International Journal of Mathematical Archive-5(6), 2014, 129-135
Available online through www.ijma.info ISSN 2229 – 5046

THE FAITHFUL EXTENSION OF CONSTANT HEIGHT AND WIDTH BETWEEN FINITE POSETS

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(Received on: 06-06-14; Revised & Accepted on: 23-06-14)

ABSTRACT

The problem of faithful extension with the condition of keeping constant height h and constant width w, i.e. for h w -inextensibility, seems more interesting than the brute extension of finite poset (partially ordered set). We shall investigate some theorems of hw-inextensive and hw-exrensive posets can be used to formulate the faithful extension problem. A theorem in its general form of hw-inextensive posets are given to implement the presented theorems.

Keywords: *Faithful extension, poset , extension, inextension, height, width, hw-extensive, hw-inextensive,*

1. INTRODUCTION

This paper is concerned the problem of *hw-extension* with height less than or equal 2 and constant width ≥ 1. The notions of *restriction*, *extension*, *isomorphism* and *mapping* between binary relations have been investigated.

Recall that a relation P is mapped into S or P$\leq$ S iff there exists an isomorphism from P onto a restriction of S. The problem of *faithful extension* (see [1]) is defined as follows:

Given two *posets* P and S such that P non$\leq$ S, then there exists a strict extension S^* of S (specially an extension with one additional element) which saves the non-mapping P non$\leq$ S^*, (For more details, see [1]).

The faithful extension among infinite *chain* or *linearly ordered sets* is studied in [2, 3] together with a result on this faithful extension has been mentioned in [7]). Faithful extension between *bipartite graphs* and alias *bivalent tableaux* has been discussed in several works such as [4, 5, 6, 8]. In this present paper, the finite posets are considered only.

2. PRELIMINARY CONCEPTS

In what follows we give important and useful definitions in the representation of the problem of concern.

Definition 1: [1] The poset P is an *extensive* iff for each poset S so that P non$\leq$ S (P not mapped into S) and there exists an extension S^* of S by adding one element to S such that P non$\leq$ S^*.

Definition 2: [1] The poset P is *inextensive* by S iff P non$\leq$ S but P $\leq$ S^* where P is mapped into each poset S^*, (the extension of S by adding one element).

Definition 3: [9] The *height* of finite poset P is the number of elements of maximal chains which are restriction of P.

Definition 4 [2, 3]: An *antichain* is a subset of a poset such that any two elements in the subset are incomparable.

Definition 5 [2, 3]: A *maximal antichain* is an antichain that is not a proper subset of any other antichain and that has cardinality at least as large as every other antichain.

Definition 6 [2, 3]: The *width w* of a poset is the cardinality of a maximum antichain.

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We can give new definitions concerning our work

Definition 7: The *faithful extension* between finite posets of constant height h and a constant width *w*, is called hw-*extension*.

Definition 8: The poset P is an *hw-inextensive* for S iff P and S have a common height h and a common width *w,* P non $\leq$ S but P is mapped into each poset S^* ($P\leq S^*$) of the same height h and width *w* where S^* is an extension of S by adding one element.

Definition 9: The poset P is an *hw-extensive* iff for every S of height h and width *w*, so that P non $\leq$ S, and there exists an extension S^* of S by adding one element with height h and width w such that P non $\leq$ S^*.

Definition 10: [1] The elements a and b of a poset $(P,\leq)$ are called *comparable* if either $a\leq b$ or $b\leq a$. When a and b are elements of P such that neither $a\leq b$ nor $b\leq a$, a and b are called *incomparable* and denoted by $a \mid b$.

Proposition 1: If P is any finite poset and L is the set of elements in P which are incomparable with the other ones, then P\L be the restriction of P to the complement of L.

Proposition 2: Let P be a poset, then all the minimal elements of P are called the elements of rank 0.

Proposition 3: Let $\mid$ be the poset with one *minimal* element of rank 0 and one element of rank 1, then the minimal element being anterior to the maximal element.

It should be noted that the notation ($\mid$ **.)** will be used to express the preceding poset with an additional incomparable element, while the notation ($\mid$ **. .**) is used for two incomparable elements.

3. REPRESENTATION OF THE PROBLEM

Many posets that are h-extensive and some posets that are 2-extensive have been studied in [9] together with one counter example of a poset of 2- inextensive has been given. In addition, it has been proved in [9] that P = ($|X|$ **. .**) is 2-inextensive by the poset S of cardinal 10.

