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FIXED POINTS FOR FOUR SELF-MAPPINGS IN CONE METRIC SPACES

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ABSTRACT

In this paper we prove a unique common fixed point theorem for four self- mappings in cone metric spaces without using the commutativity condition in cone metric spaces. Our results generalize and extend several results existing in the literature.

***Key words**: Coincidence points, Common fixed point, Cone metric space.*

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1. INTRODUCTION AND PRELIMINARIES

The study of common fixed points of mappings satisfying certain contractive conditions has been at the centre of vigorous research activity, being the applications of fixed point very important in several areas of Mathematics. Huang and Zhang [12] have generalized the concept of a metric space, replacing the set of real numbers by an ordered Banach space and obtained some fixed point theorems for mapping satisfying different contractive conditions. Subsequently, Abbas and Jungck [1] and Abbas and Rhoades [2], Ilic and Rakocevic [13], Rezapour and Hamlbarani[14] and Vetro [17] studied fixed point theorems for contractive type mappings in cone metric spaces .

In this paper we prove a unique common fixed point theorem for four self-mappings in cone metric spaces without using the commutativity condition. Our result generalized and extends the results of [11].

In all that follows B is a real Banach Space, and θ denotes the zero element of B.

For the mapping f, g: X $\rightarrow$ X, let C (f, g) denote the set of coincidence points of f and g, that is
C(f, g) = {z $\in$ X: fz = gz}.

The following definitions are due to Huang and Zhang [12].

Definition 1.1: Let B be a real Banach Space and P a subset of B .The set P is called a cone if and only if:
(a). P is closed, non –empty and P $\neq \{\theta\}$;
(b). a, b $\in R$, a, b ≥ 0 , x, y $\in P$ implies ax+by $\in P$;
(c). x $\in$ P and -x $\in P$ implies x = θ .

Definition 1.2: Let P be a cone in a Banach Space B, define partial ordering '$\leq$' with respect to P by x $\leq$ y if and only if y-x $\in P$. We shall write x<y to indicate $x \leq y$ but $x \neq y$ while X<<y will stand for y-x $\in$ Int P, where Int P denotes the interior of the set P. This Cone P is called an order cone.

Definition 1.3: Let B be a Banach Space and P $\subset$ B be an order cone. The order cone P is called normal if there exists L>0 such that for all x, y $\in$ B,
$\theta \leq x \leq y$ implies $\|x\| \leq L \|y\|$.

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The least positive number L satisfying the above inequality is called the normal constant of P.

Definition 1.4: Let X be a nonempty set of B .Suppose that the map d: $X \times X \to B$ satisfies:

(d1). $\theta \le d(x, y)$ for all x, y $\in X$ and d(x, y) = θ if and only if x = y;

(d2).d(x, y) = d(y, x) for all x, y$\in X$;

(d3).d(x, y)$\le$d(x, z) + d(y, z) for all x, y, z$\in X$.

Then d is called a cone metric on X and (X, d) is called a cone metric space.

The concept of a cone metric space is more general than that of a metric space.

Example 1.5: ([12]). Let E =R^2, P = {(x, y)$\in$E such that : x, y $\ge$ 0} $\subset$ R^2, X = R and d: X × X→E such that d(x, y) = (│x - y│, α │x - y│), where α ≥ 0 is a constant. Then (X, d) is a cone metric space.

Definition 1.6: Let (X, d) be a cone metric space. We say that {x_n} is

(i) a Cauchy sequence if for every c in B with c>>θ, there is N such that for all n, m>N, d(x_n, x_m)<<c;

(ii) Convergent sequence if for any c >>θ, there is an N such that for all n>N, d(x_n, x)<<c, for some fixed x in X. We denote this $x_n \to x$ (as $n \to \infty$) .

Lemma 1.7: Let (X, d) be a cone metric space, and let P be a normal cone with normal constant K.
Let {x_n } be a sequence in X .Then

(i). {x_n} converges to x if and only if d(x_n ,x)→ 0 ($n \to \infty$).

(ii). {x_n} is a Cauchy sequence if and only if d (x_n , x_m)→0 (as n, m→∞).

