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ON CHARACTERISTIC FUNCTION OF WRAPPED POISSON DISTRIBUTION

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ABSTRACT

In this paper, we derive the characteristic function of Wrapped Poisson Distribution and the population characteristics are studied and the graph of probability mass function is also drawn for various values of parameters.

Keywords: *Characteristic function, Circular models, trigonometric moments, wrapped Models.*

1. INTRODUCTION

In many diverse scientific fields, the observations are ‘directions’. For instance a biologist may be interested in the direction of flight of a bird or the orientation of an animal while a geologist may be measuring the direction of earth’s magnetic pole. Such directions may be in two dimensions or in three dimensions. A set of such observations on directions is referred to as ‘DIRECTIONAL DATA’, in particular, directional data of two dimensions is called ‘CIRCULAR DATA’.

Dattatreya Rao *et al* (2007) derived a good number of wrapped circular models. A new method of generation of circular models by using the Rising Sun function was developed by Girija (2010). Stereographic circular models (Phani (2013)) and Offset circular models (Radhika (2014)) provide a rich and very useful class of models for circular as well as *l*-axial data. Mardia and Jupp (2000) made a mention of discrete circular models by applying a method wrapping on existing linear discrete model and the probability mass function of the Wrapped Poisson models was also placed. It is identified that population characteristics were not derived so far. Hence an attempt is made to derive the population characteristics of the Wrapped Poisson model.

2. CIRCULAR DISTRIBUTIONS

A circular distribution is a probability distribution whose total probability is concentrated on the circumference of a unit circle. Since each point on the circumference represents a direction, it is a way of assigning probabilities to different directions or defining a directional distribution. The range of a circular random variable θ measured in radians, may be taken to be $\left(0,2\pi\right]$ or $\left[-\pi,\pi\right]$.

Circular distributions are of two types: they may be discrete-assigning probability masses only to a countable number of directions, or may be absolutely continuous. In the latter case, the probability density function $f\left(\theta\right)$ exists and has the following basic properties.

1) $f\left(\theta\right)\geq 0$

2) $\int_0^{2\pi} f\left(\theta\right)d\theta = 1$

3) $f\left(\theta\right)= f\left(\theta+2\pi k\right)$, for any integer *k*, That is $f\left(\theta\right)$ is periodic with period 2π

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Wrapped Discrete Circular Random Variables

If *X* is a discrete random variable on the set of integers, then reduction modulo $2\pi m\left(m \in \mathbb{Z}^{+}\right)$ wraps the integers on to the group of m^{th} roots of unity which is a sub group of unit circle.

i.e. $\Phi = 2\pi x\left(\bmod 2\pi m\right)$

More precision Φ is a mapping from a set of integers (*G*) which is a group with respect to '+' to the set of m^{th} roots of unity G' which is a group with respect to '.' is defined as $\Phi(x) = e^{\frac{2\pi ix}{m}}$ where $x \in G$, $e^{\frac{2\pi ix}{m}} \in G'$ then Φ is called wrapped discrete circular random variable.

Clearly Φ is a homomorphism

i.e. (1) $\Phi(x+y) = e^{\frac{2\pi i(x+y)}{m}}$

$$= e^{\frac{2\pi ix}{m}} \cdot e^{\frac{2\pi iy}{m}}$$

$$= \Phi\left(x\right).\Phi\left(y\right)$$

(2) $\Phi(0) = e^{\frac{2\pi i(0)}{m}}$

$= e^0 = 1$ where $0 \in G$, $1 \in G'$

Since Φ contains a finite number of elements they are denoted by $\Phi = \left\{ \frac{2\pi r}{m} \middle/ r = 0,1,2.....m-1 \right\}$ which is lattice on the unit circle.

Probability Mass Function

Suppose if θ is a wrapped discrete circular random variable then probability mass function of θ is denoted by $pr\left(\theta = \frac{2\pi r}{m}\right)$ which is defined as

$$pr\left(\theta = \frac{2\pi r}{m}\right) = \sum_{k=-\infty}^{\infty} p(r+km) \text{ Where } r = 0,1,2,3,...,m-1 \text{ and } m \in \mathbb{Z}^{+}$$

But to exist the probability mass function it has satisfies the following properties.

1. $pr\left(\theta = \frac{2\pi r}{m}\right) \geq 0$
2. $\sum_{r=0}^{m-1} pr\left(\theta = \frac{2\pi r}{m}\right) = 1$
3. $pr\left(\theta\right) = pr\left(\theta + 2\pi k\right)$ for any integer k i.e. pr is a periodic function.

