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FIXED POINT OF GERAGHTY'S CONTRACTIVE MAPPING IN 2-METRIC SPACES

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ABSTRACT

In this paper a necessary and sufficient condition for the existence of fixed points for a class of Geraghty contractive mappings on a complete 2-metric space have been obtained. Our results generalize and improve several fixed point theorems for contractive mappings on a complete 2-metric space as available in the literatures.

Key words and phrases: *complete 2-metric space, fixed point, Geraghty function, convergence of iteration.*

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1. INTRODUCTION:

The notion of a 2-metric and its basic properties were initially introduced by Gähler in a series of papers ([12]-[13]). Then after a decade, Iseki [5] started investigation of fixed point of mappings in a setting of 2-metric space. Subsequently many researchers like Rhoades [1], Khan et al. [10], Imdad et al.[8] and Saha et al.[9] had worked on several generalized version of fixed point theorems for different types of contractive type mappings in 2-metric spaces. Let X be a complete 2-metric space and $f: X \to X$ is said to be contractive if for each $a \in X$,

$$d(f(x), f(y), a) < d(x, y, a), \quad for\ all\ x, y \in X.$$

Let x_0 be a chosen point in X and set Picard's iteration $x_n = f(x_{n-1})$, $n > 0$, Criteria for the sequence of iterates to be Cauchy are then of interests for the existence of fixed point, for if it were Cauchy one can easily prove that it converges to a unique fixed point for the map f on X. Usual technique namely Cauchy criterion for the convergence of a sequence have been proceeded for achieving fixed point of contractive mappings (see for details [5], [6]). However, Hsiao [3] had shown that these kinds of contractivity conditions don't have a wide range of applications, since they imply collinearity of the sequence of iterates starting with any point. Though his remarks is debated we replace here the Cauchy criterion for convergence of a sequence by an equivalent form due to Geraghty [7] in a complete 2-metric space.

2. PRELIMINARIES:

Definition 2.1: *For any non-empty set X, a real valued function d on $X \times X \times X$ is said to be a 2-metric on X if*

1. Given two distinct elements $x, y \in X$, there exists an element z of X such that $d(x, y, z) \neq 0$.
2. $d(x, y, z) = 0$, when at least two of x, y, z are equal.
3. $d(x, y, z) = d(x, z, y) = d(y, z, x)$ for all $x, y, z \in X$.
4. $d(x, y, z) \leq d(x, y, w) + d(x, w, z) + d(w, y, z)$ for all $x, y, z, w \in X$.

   When d is a 2-metric on X, then the ordered pair (X, d) is called 2-metric space.

Definition 2.2: *A sequence $\{x_n\}$ in X is called Cauchy, if for each $a \in X$, $d(x_n, x_m, a) \to 0$, as $n, m \to \infty$.*

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Definition 2.3: *A sequence* $\{x_n\}$ *in* X *is said to converge to an element* $x \in X$ *if for each* $a \in X$, $\lim_{n\to\infty} d(x_n, x, a) = 0$.

Definition 2.4: *A* 2*-metric space* (X, d) *is said to be complete if every Cauchy sequence in* X *converges to an element of* X.

3. MAIN RESULTS:

For any pair of sequence x_n and y_n in X with $x_n \neq y_n$ we take for each $a \in X$,

$$d_n = d(x_n, y_n, a), \quad where\ x_n \neq y_n \neq a \ \text{ and } \ \Delta_n = \frac{d(f(x_n), f(y_n), a)}{d_n}$$

Theorem 3.1: *Let* X *be a complete* 2*-metric space and let* $f : X \to X$ *with* $d(f(x), f(y), a) < d(x, y, a)$, *for all* $x, y, a \in X$. *Take* $x_0 \in X$ *and set* $x_n = f(x_{n-1})$ *for* $n > 0$. *Then* $\{x_n\}$ *converges to* x^* *in* X *with* x^* *be a unique fixed point of* f *in* X *if and only if for any two subsequences* x_{h_n} *and* x_{k_n} *with* $x_{h_n} \neq x_{k_n}$, *we have*

$\Delta_n \to 1$ *only if* $d_n \to 0$ *as* $n \to \infty$.