The problem of extension is an open point of discussion for posets as well as for bipartite graphs, (see [1]). The problem addressed in the present paper as follows:
"*Does there exist a finite poset P of a constant height h and constant width w which is hw-inextensive for infinitely many finite poset S** (considered up to isomorphism, equivalently having unlimited finite cardinals).

4. THE *hw*-INEXTENSIVE PROBLEM

The essential and important purpose of the work presented in this paper is to give some theorem on *hw*-extensive and another theorems of a poset which is *hw*-inextensive.

Theorem 1: If a poset P contain only one element then P is *hw*-inextenive

Proof: We take an empty set S then P $\leq$ S^* since we add one element to S. So P is *hw*-inextesive.

Theorem 2: If a poset P contain two elements incomparable to each other, then P is *hw*-inextenive

Proof: By taking S with one element then P $\leq$ S^* since we add one element to S incomparable to the other one (if not then S will be of height 2: contradiction because P of height 1) since S^* must be same height h and same width *w*. So P is *hw*-inextesive

Theorem 3: The *poset* P ($\mid$) is *hw*-inextensive.

Proof: We take the poset S with one element, if we add another element to S^* then this element must be comparable to the other element (if not S^* will be antichain of cardinality 2: contradiction because P is an antichain with cardinality 1), then P $\leq$ S^*. So P is *hw*-inextesive

In contrast we shall prove that P ($\Big|$.) is *hw*-extensive.

Theorem 4: The poset P = ($\Big|$**.)** is *hw*-extnesive.

Proof: We shall treat two cases:

Case – 1: If P/L is non$\leq$ S, then S must be an antichain of cardinal 2 so we add one element comparable to these elements, (if not S will be an antichain with cardinal > 3), then P non $\leq$ S^*.

Case - 2: If P/L$\leq$ S, if S is an antichain of cardinality 1 and height 2 then we add an element s greater than the element in rank 0 or s less than the element in rank 1, then P non $\leq$ S^*. If S is an antichain of cardinality 2 and height 2, then we add an element s posterior to the element of maximal antichain or s anterior the element of maximal antichain, then P non $\leq$ S^*. So *hw*-extnesive.

We see in the following theorem that the precedent poset P = $\left(\Big| \ \right)$ with L= 2, that is P= ($\Big|$**..)** is *hw*-inextnesive.

Theorem 5: The poset P = ($\Big|$**..)** is *hw*-inextnesive.

Proof: First, we construct the poset S to be in form of one plate isomorphic to $|X|$. Let a, b the two minimal elements (of rank 0) of these plates and a', b' the two maximal elements (of rank 1). The two minimal a, b and the two maximal a', b' can be expressed by a figure in which we put a' vertically to a, and similarly b' for b. The plate is formed by a, b, a', b'. Moreover we add one element e of rank 0 (or one element e' of rank 1) incomparable to all other elements.

It should be verified that P non $\leq$ S then we have the following:

If the additional element s is from rank 1 and s > e (if s incomparable to e then the antichain will be of cardinality 4: contradition), then P $\leq$ S^* by s, e, a', b' which covers the four cases: s > e, ; then s > a, e; then s > b, e; then s > a, b, e. then P $\leq$ S^*. So P is *hw*-inextesive

Now if the additional element from rank 0, the proof is similar to the above one and we leave to the reader for completing the proof.

We extend our work with the precedent poset P = ($\Big|$ ..) with L= 3, that is P = $\left(\ \Big|...\right)$ is *hw*-inextnesive.

Theorem 6: The poset P = $\left(\ \Big|...\right)$ is *hw*-inextnesive.

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Proof: First, we construct the poset S to be in form of two plates; each plate is isomorphic to $|X|$. Let a, b, c the three minimal elements (of rank 0) of these plates and a', b', c' the three maximal elements (of rank 1). The three minimal a, b, c and the three maximal a', b', c' can be expressed by a figure in which we put a' vertically to a, and b' for b, and similarly c for c'. The first plate is formed by a, b, a', b' and the second plate is formed by b, c, b', c'. Moreover we add one element e of rank 0 (or one element e' of rank 1) incomparable to all other elements.