Distribution Function

Suppose if θ is a wrapped discrete circular random variable then distribution function of θ is denoted by $F_w\left(\theta\right)$ and it is defined as

$$F_w(\theta) = \sum_{r=0}^{k}\left[\sum_{k=-\infty}^{\infty} p(r+km)\right]$$

Wrapped Poisson Distribution

Suppose if θ is a wrapped discrete circular random variable then the probability mass function of θ is denoted by $pr\left(\theta = \frac{2\pi r}{m}\right)$ which is defined as

$$pr\left(\theta = \frac{2\pi r}{m}\right) = \sum_{k=-\infty}^{\infty} p(r+km) \quad r = 0,1,2,3,\ldots,m-1 \text{ and } m \in \mathbb{Z}^+$$

Where $p(x)$ is a probability of x

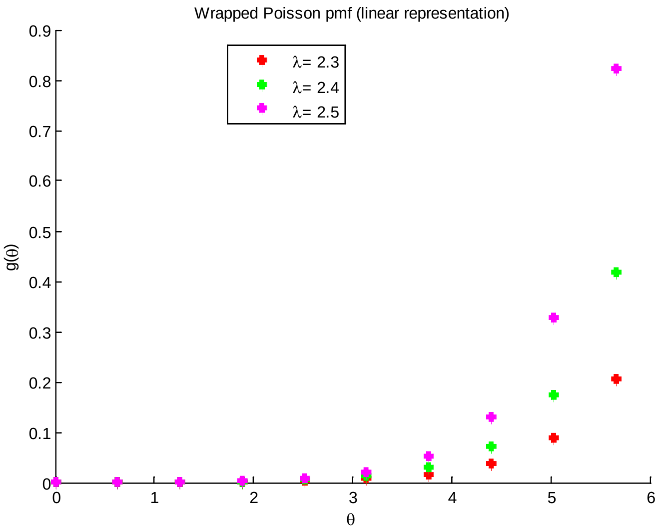
Suppose if X follows Poisson distribution with mean $\lambda > 0$ then the probability mass function of the wrapped Poisson distribution is defined as

$$pr\left(\theta = \frac{2\pi r}{m}\right) = \sum_{k=0}^{\infty} p(r+km)$$

Now the distribution function of the wrapped Poisson distribution is defined as

$$F_w(\theta) = \sum_{r=0}^{k} e^{-\lambda}\left[\frac{\lambda^r}{r!} + \frac{\lambda^{r+m}}{(r+m)!} + \frac{\lambda^{r+2m}}{(r+2m)!} + \ldots\ldots\ldots\right] \text{ where } \lambda > 0\,,\ m \text{ are parameters}$$

Graph of the probability mass function of the Wrapped Poisson distribution



Characteristic Function of Wrapped Poisson Distribution

Since X follows Poisson distribution with mean λ then it is well known that the characteristic function of the Poisson distribution is defined as

$\phi_X(t) = e^{-\lambda\,(1-e^{it})}$ where t is a real number

But at a integer p the characteristic function of a unwrapped distribution $\phi_X(p)$ is equal to characteristic function of a wrapped distribution φ_p . More precisely if $G_\Phi(\theta)$ and $F_X(x)$ are distribution functions of Φ and X . Then

$$\begin{aligned}\phi_p &= \int_0^{2\pi} e^{ip\theta}\, d\left(G_\Phi(\theta)\right) \\ &= \sum_{k=-\infty}^{\infty} \int_{2\pi k}^{2\pi(k+1)} e^{ip\theta}\, d\left(F_X(\theta)\right) \\ &= \int_{-\infty}^{\infty} e^{ipx}\, d\left(F_X(x)\right)\end{aligned}$$

$$= \phi_X\left(p\right)$$

$$\therefore\ \varphi_p = \phi_X\left(p\right)$$

$$\Rightarrow \varphi_p = e^{-\lambda\left(1-e^{\frac{2\pi ip}{m}}\right)}$$

φ_p is also called p^{th} trigonometric moment of θ.

Clearly

$$\varphi_p = E(\mathrm{e}^{ip\theta}) = \rho_p e^{i\mu_p}$$

$$= \alpha_p + i\beta_p = e^{-\lambda\left(1-e^{\frac{2\pi ip}{m}}\right)}$$

$$= e^{-\lambda+\lambda\, e^{\frac{2\pi ip}{m}}} = e^{-\lambda}.e^{\lambda\, e^{\frac{2\pi ip}{m}}}$$

$$= e^{-\lambda}.e^{\lambda\left[\cos\frac{2\pi p}{m}+i\sin\frac{2\pi p}{m}\right]}$$

$$= e^{-\lambda}.e^{\lambda\cos\frac{2\pi p}{m}}.e^{i\lambda\sin\frac{2\pi p}{m}}$$

$$\varphi_p = e^{-\lambda\left[1-\cos\frac{2\pi p}{m}\right]}\left[\cos\left(\lambda\sin\frac{2\pi p}{m}\right)+i\sin\left(\lambda\sin\frac{2\pi p}{m}\right)\right]$$

where $\alpha_p = e^{-\lambda\left(1-\cos\frac{2\pi p}{m}\right)}\cos\left(\lambda\sin\frac{2\pi p}{m}\right)$

$$\beta_p = e^{-\lambda\left(1-\cos\frac{2\pi p}{m}\right)}\sin\left(\lambda\sin\frac{2\pi p}{m}\right)$$