Proof: Let $x_n \to x^*$ in X and x^* be a unique fixed point of f. Again let x_{h_n} and x_{k_n} be two subsequences. It is clear that for each $a \in X$, $d_n = (x_{h_n}, x_{k_n}, a) \to 0$ as $n \to \infty$. So, $d_n \to 0$ is satisfied.

Next assume that for a given initial point $x_0 \in X$ the condition is satisfied. Then for each $a \in X$, $\{d_n = d(x_n, x_{n+1}, a)\}$ be a decreasing sequence of non-negative real numbers which is bounded below and hence convergent. Let $d_n \to \varepsilon$ as $n \to \infty$, for some $\varepsilon \geq 0$. If possible, let $\varepsilon > 0$. If we take $h_n = n$ and $k_n = n+1$, we get $d(x_n, x_{n+1}, a) \to \varepsilon > 0$ as $n \to \infty$ while $\Delta_n \to 1$ as $n \to \infty$, a contradiction. So, $d_n \to 0$, as $n \to \infty$.

Conversely, let $\{x_n\}$ be a sequence of X which is not Cauchy. Then there exists some $\varepsilon > 0$ such that for any positive integer N the sequence $\{x_n\}_{n \geq N}$ satisfies

$D_N = \sup_{n,m \geq N} d(x_n, x_m, a) > \varepsilon$ for all $a \in X$. For this ε, we can construct two subsequences.

For any $n > 0$ let N_n be so large such that $d(x_m, x_{m+1}, a) < 1/n$ for all $m \geq N_n$ and for each $a \in X$ as possible, since $d(x_m, x_{m+1}, a) \to 0$ as $m \to \infty$.

Let $h_n \geq N_n$ be the lowest positive integer such that for some $k_n > h_n$, $d(x_{h_n}, x_{k_n}, a) > \varepsilon$. Such pair exists as $D_N > \varepsilon$. Choose k_n to be least integer above h_n so that $k_n = h_n + 1$ or $d(x_{h_n}, x_{k_{n-1}}, a) \leq \varepsilon$. In both the cases we get,

$$\varepsilon \leq d_n = d(x_{h_n}, x_{k_n}, a) < \varepsilon + 2/n$$

Moreover,

$$1 \geq \Delta_n = \frac{d(f(x_{h_n}), f(x_{k_n}), a)}{d(x_{h_n}, x_{k_n}, a)} \geq \frac{d_n - 6/n}{d_n}$$

So we get that $\Delta_n \to 1$ while $d_n \to \varepsilon > 0$ again contradicting the condition. So $\{x_n\}$ must be a Cauchy sequence in X and as X is complete we have $x_n \to x^*$ for some x^* in X. The uniqueness of x^* is also clear. So x^* be the unique fixed point of f and the proof is complete.

Now we replace the criterion for convergence of an iterative sequence by an equivalent form obtained by Geraghty function [7].

Definition 3.2: *(Geraghty function [7]) Let* $\mathcal{S}$ *denotes the class of functions* $\alpha : \mathbb{R}^+ \to [0,1)$ *which satisfies the condition* $\alpha(t_n) \to 1$ *implies* $t_n \to 0$.

Theorem 3.3: *Let* $f : X \to X$ *be a contraction on a complete* 2*-metric space. Let* $x_0 \in X$ *and* $x_n = f(x_{n-1})$ *for* $n > 0$. *Then* x_n *converges to* x^* *in* X *with* x^* *be unique fixed point of* f *in* X *, if and only if there exists an* α *in* $\mathcal{S}$ *such that for all* n, m

$$d(f(x_n), f(x_m), a) \le \alpha(d(x_n, x_m, a))d(x_n, x_m, a) \tag{1}$$

for all $a \in X$.