It should be verified that P non $\leq$ S then we have the following:

If the additional element s is from rank 1 and s > e (if s incomparable to e then the antichain will be of cardinality 5:contradition), then P$\leq$ S^* by s, e, a', b', c' which covers the eight cases: s $>$ e,; then s$>$a, e; then s$>$b, e; s$>$c, e; then s$>$a, b, e; then s $>$a, c, e; then s $>$b, c, e; then s $>$a, b, c, e. So P is *hw*-inextesive

Now if the additional element from rank 0, the proof is similar to the above one and we leave to the reader for completing the proof.

In the following theorem we see that the same poset P = $\left(\Big| \ \right)$ but with L= 4, that is P = $\left(\ \Big|....\right)$is *hw*-inextnesive.

Theorem 7: The poset P = $\left(\,|....\right)$ is *hw*-inextnesive.

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Proof: First, we construct the poset S to be in form of three plates; each plate is isomorphic to $|X|$. Let a, b, c, d the four minimal elements (of rank 0) of these plates and a', b', c', d' the four maximal elements (of rank 1). The four minimal a, b, c, and d and the four maximal a', b', c', d' can be expressed by a figure in which we put a' vertically to a, and b' for b, and similarly c for c' and d for d'. The first plate is formed by a, b, a', b' and the second plate is formed by b, c, b', c', the third plate is formed by c, d, c', d' . Moreover we add one element e of rank 0 (or one element e' of rank 1) incomparable to all other elements.

It should be verified that P non $\leq$ S then we have the following:

If the additional element s is from rank 1 and $s > e$ (if s incomparable to e then the antichain will be of cardinality 6:contradition), then P $\leq$ S^* by s, e, a', b', c', d' which sixteen covers the cases: $s > e$, ; then $s > a$, e; then $s > b$, e; then $s > c$, e; then $s > d$, e; then $s > a$, b, e; then $s > a$, c, e; then $s > a$, d, e; then $s > b$, c, e; then $s > b$, d, e; then $s > c$, d, e; then $s > a$, b, c, e, ; then $s > a$, b, d, e,; then $s > a$, c, d, e; then $s > b$, c, d, e,; then $s > a$, b, c, d, e. So P is *hw*-inextesive

Now if the additional element from rank 0, the proof is similar to the above one and we leave to the reader for completing the proof.

5. GENERAL FORM OF *hw*-INEXTENSIVE OF THE POSETS P ($|\ldots\ldots$ **.**) L = n ($n \geq 2$)

We can give the general case of poset P ($|\ldots\ldots$ **.**) with L = n ($n \geq 2$) which is *hw*-inextensive

Theorem 8: The posets P ($|\ldots\ \ldots$ **.**) with L = n ($n \geq 2$) are *hw*-inextensive

Proof: Put P **= (** $|\ldots\ \ldots$ **.)** and construct the poset S to be in the form of n plates, each plate is isomorphic to $|X|$; we call $a_1, a_2, \ldots, a_n$ are the n minimal elements of rank 0 of these plates, and $a_1', a_2', a_3', \ldots, a_n'$ are the n maximal of rank 1. In the figure of the plates, we put a_1' vertically to a_1, and similarly for a_2' and a_2, and so on a_n' and a_n. The first plate is formed by a_1, a_2, a_1', a_2'; the second plate by a_2, a_3, a_2', a_3'; and so on the last one by a_{n-1}, a_n, a_{n-1}', a_n';. Moreover we add one element e from rank 0, (or one element e' from rank 1).

It should be verified that P non $\leq$ S.

If the additional element s is from rank 1 and s $>$ e (if s incomparable to e then the antichain will be of cardinality n+1: contradiction), then P $\leq$ S^* by s, e, $a_1', a_2', a_3', \ldots, a_n'$ which covers all the 2^n cases (we leave to the reader to complete the proof)**.** So P is *hw*-inextensive.

If the additional element s from rank 1; prove analogous to precedent, so the theorem is proved.

6. WE PRESENT ANOTHER FORM OF THE POSETS P WHICH ARE *hw*-INEXTENSIVE

Theorem 9: The poset P with one element in rank 1 posterior to two incomparable elements in rank 0 is *hw*-inextnesive.

Proof: First, we construct the poset S to be in form of two elements in rank 1 posterior to one element in rank 0. Let a the minimal element (of rank 0), and a', b' the maximal element (of rank 1).