2. MAIN RESULT

In this section we prove, a unique common fixed point theorem for four self-mappings in cone metric spaces without commutativity condition. Our result generalizes and extends the results of Guangxing Song *et.al* [11].

The following Theorem is extend and improve the Theorem 2.1of Guangxing Song *et.al* [11].

Theorem 2.1: Let (X, d) be a complete cone metric space and normal cone with normal constant L. Suppose that the mappings S, T, f, g are four self-maps such that T(x)$\subseteq$f(x), S(x)$\subseteq$g(x) and S(X) = T(X). Satisfying

$$d(Sx, Ty) \le a_1 d(fx, gy) + a_2 [d(Sx, fx)+ a_3 d(Ty, gy)] + a_4[d(fx,Ty)+ a_5 d(Sx, gy)] \quad \ldots \quad (1)$$

for all x, y$\in$X, where $a_i \ge 0$ (i= 1,2,3,4,5)be constants with $a_1+ a_2 + a_3 +2 a_4+ a_5<1$.

If {S, f} and {T, g} have a coincidence point. Then S, T, f and g have a unique common fixed point.

Proof: Suppose x_0 is an arbitrary point of X and define the sequence {y_n} in X such that

$y_{2n} = Sx_{2n} = gx_{2n+1}$,

$y_{2n+1} = Tx_{2n+1} = fx_{2n+2}$.

By (1) we have

$$\begin{aligned} d(y_{2n}, y_{2n+1}) &= d(Sx_{2n}, Tx_{2n+1}) \\ &\le a_1 d(fx_{2n}, gx_{2n+1}) + a_2 d(Sx_{2n} ,fx_{2n}) + a_3 d(Tx_{2n+1} , gx_{2n+1}) + a_4 d(fx_{2n}, Tx_{2n+1})+ a_5 d(Sx_{2n} , gx_{2n+1}) \\ &\le a_1 d(y_{2n-1}, y_{2n}) + a_2 d(y_{2n}, y_{2n-1}) + a_3 d(y_{2n+1}, y_{2n}) + a_4 d(y_{2n-1},y_{2n+1}) + a_5 d(y_{2n},y_{2n}) \\ &\le a_1 d(y_{2n-1}, y_{2n}) + a_2 d(y_{2n}, y_{2n-1}) + a_3 d(y_{2n+1}, y_{2n}) + a_4 [d(y_{2n-1},y_{2n}) +d(y_{2n}, y_{2n+1})]+ a_5 d(y_{2n},y_{2n}), \\ &\le (a_1 + a_2 + a_4) d(y_{2n-1},y_{2n}) +(a_3 + a_4)d(y_{2n}, y_{2n+1}), \end{aligned}$$

which implies that

$$d(y_{2n}, y_{2n+1}) \le \frac{a_1 + a_2 + a_4}{1-(a_3 + a_4)} d(y_{2n-1},y_{2n})$$

(or)
$d(y_{2n}, y_{2n+1}) \le \lambda\, d(y_{2n-1}, y_{2n})$.

Where, $\lambda = \dfrac{a_1 + a_2 + a_4}{1-(a_3 + a_4)} < 1$. Since, $a_1 + a_2 + a_3 + 2a_4 + a_5 < 1$.

Similarly, it can be shown that

$d(y_{2n+1}, y_{2n+2}) \le \lambda\, d(y_{2n}, y_{2n+1})$.

Therefore, for all n,

$d(y_{n+1}, y_{n+2}) \le \lambda\, d(y_n, y_{n+1}) \le \ldots \le \lambda^{n+1} d(y_0, y_1)$.