Proof: Only we have to show the existence of such type of α in $\mathcal{S}$ which is equivalent to the convergence of an iterated sequence in Theorem 3.1.

Assume that such α exists. Let x_{h_n} and x_{k_n} be two subsequences of $\{x_n\}$ with $x_{h_n} \ne x_{k_n}$. Assume $\Delta_n \to 1$. Then from the inequality (1) and using the definition of α we get $\alpha(d(x_{h_n}, x_{k_n}, a)) \to 1$. But $\alpha \in \mathcal{S}$, so $d(x_{h_n}, x_{k_n}, a) \to 0$ as $n \to \infty$.

So by theorem 3.1., f has a unique fixed point in X .

Conversely, let the sequential condition holds. Define $\alpha : \mathbb{R}^+ \to \mathbb{R}$ as follows

$$\alpha(t) = \sup\left\{ \frac{d(f(x_n), f(x_m), a)}{d(x_n, x_m, a)} : d(x_n, x_m, a) \ge t \right\}$$

As f is a contraction $\alpha(t) \le 1$ for all $t > 0$. So α is defined for all $t > 0$ and $\alpha \le 1$.Now assume that $\alpha(t_n) \to 1$ as $n \to \infty$ for $t_n \in \mathbb{R}^+$. So we can say $1 - \frac{1}{n} < \alpha(t_n) \le 1$. Then we have to show $t_n \to 0$ as $n \to \infty$.

But $\alpha(t_n)$ is above the least upper bound. So for $n > 0$ a pair of subsequences $\{x_{h_n}\}$ and $\{x_{k_n}\}$ in $\{x_n\}$ with

$$d(x_n, x_m, a) \ge t_n, \quad for\ each\ a \in X,$$

$$1 - \frac{1}{n} < \frac{d(f(x_{h_n}), f(x_{k_n}), a)}{d(x_{h_n}, x_{k_n}, a)} \le \alpha(t_n) \le 1$$

So the sequence $\Delta_n \to 1$ as $n \to \infty$, but by the sequential convergence in Theorem 3.1., $d(x_{h_n}, x_{k_n}, a) \to 0$. So $t_n \to 0$ as $n \to \infty$. This completes the proof.

We now cite an example in support of the Theorem 3.3.

Example 3.4: Let $X = \mathbb{R}^+ \times \mathbb{R}^+$ and $d : X \to \mathbb{R}^+ \cup \{0\}$ where d be the 2-metric which expresses $d(x, y, a)$ as the area of the Euclidean triangle. Then (X, d) is a complete 2-metric space [14]. Let $f : X \to X$ be defined by $f(x, y) = \frac{1}{3}(x, y)$, for all $(x, y) \in X$.

Clearly f is a contraction map.Define $z_n = f(z_{n-1})$

where $z_0 = (x_0, y_0)$, $z_1 = \left(\frac{1}{3}x_0, \frac{1}{3}y_0\right)$,.... $z_n = \left(\frac{1}{3^n}x_0, \frac{1}{3^n}y_0\right)$, .

Let $p=(a,b)\in X\times X$ be any point. Then for $m>n$

$$d(z_m,z_n,p)=d\left(\left(\frac{1}{3^m}x_0,\frac{1}{3^m}y_0\right),\left(\frac{1}{3^n}x_0,\frac{1}{3^n}y_0\right),(a,b)\right)$$
$$=\left|\frac{1}{2}(bx_0-ay_0)\left(\frac{1}{3^n}-\frac{1}{3^m}\right)\right|$$

Similarly,

$$d(f(z_m),f(z_n),p)=d\left(\left(\frac{1}{3^{m+1}}x_0,\frac{1}{3^{m+1}}y_0\right),\left(\frac{1}{3^{n+1}}x_0,\frac{1}{3^{n+1}}y_0\right),(a,b)\right)$$
$$=\left|\frac{1}{2}\times\frac{1}{3}(bx_0-ay_0)\left(\frac{1}{3^n}-\frac{1}{3^m}\right)\right|$$

