International Journal of Mathematical Archive-5(7), 2014, 231-235

IJMA Available online through www.ijma.info ISSN 2229 – 5046

COMMON FIXED POINT ON MENGER SPACE ON FUZZY METRIC SPACE

M. L. L. Phanikanth[1], Venugoplam[2] and Vijaya Kumar*[3]

Asst. Professor in mathematics, college: Sri Vasavi Institute of Engineering and Technology, India.

(Received On: 08-07-14; Revised & Accepted On: 24-07-14)

ABSTRACT

In this paper, the concept of semi-compatibility and weak compatibility has been applied to prove common fixed point theorem on menger space in metric space,

Mathematics Subject Classification: *47H10, 54H25.*

Keywords: *Common fixed points,, menger space, metric space, Semi-compatible maps, Weak compatible maps, and Compatible maps.*

1. INTRODUCTION

K. Menger [20] introduced the notion of probabilistic metric spaces as the generalization of core notion of metric space. In this space one thinks of the distance between points as being probabilistic with $F_{x,y}(t)$ representing the probability that the distance between x and y is less than t. Recently, the study of fixed point theorems in probabilistic metric spaces is also a topic of recent interest and forms an active direction of research. Sehgal [30] made the first ever effort in this direction. Since then several authors have already studied fixed point and common fixed point theorems in PM spaces, we refer to [28, 32, 22, 17, 15, 7, 31, 13, 27, 21, 30] and others have recently initiated work along these lines.

The first ever notion of the compatible maps in Menger spaces appears to be made by Mishra [21]. Further Singh and Jain [32] generalized the notion of compatible maps by introducing the notion of weakly compatible maps. Several authors have studied and given many results in probabilistic settings which include [33, 16, 18, 19, 30]. The study of common fixed point of non-compatible mappings is also equally interesting which has been initiated by Pant [24, 25, 26, 23].

In 2008, Al-Thaga and Shahzad [2] weakened the notion of weakly compatible maps by introducing occasionally weakly compatible maps. It is worth to mention that every pair of commuting self-maps is weakly commuting, each pair of weakly commuting self-maps is compatible, each pair of compatible self-maps is weak compatible and each pair of weak compatible self-maps is occasionally weak compatible but the reverse is not always true. Many other results using owc on the theory of Menger PM-spaces exist in the literature, for more details, we refer the reader to Abbas and Rhoades [1], Al-Thaga and Shahzad [3], Bhatt *et al*. [5], Chandra [8], Bouhadjera *et al*. [6], Chouhan and Pant [9], Ciric *et al*. [10], and Vetro [34].

2. PRELIMINARIES

Definition 2.1: [29] A mapping Δ: $[0, 1] \times [0, 1] \to [0, 1]$ is called a triangular norm (shortly t-norm) if:
(i) $(a, 1) = a$, $\Delta(a, 0) = 0$,
(ii) $(a, b) = \Delta(b, a)$,
(iii) $(a, b) \le \Delta(c, d)$ for $a \le c$ and $b \le d$,
(iv) $\Delta(\Delta(a, b), c) = \Delta(a, \Delta(b, c))$ for all $a, b, c \in [0, 1]$.

Definition 2.2: [29] A mapping $F: R \to R^+$ is called distribution function if it is non-decreasing, left continuous with
$\inf \{F(t) : t \in R\} = 0$,

$\sup \{F(t) : t \in R\} = 1$.

Corresponding author: Vijaya Kumar*[3]

We shall denote by $\Im$ by the set of all distribution function while H will always denote the specific distribution function defined by

$$H(t)=\begin{cases}0 \ if \ \ t\le 0\\ 1 \ if \ \ t>0\end{cases}$$

If X is a non-empty set, $F: X\times X\to\Im$ is called the probabilistic distance on X and the value of F at $(x,y)\in X\times X$ is represented by $F_{x,y}$.

Definition 2.3: [29] The ordered pair (X, F) is called a PM-space if X is a non empty set and F is a probabilistic distance satisfying the following conditions: for all x, y, z $\in$ X and s, t $\geq$ 0,
(1) $F_{x,y}(t) = H(t)$ if and only if x = y,
(2) $F_{x,y}(t) = F_{y,x}(t)$,
(3) $F_{x,y}(t) = 1$ and $F_{y,z}(s) = 1$ then $F_{x,z}(t + s) = 1$.