Now, for any m > n,

$$d(y_n, y_m) \le d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2}) + \ldots + d(y_{m-1}, y_m)$$
$$\le [\lambda^n + \lambda^{n+1} + \ldots + \lambda^{m-1}]\, d(y_0, y_1)$$
$$\le \frac{\lambda^m}{1-\lambda} d(y_0, y_1).$$

By the definition of (1.3), we have
$$\| d(y_n, y_m) \| \le \frac{\lambda^m}{1-\lambda} L \| d(y_0, y_1) \|.$$

Which implies that $d(y_n, y_m) \to 0$ as $n, m \to \infty$. Hence $\{y_n\}$ is a Cauchy sequence in T(X) which is complete, there exists $z \in T(X)$ such that $y_n \to z$. Since $T(X) \subset f(X)$, then there exist a point $p \in X$ such that z = fp. Let us prove that z = Sp. Then by triangle inequality and (1), we have

$$d(Sp, z) \le d(Sp, Tx_{2n-1}) + d(Tx_{2n-1}, z)$$
$$\le a_1 d(fp, gx_{2n-1}) + a_2 d(Sp, fp) + a_3 d(Tx_{2n-1}, gx_{2n-1}) + a_4 d(fp, Tx_{2n-1}) + a_5 d(Sp, gx_{2n-1}) + d(z, Tx_{2n-1})$$

By (1.3), we have
$$\| d(Sp, z) \| \le L(a_1 \| d(fp, gx_{2n-1}) \| + a_2 \| d(Sp, fp) \| + a_3 \| d(Tx_{2n-1}, gx_{2n-1}) \| + a_4 \| d(fp, Tx_{2n-1}) \|$$
$$+ a_5 \| d(Sp, gx_{2n-1}) \| + \| d(z, Tx_{2n-1}) \|)$$

Letting $n \to \infty$,

$$\| d(Sp, z) \| \le L(a_1 \| d(z, z) \| + a_2 \| d(Sp, z) \| + a_3 \| d(z, z) \| + a_4 \| d(z, z) \| + a_5 \| d(Sp, z) \| + \| d(z, z) \|)$$
$$\le L(a_2 + a_5) \| d(Sp, z) \|, \text{ which is a contradiction .}$$

Since, $a_1 + a_2 + a_3 + 2a_4 + a_5 < 1$.

Hence, Sp = z.

Therefore, Sp = fp = z, p is a coincidence point of {S, f}. (2)

Since, $S(X) \subseteq g(X)$ there exists a point $q \in X$ such that z = gq. We shall show that Tq = z.

Then by (1), we have
$$d(z, Tq) = d(Sp, Tq)$$
$$\le a_1 d(fp, gq) + a_2 d(fp, Sp) + a_3 d(Tq, gq) + a_4 d(fp, Tq) + a_5 d(Sp, gq),$$
$$\le a_1 d(z, z) + a_2 d(z, z) + a_3 d(Tq, z) + a_4 d(z, Tq) + a_5 d(z, z),$$
$$\le (a_3 + a_4)\, d(z, Tq),$$

which is a contradiction. Since, $a_1+a_2+a_3+2a_4+a_5<1$.

Implies $Tq = z$.

Therefore, $Tq = gq$ (=z), q is a coincidence point of {T, g}. (3)

From (2) and (3) it follows that

$Sp = fp = Tq = gq$ (=z). … (4)

Now using (1), we have

$$d(ffp, fp) = d(ffp, y_{2n+1}) + d(y_{2n+1}, fp)$$

$$= d(ffp, y_{2n+1}) + d(Tx_{2n+1}, Sp)$$

$$= d(ffp, y_{2n+1}) + d(Sp, Tx_{2n+1})$$

$$\leq d(ffp, y_{2n+1}) + a_1 d(fp, gx_{2n+1}) + a_2 d(Sp, fp) + a_3 d(Tx_{2n+1}, gx_{2n+1}) + a_4 d(fp, Tx_{2n+1}) + a_5 d(Sp, gx_{2n+1}),$$

$$\leq d(ffp, y_{2n+1}) + a_1 [d(fp, fp)+d(fp, gx_{2n+1})] + a_2 d(Sp, fp) + a_3 d(Tx_{2n+1}, gx_{2n+1}) + a_4 [d(fp,Sp)+d(Sp, Tx_{2n+1})] + a_5 [d(Sp,fp)+d(fp, gx_{2n+1})],$$

$$\leq d(ffp, y_{2n+1}) + a_1 d(fp, fp) + (a_2+a_4+a_5) d(fp,Sp) + a_1 d(fp, gx_{2n+1}) + a_3 d(Tx_{2n+1}, gx_{2n+1}) + a_4 d(Sp, Tx_{2n+1}) + a_5 d(fp, gx_{2n+1}).$$