Firstly we verify that P non $\leq$S.

If we add one minimal element s of rank 0 (otherwise: contradiction with the width will be > 2) such that $s < a'$ then P $\leq$ S^* by s, a, a'**;** which cover the two cases $s < b'$ and $s < a', b'$. So P is *hw*-inextesive

Theorem 10: The poset P with two maximal incomparable elements in rank 1 posterior to one element in rank 0 is *hw*-inextnesive.

Proof: First, we construct the poset S to be in form of one maximal element in rank 1 posterior to two incomparable minimal elements in rank 0 . Let a, b the minimal elements (of rank 0) and a' the maximal element (of rank 1).

We verify that P non $\leq$ S.

If we add one maximal element s of rank 1 (otherwise: contradiction with the width will be > 2) such that $s < a$ then $P \leq S^*$ by s, a', a; which cover the two cases $s < b$ and $s < a, b$. So P is *hw*-inextesive

Theorem 11: The poset P with one maximal element in rank 1 and three incomparable elements in rank 0 such that the maximal element posterior to two incomparable elements in rank 0 and incomparable to third one is *hw*-inextnesive.

Proof: First, we construct the poset S to be in form of three elements a', b', c' in rank 1 and two elements a, b in rank 0 such that a' posterior to a, b' posterior to a and b and c' posterior to b .

We verify that P non $\leq$ S.

If we add one minimal element s of rank 0 (otherwise: contradiction because the width will be > 3) such that $s < a'$ then $P \leq S^*$ by s, a, a', b; which cover the three cases $s < a', b'$; $s < a', c'$; and $s < a', b', c'$. So P is *hw*-inextesive.

If $s < b'$ then $P \leq S^*$ by s, b, a', b' which cover $s < b', c'$. So P is *hw*-inextesive.

If $s < c'$ then $P \leq S^*$ by s, b, a', c'. So P is *hw*-inextesive.

Theorem 12: The poset P with one minimal element in rank 0 and three incomparable elements in rank 1 such that the minimal element anterior to two incomparable elements in rank 1 and incomparable to third one is *hw*-inextnesive.

Proof: First, we construct the poset S to be in form of two elements a', b' in rank 1 and three elements a, b, and c in rank 0 such that a' posterior to a, and b; b' posterior to b and c .

We verify that P non $\leq$ S.

If we add one maximal element s of rank 1 (otherwise: contradiction because the width will be of cardinal > 3) such that $s > a$ then $P \leq S^*$ by s, a, a', b'; which cover the three cases $s > a, b$; $s > a, c$; and $s > a, b, c$. So P is *hw*-inextesive.

If $s > b$ then $P \leq S^*$ by s, b, b', a which cover $s > b, c$. So P is *hw*-inextesive.

If $s > c$ then $P \leq S^*$ by s, c, a', b'. So P is *hw*-inextesive.

Theorem 13: The poset P with one maximal element in rank 1 and four incomparable elements in rank 0 such that the maximal element posterior to two incomparable elements in rank 0 and incomparable to the third and fourth one is *hw*-inextnesive.

Proof: First, we construct the poset S to be in form of four elements a', b', c', and d' in rank 1 and three elements a, b, and c in rank 0 such that a' posterior to a only ; b' posterior to a and b; c' posterior to b, and c; and d' posterior to c only.

We verify that P non $\leq$ S.

If we add one minimal element s of rank 0 (otherwise: contradiction because the width will be of cardinal > 4) such that $s < a'$ then $P \leq S^*$ by s, a, a', b, c; which cover the seven cases $s < a', b'$; $s < a', c'$; $s < a', d'$; $s < a', b', c'$; $s < a', b', d'$; $s < a', c', d'$; $s < a', b', c', d'$. So P is *hw*-inextesive.

If $s < b'$ then $P \le S^*$ by s, b, b', c, a' which cover the cases $s < b', c'$; $s < b', d'$; $s < b', c', d'$. So P is *hw*-inextesive.

If $s < c'$ then $P \le S^*$ by s, c, c', a', b' which cover the case $s < c', d'$. So P is *hw*-inextesive.

If $s < d'$ then $P \le S^*$ by s, c, d', a', b'. So P is *hw*-inextesive.