Here α_p, β_p are called p^{th} trigonometric moments

Clearly $\rho_p = \sqrt{\alpha_p^2+\beta_p^2} = e^{-\lambda\left(1-\cos\frac{2\pi p}{m}\right)}$

and $\mu_p = \tan^{-1}\left(\dfrac{\beta_p}{\alpha_p}\right)$

$$= \tan^{-1}\left[\tan\left(\lambda\sin\frac{2\pi p}{m}\right)\right]$$

$$\mu_p = \lambda\sin\frac{2\pi p}{m}$$

Now the circular mean direction is defined as

$$\mu_1 = \lambda\sin\frac{2\pi}{m}$$

If μ_1 is denoted by μ then $\mu = \lambda\sin\dfrac{2\pi}{m}$

where λ is a parameter and ρ_1 represents concentration towards mean direction which is defined as

$$\rho_1 = e^{-\lambda\left(1-\cos\frac{2\pi}{m}\right)}$$

If ρ_1 is denoted by ρ then $\rho = e^{-\lambda\left(1-\cos\frac{2\pi}{m}\right)}$, $(0 \le \rho \le 1)$

Now the circular variances is denoted by V_0 and it is defined as

$$V_0 = 1-\rho$$

$$\mathrm{V}_0 = 1-e^{-\lambda\left(1-\cos\frac{2\pi}{m}\right)}$$

and standard deviation is denoted by σ_0 and it is defined as

$$\sigma_0 = \sqrt{-2\log(1-\mathrm{v}_0)}$$

$$= \sqrt{-2\log\left(1-1+e^{-\lambda\left(1-\cos\frac{2\pi}{m}\right)}\right)}$$

$$= \sqrt{-2\log\left(e^{-\lambda\left(1-\cos\frac{2\pi}{m}\right)}\right)}$$

$$\sigma_0 = \sqrt{2\lambda\left(1-\cos\frac{2\pi}{m}\right)}$$

Central Trigonometric Moments

The p^{th} central trigonometric moment of θ is defined as

$$\varphi_p^* = E\left[e^{ip(\theta-\mu)}\right]$$

$$= E\left[e^{ip\theta}.e^{-ip\mu}\right]$$

$$= e^{-ip\mu}E\left[e^{-ip\theta}\right]$$

$$= e^{-ip\mu}e^{-\lambda\left(1-\cos\frac{2\pi p}{m}\right)}\left[\cos\left(\lambda\sin\frac{2\pi p}{m}\right)+i\sin\left(\lambda\frac{2\pi p}{m}\right)\right]$$

$$= e^{-ip\mu}e^{-\lambda\left(1-\cos\frac{2\pi p}{m}\right)}\left[\cos\mu_p + i\sin\mu_p\right]$$

$$= e^{-ip\mu}e^{-\lambda\left(1-\cos\frac{2\pi p}{m}\right)}.e^{i\mu_P}$$

$$= e^{-\lambda\left(1-\cos\frac{2\pi p}{m}\right)}e^{i(\mu_p-p\mu)}$$

$$= e^{-\lambda\left(1-\cos\frac{2\pi p}{m}\right)}\left[\cos(\mu_p-p\mu)+i\sin(\mu_p-p\mu)\right]$$

Where $\alpha_p{*} = e^{-\lambda\left(1-\cos\frac{2\pi p}{m}\right)}\cos(\mu_p-p\mu)$

$$\beta_p{*} = e^{-\lambda\left(1-\cos\frac{2\pi p}{m}\right)}\sin(\mu_p-p\mu)$$

Which are called p^{th} central trigonometric moments. Now the circular skewness for wrapped Poisson distribution is denoted by γ_1 and it is defined as $\gamma_1 = \frac{\beta_2{*}}{v_0^{\frac{3}{2}}} = \frac{e^{-\lambda\left(1-\cos\frac{4\pi}{m}\right)}\sin(\mu_2-2\mu)}{\left[1-e^{-\lambda\left(1-\cos\frac{2\pi}{m}\right)}\right]^{\frac{3}{2}}}$ and circular kurtosis for wrapped

Poisson distribution is denoted by γ_2 and it is defined as

$$\gamma_2 = \frac{\alpha_2 * -(1-v_0)^4}{{v_0}^2}$$

$$= \frac{e^{-\lambda\left(1-\cos\frac{4\pi}{m}\right)}\cos(\mu_2 - 2\mu) - e^{-4\lambda\left(1-\cos\frac{2\pi}{m}\right)}}{\left[1-e^{-\lambda\left(1-\cos\frac{2\pi}{m}\right)}\right]^2}$$

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