z_2)$ then $|\mu_n, \nu_n\rangle$ is orthogonal to $|f^*\rangle$.

The number state

$$f(z_1, z_2) = \frac{z_1^n z_2^m}{\sqrt{n!m!}}. \tag{38}$$

has exactly $n+m$ zeros, and has the zero $(\mu_n, \nu_n) = (0.0)$ (with multiplicity $n+m$).

As an example we consider the function

$$f(z_1, z_2) = \sum_{n,m=0}^{1} \frac{z_1^n z_2^m}{\sqrt{n!m!}}., \tag{39}$$

where the coefficients $f_{n,m}$ are

$$H = \begin{bmatrix} 0.5147 - 0.3337i & 0.2203 - 0.2139i \\ 0.9354 - 0.6228i & -0.0599 - 0.3751i \\ & \end{bmatrix} \tag{40}$$

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