Theorem 14: The poset P with one minimal element in rank 0 and four incomparable elements in rank 1 such that the minimal element anterior to two incomparable elements in rank 1 and incomparable to third and fourth one is *hw*-inextnesive.

Proof: First, we construct the poset S to be in form of three elements a', b', c' in rank 1 and four elements a, b, c, and d in rank 0 such that a' posterior to a, and b; b' posterior to b and c, and c' posterior to c and d.

We verify that P non $\le$ S.

If we add one maximal element s of rank 1 (otherwise: contradiction because the width will be of cardinal > 4) such that $s > a$ then $P \le S^*$ by s, a, a', b', c'; which cover the three cases $s > a, b$; $s > a, c$; $s > a, d$; $s > a, b, c$; $s > a, b, d$; $s > a, c$, d; $s > a, b, c$, d. So P is *hw*-inextesive.

If $s > b$ then $P \le S^*$ by s, b, b', a, c' which cover $s > b, c$; $s > b, d$; $s > b, c, d$. So P is *hw*-inextesive.

If $s > c$ then $P \le S^*$ by s, c, c', b', a which cover $s > c$, d. So P is *hw*-inextesive.

If $s > d$ then $P \le S^*$ by s, d, c', a', b'. So P is *hw*-inextesive.

7. GENERAL FORM OF THE POSETS P ($\vee$) and P ($\wedge$) with L = n (n$\ge$1) WHICH IS hw-INEXTENSIVE

Finally, we can conclude our working by present another theorems that are also hw-inextensive .

At the end we give the general cases of poset P ($\wedge$. . . . … .) with L = n (n$\ge$1) and P ($\wedge$. . . . … .) with L = n (n$\ge$1) which is *hw*-inextensive

Theorem 15: The poset P ($\vee$. . . . … .) with one maximal element in rank 1 and L= n$\ge$1 incomparable elements in rank 0 such that the maximal element posterior to two incomparable elements in rank 0 and incomparable to the L= n$\ge$1elements is *hw*-inextnesive.

The proof is similar to the previous theorems

Theorem 16: The poset P ($\vee$. . . . … .) with one minimal element in rank 0 and L= n$\ge$1 incomparable elements in rank 1 such that the minimal element anterior to two incomparable elements in rank 1 and incomparable to the L= n$\ge$1 elements is *hw*-inextnesive.

The proof is similar to the previous theorems

8. CONCLUSION

We find in this work some posets which are hw-extensive and some posets which are hw-inextensive.

We can conclude that we can find many other posets which are hw-inextensive and many other posets which are hw-extensive.

9. ACKNOWLEDGEMENT

'This publication was made possible by support of Qatar University. Its contents are solely the responsibility of the authors, and do not necessarily represent the official views of Qatar University'

REFERENCES

[1] R.FRAISSE, "*Theory of Relation*", Studies in Logic and the Foundation of Mathematics Number, N^o 145, North Holland, Elsevier Science Publisher (1986).

[2] J. HAGENDORF, "Extension immediate et respectueuses de chaines et de relation", *C. R. Acad. Sci. Paris*, 275, serie A, p. 949-950 (1972).

[3] J. HAGENDORF, "Extension de chaines", *Z. Math. Logik Grundlag. Math*., 25, p. 423-444 (1979).

[4] G. LOPEZ, "Problem d'extension respectueuse", *C. R. Acad. Sci. Paris,* 277, serie A, p. 567-569 (1973).

[5] G. LOPEZ, "La p-extensivite d'un tableau bivalent pour $p \geq 5$", *C. R. Acad. Sci. Paris*, 284, serie A, p.1245-1248 (1977).

[6] C. RAUZY, "Sur l'extensivite des tableau bivalents a deux colonnes", *C. R. Acad. Sci. Paris*, 303, serie I, p. 721-724 (1986).

[7] J. ROSENSTEIN, "*Linear ordering*", Academic Press, (1982).

[8] WALIED H. SHARIF, "On the *p*-inextensivity of bivalent tables with two columns". Far East J. Math. Sci (FJMS), Vol. 3, No. 6, (2001).

[9] R. FRAISSE, and WALIED H. SHARIF, "L' extension respectueuse a hauteur constante entre posets finis". *C. R. Acad. Sci. Paris*, t. 316, serie I, p. 637-642, (1993).

Source of support: Nil, Conflict of interest: None Declared