The ordered triplet (X, F, Δ) is called a Menger space if (X, F) is a PM-space, is a t-norm and the following conditions holds: for all x, y, z $\in$ X and s, t $\geq$ 0

$$F_{x,z}(t + s) \geq \Delta(F_{x,y}(t), F_{y,z}(t))$$

Lemma 2.4: [30] Let (X, d) be a metric space. Define a mapping F: X × X → $\Im$ by

$$F_{x,y}(t) = H(t - d(x, y))$$

for all x, y$\in$X and t > 0. Then (X,F, min) is a Menger space, is called the induced Menger space by (X, d) and it is complete if (X, d) is complete.

Definition 2.5: [11] Let $F_1, F_2\in\Im$. The algebraic sum $F_1\oplus F_2$ of F_1 and F_2 is defined by

$$(F_1\oplus F_2)(t)=\sup_{t_1+t_2=t}\min\{F_1(t_1),F_2(t_2)\}$$

for all t$\in$R.

Obviously,

$$(F_1\oplus F_2)(2t)\ge\min\{F_1(t),F_2(t)\}$$

for all t $\geq$ 0.

Definition 2.6: [12, 13] A t-norm Δ is said to be of H-type if the family of functions $\{\Delta^m(t)\}_{m=1}^{\infty}$ is equicontinuous at t = 1, where

$$\Delta^1(t) = \Delta(t, t),\ \Delta^m(t) = \Delta(t, \Delta^{m-1}(t)),\ m = 1, 2, \dots, t\in [0, 1]$$

The t-norm Δ_M = min is a trivial example of t-norm of H-type, but there are t-norm of H-type with $\Delta = \Delta_M$.

Definition 2.7: [32] Two self maps A and B of a non empty set X are said to be weakly compatible it they commute at their coincidence points, i.e., Ax = Bx for some x$\in$X, then ABx = BAx

Definition 2.8: [2, 17] Two self maps A and B of a non empty set X are said to be occasionally weakly compatible if there is a point x$\in$X which is coincidence point of A and B at which A and B commute.

Example 2.9: Let X be a non empty set, where X = [0, 1). Let A, B: X → X be maps defined by

$$A(x)=\begin{cases}1 \ if \ \ 0\le x\le 1\\ 3x \ if \ \ x>1;\end{cases}$$

$$B(x)=\begin{cases}\dfrac{x+1}{2} \ if \ \ 0\le x\le 1\\ x^2 \ \ if \ \ x>1\end{cases}$$

Here 1 and 3 are two coincidence points of the self maps A and B. It is noted that AB(1) = A(1) = 1 = B(1) = BA(1) but AB(3) $\neq$ BA(3). Thus self maps A and B are owc but not weakly compatible.

3. MAIN THEOREM

Theorem 3.1: Let (X, F,Δ) be a Menger Space. Let A, B, S and T be self mappings on X and the pair (A, T) and (B, S) are each occasionally weakly compatible. If there exists that is

$$F_{(M(Ax,\, By,)}\ (t)^*{}_{M(Sx,\, Ty,)}\ (t)^*{}_{M(Sx,\, Ax},(\ t)^*{}_{M(By,\, Ty},(\ t) \geq 0, \tag{3.1}$$

holds for all x, y, ∈ X, and t > 0. Then, if the pairs (A, T) and (B, S) are each owc, there exists a unique point w ∈ X such that Aw = Tw = w and a unique point z ∈ X such that Bz = Sz = z. Moreover z = w so that there is a unique fixed point of A, B, S and T.

Proof: Since the pairs (A; T) and (B; S) are each occasionally weakly compatible, there exists points a, b ∈ X such that Aa = Ta, ATa = TAa, Bb = Sb and BSb = SBb.

Now we show that Aa = Bb..Let

$$\lim_{n\to+\infty} Ax_n = \lim_{n\to+\infty} Sx_n = \lim_{n\to+\infty} By_n = \lim_{n\to+\infty} Ty_n = z \tag{3.2}$$

for all t > 0. This implies Aa = Bb. Therefore Aa = Ta = Bb = Sb. Moreover if there exists another point u such that Au = Tu. Then using inequality (3.1), it follows that Au = Tu = Bb = Sb and Aa = Au. Hence v = Aa = Ta is the coincidence point of A and T. if v is a unique common fixed point of A and T. therefore ,there exist a point u ∈ X such that Su =z . Then by (3.1).we have, we get

$$F(_{M(Au,\, Byn},\ (t),\ ^*{}_{M(Su,\, Tyn},(\ t)^*{}_{M(Su,\, Au,}\ (t)\ ^*M_{(Byn,\, Tyn,)}\ (t)) \geq 0, \tag{3.3}$$