Let $\alpha:[0,\infty)\to[0,1)$ be such that

$$\alpha(t)=\exp(-2t),\quad t>0$$
$$=\alpha(0)\in[0,1)$$

Clearly α is a Geraghty function. Also it can be shown that all the condition (1) of Theorem 3.3. is satisfied. Now $z_n=\left(\frac{1}{3^n}x_0,\frac{1}{3^n}y_0\right)\to(0,0)\in X$ as $n\to\infty$ with $(0,0)$ is the unique fixed point of f. The converse part is clear. In fact $(0,0)$ is the unique fixed point for f where $z_n=\left(\frac{1}{3^n}x_0,\frac{1}{3^n}y_0\right)\to(0,0)\in X$ as $n\to\infty$. By Theorem 3.3. we can at once get a function $\alpha\in\mathcal{S}$ satisfying the condition (1) of Theorem 3.3.

This example supports the necessary and the sufficient condition as in Theorem 3.3.

Example 3.5: If f is not contraction then there is no grantee to ensure the existence of Geraghty function so that condition (1) of Theorem 3.3. is satisfied for f, though the function f have a unique fixed point. For this we cite an example. Let X and d be considered as shown in the previous example. Take $f(x,y)=3(x,y)$ and $z_0=(x_0,y_0)$ and $z_n=3^n z_0$ satisfying $z_n=f(z_{n-1})$. Now it can be seen that f is not a contraction mapping with a fixed point $(0,0)$. Also f does not satisfy the condition (1) of the Theorem 3.3. for any Geraghty function $\alpha\in\mathcal{S}$.

We may now apply the above result by an equivalent convergence form of an iterated sequence for finding unique fixed point. The following theorem is as follows:

Theorem 3.6: *Let $f:X\to X$ be a contraction map of a complete 2-metric space satisfying the condition*

$$d(f(x),f(y),a)\le\alpha(d(x,y,a))d(x,y,a),\ \text{for all } x,y,a\in X, \tag{2}$$

where $\alpha\in\mathcal{S}$. Then for any initial point x_0 the iterated sequence $\{x_n\}$ defined by $x_n=f(x_{n-1})$ for $n>0$, converges to unique fixed point x^* of f in X.

Proof: Taking any initial point x_0 and by applying Theorem 3.3., the proof follows.
Similarly one can also prove following iterative analogue results due to Rakotch [4] and Boyd-Wong [2] in a setting of 2-metric space.

Theorem 3.7: *(Analogue of Rakotch [4]) If $f:X\to X$ be a contraction map of a complete 2-metric space satisfying the condition*

$$d(f(x),f(y),a)\le\alpha(d(x,y,a))d(x,y,a),\ \text{for all } x,y,a\in X, \tag{3}$$

where $\alpha : \mathbb{R}^+ \to [0,1)$ and is monotone decreasing. Then for any choice of $x_0 \in X$, the sequence $\{x_n\}$ defined by $x_n = f(x_{n-1})$ for $n > 0$ converges to a unique fixed point x^* of f in X.

Proof: We can verify that such an α is clearly in $\mathcal{S}$ and the proof is clear.

Remark 3.8: *If* α *is monotonically increasing then theorem 3.7 also hold.*

Theorem 3.9: *(Analogue of Boyd-Wong[2]) If* $f : X \to X$ *be a contraction map of a complete* 2*-metric space satisfying the condition*

$$d(f(x), f(y), a) \le \alpha(d(x, y, a))d(x, y, a), \text{ for all } x, y, a \in X, \tag{4}$$

where $\alpha : \mathbb{R}^+ \to [0,1)$ is continuous, Then for any choice of $x_0 \in X$, the sequence $\{x_n\}$ where $x_n = f(x_{n-1})$ for $n > 0$, converges to a unique fixed point x^* of f in X.

Proof: Applying the Theorem 3.1. one can prove it easily.

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