Which from (1.3), implies

$$\|d(ffp, fp)\| \leq L\{\|d(ffp, y_{2n+1})\| + a_1\|d(fp, fp)\| + (a_2+a_4+a_5)\|d(fp,Sp)\| + a_1\|d(fp, z)\| + a_3\|d(z, z)\| + a_4\|d(Sp, z)\| + a_5\|d(fp, z)\|\}$$

Letting $n\to\infty$, we get that

$$\|d(ffp, fp)\| \leq L\{\|d(ffp, z)\| + a_1\|d(fp, fp)\| + (a_2+a_4+a_5)\|d(fp, Sp)\| + a_1\|d(fp, z)\| + a_3\|d(z, z)\| + a_4\|d(Sp,z)\| + a_5\|d(fp, z)\|\},$$

$$\leq L\{\|d(ffp, fp)\| + a_1\|d(z, z)\| + (a_2+a_4+a_5)\|d(z, z)\| + a_1\|d(z, z)\| + a_3\|d(z, z)\| + a_4\|d(z, z)\| + a_5\|d(z, z)\|\},$$

$$\leq L\|d(ffp, fp)\|.$$

Implies, $(1-L)\|d(ffp, fp)\| \leq 0$.

Hence, $ffp = fp$ (=z).

Therefore, fp(=z) is a fixed point of f. … (5)

$$d(SSp,Sp) = d(SSp, Tq)$$

$$\leq a_1 d(fSp, gq) + a_2 d(SSp, fSp) + a_3 d(Tq, gq) + a_4 d(fSp,Tq) + a_5 d(SSp, gq),$$

$$\leq a_1 d(SSp, Sp) + a_2 d(SSp, Sp) + a_2 d(Sp, SSp) + a_3 d(Tq, Tq) + a_4 d(SSp,Sp) + a_5 d(SSp, Sp),$$

$$\leq (a_1+2a_2+a_4+a_5) d(SSp, Sp).$$

Which, using the definition of partial ordering of cone P, gives $d(SSp, Sp) = 0$ and $SSp = Sp$(=z).

Therefore, $SSp = Sp$(= z), Sp(=z) is a fixed point of S. … (6)

Now,

$d(ggq, gq) = d(ggq, y_{2n}) + d(y_{2n} ,gq)$

$= d(ggq, y_{2n}) + d(Sx_{2n} ,Tq)$

$\leq d(ggq, y_{2n}) + a_1 d(fx_{2n}, gq) + a_2 d(Sx_{2n}, fx_{2n}) + a_3 d(Tq, gq) + a_4 d(fx_{2n}, Tq) + a_5 d(Sx_{2n}, gq),$

$\leq d(ggq, y_{2n}) + a_1 d(fx_{2n}, gq) + a_2 [d(Sx_{2n}, gq)+d(gq, fx_{2n})]+ a_3 d(Tq, Tq) + a_4 d(fx_{2n}, Tq) + a_5 d(Sx_{2n}, gq),$

$\leq d(ggq, y_{2n}) + (a_1 + a_2)d(fx_{2n}, gq) + (a_2 + a_5)d(Sx_{2n}, gq) + a_4 d(fx_{2n}, Tq).$

Which from (1.3) implies that

$\|d(ggq, gq)\| \leq L\{\| d(ggq, y_{2n})\| + (a_1 + a_2)\|d(fx_{2n}, gq)\| + (a_2 + a_5)\|d(Sx_{2n}, gq)\| + a_4 \|d(fx_{2n}, Tq)\|\}$.

Letting, $n\to\infty$, we get that

$\|d(ggq, gq)\| \leq L\{\| d(ggq, z)\| + (a_1 + a_2)\|d(z, gq)\| + (a_2 + a_5)\|d(z, gq)\| + a_4 \|d(z, Tq)\|\}$

$\leq L\| d(ggq, gq)\|$.

Hence, $ggq = gq(= z)$, $gq(= z)$ is a fixed point of g. … (7)

Now,

$d(Tq,TTq) = d(Sp, TTq)$

$= d(SSp,TTq)$

$\leq a_1 d(fSp, gTq)+ a_2 d(SSp, fSp) + a_3 d(TTq, gTq) + a_4 d(fSp,TTq) + a_5 d(SSp, gTq),$

$\leq a_1 d(gTq, gTq)+ a_2 d(SSp, SSp) + a_3 d(gTq, gTq) + a_4 d(SSp, SSp) + a_5 d(SSp, SSp,$

$\leq 0.$

Hence, $TTq = Tq(= z)$, $Tq (=z)$ is a fixed point of T. …. (8)

Since, $Sp = fp = Tq = gq(= z)$.

Therefore, from (5), (6), (7) and (8), we get that

S, T, f, and g have a common fixed point namely z.

Uniqueness, let W be another common fixed point of S, T, f and g, then

$d(z, w) = d(Sz, Tw)$

$\leq a_1 d(fz, gw) + a_2 d(Sz, fz) + a_3 d(Tw, gw) + a_4 [d(fz, Tw) + a_5 d(Sz, gw),$

$\leq a_1 d(z, w) + a_2 d(z, z) + a_3 d(w, w) + a_4 d(z, w) + a_5 d(z, w),$

$\leq (a_1+ a_4 + a_5)d(z, w)$, since, $a_1+ a_2 + a_3 +2 a_4+ a_5<1$.

From the above it is easy to obtain that $d(z,w) = d(Sz, Tw) = 0$. That is, $z = w$.

Therefore, S, T, f and g have a unique common fixed point in X.

REFERENCES

1. M.Abbas and G.Jungck, Common fixed point results for non commuting mappings without continuity in cone metric spaces,J.math.Anal.Appl. 341(2008)416-420.

2. M.Abbas and B.E.Rhoades, Fixed and periodic point results in cone metric spaces, Appl. Math. Lett. 22(2009)511-515.

3. I.Altun, G.Durmaz, Some fixed point theorems on ordered cone metric spaces, Rend. circ. Mat. palermo 58(2009)319-325.

4. I.Altun, B.Damjanovi, Dragan Djoric, Fixed point and common fixed point theorems on ordered cone metric spaces, Appl. Math. Lett. 23(2010)310-316.

5. M.Amari, D.El Moutawakil, Some new common fixed point theorems under strict contractive conditions, J. Math. Anal. Appl. 270(2002)181-188.

6. M.Arshad, A.Azam, P.vetro, Some common fixed point results on cone metric spaces, Fixed Point Theory Appl.(2009)11, Article ID 493965.

7. A.Azam, M.Arshad, I.Beg, Common fixed points of two maps in cone metric spaces, Rend. Circ.Mat. palermo 57(2008)433-441.

8. I. Beg, M.Abbas, Coincidence point and invariant approximation for mappings satisfying generalized weak contractive condition, Fixed Point Theory Appl.(2006)7pages, Article ID 74503.

9. B.S. Choudhry, N. Metiya, The point of coincidence and common fixed point for a pair of mappings in cone metric spaces, Computers and Mathematics with Applications 60(2010)1686-1695.

10. B.Fisher, Four mappings with a common fixed point (Arabic summary), J. Univ. Kuwait Sci.(1981) 131- 139.

11. Guangxing Song, Xiaoyan Sun, Yian Zhao, Guotao Wang, New common fixed point theorems for maps on cone metric spaces, Appl. Math. Lett. 32(2010)1033-1037.

12. L.G.Huang, X.Zhang, Cone metric spaces and fixed point theorems of contractive mappings, J. Math. Anal. Appl. 332(2)(2007)1468-1476.

13. D.IIic, V.Rakocevic, Common fixed points for maps on cone metric space, J.Math. Anal. Appl. 341(2008)876-882.

14. S.Rezapour and Hamlbarani, Some notes on the paper cone metric spaces and fixed point theorem of contractive mappings, J. Math. Anal. Appl. 345(2008)719-724.

15. B.E.Rhoars, A comparison of various definitions of contractive mappings, Trans. Amer. Math. Soc. 26(1977)257-290.

16. S.L.Singh, Apichai Hematulin and R.P.Pant, New coincidence and common fixed point theorem, Applied General Topology 10(2009),no.1, 121-130.

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