Which on making n→ +∞ reduces to

$$F_{(M(Au,\, z\ \ ,}\ (t)^*{}_{M(Su,\, z,}\ \ (\ t)^*{}_{M(Su,\, Au,}\ (\ t)^*{}_{M(z,\, z},\ (t)) \geq 0, \tag{3.4}$$

Or, equivalently,

$$F_{(M(Au,\, z},)\ \ (\ t),\ ^*1,\ _{M(Au,\, z,\,)}\ \ (t),\ ^*1 \geq 0, \tag{3.5}$$

Which gives M(Au, z,t) = 1 for all t >0, that is Au = z. Hence, Au =Su. Therefore, u is a point of coincidence of the pair (A,S).

Since T(x) is a closed subset of X, Then $\lim_{n\to+\infty} Ty_n$ = z. ∈ T(X). therefore, there exists a point w∈ X such that T w= z.

Now, we assert that Bw =z. indeed, again using (3.1), we have

$$F_{(M(Axn,\, Bw),}\ \ (t)^*{}_{M(Sxn,\, Tw)},\ (t)^*{}_{M(Sxn,\, Axn,)}\ \ \ (t)^*{}_{M(Bw,\, z,)}\ \ (t)) \geq 0. \tag{3.6}$$

$$F_{(M(z,\, Bw),}\ (t)^*{}_{M(z,\, z),}\ (t)^*{}_{M(z,\, z,)}\ (t)^*{}_{M(Bw,\, z),}\ (t)) \geq 0, \tag{3.7}$$

That is

$$F_{(M(z,\, Bw),}\ (t)^*\ 1^*\ 1^*,_{M(z,\, Bw),}\ (t)) \geq 0, \tag{3.8}$$

Implying there by that M(z,Bw,t) > 1 for all t>0. Hence Tw=Bw=Z , which shows that w is a point of coincidence of the point (B,T) since the pair (A,S) is weakly compatible and Au =Su, we deduce that Az =Asu=SAu =Sz.

Now, we assert that z is a common fixed point of the pair (A,S) using (3.1), we have

$$F_{(M(Az,\, Bw),}\ (t),^*{}_{M(Sz,\, Tw)},\ (t),^*{}_{M(Sz,\, Az)},\ \ (t),^*{}_{M(Bw,\, Tw),}\ (t)) \geq 0, \tag{3.9}$$

That is

$$F_{(M(Az,\, z),}\ (t),^*{}_{M(Az,z)},\ (t),^*\ 1,^*\ 1 \geq 0,$$

Hence M(Az, Z, t) =1 for all t >0 and therefore Az= z.

Now, using the notion of weakly compatibility of the pair (B,T) and (3.1) , we get Bz=z =Tz. Hence, z is a common fixed point of both the pair (A,S) and (B,T) uniqueness of z is an easy consequence of (3.1).

Example 3.2: Let X = [0, 6] with the metric d defined by d(x, y) = [x, y] and for each t ∈ [0, 1], define

$$F_{x,y}(t) = \begin{cases} \dfrac{t}{t+|x-y|} if\ t > 0 \\ 0 if\ t = 0; \end{cases}$$

for all x, y$\in$X. Then (X, F, Δ) be a Menger space, where Δ is a continuous t-norm.

Define $\phi(t) = kt,$

where $k \in (0,1), \beta = 1$ and the self maps A, B, S and T by for all x, y $\in$ X. Then (X, F, Δ) be a Menger space, where Δ is a continuous t-norm.

Define $\phi(t) = kt,$ where k $\in (0,1), \beta = 1$ and the self maps A, B, S and T by

$$A(x)=\begin{cases} 0 if\ 0 \le x \le 3 \\ 2 if\ 3 < x \le 6; \end{cases}$$

$$B(x)=\begin{cases} 0 if\ 0 \le x \le 3 \\ 4\ if\ 3 < x \le 6; \end{cases}$$

$$T(x)=\begin{cases} 0 if\ 0 \le x \le 3 \\ 3 if\ 3 < x \le 6; \end{cases}$$

$$S(x)=\begin{cases} 0 if\ 0 \le x \le 3 \\ 4 if\ 3 < x \le 6; \end{cases}$$

Then A, B, S and T satisfy all conditions of 3.1. Notice that AT(0) = A(0) = 0 = T(0) = TA(0) and
BS(0) = B(0) = 0 = S(0) = SB(0), the pairs (A, T) and (B, S) are owc. Hence 0 is the unique common fixed point of A, B, S and T.

REFERENCE

[1] Cho, Y. J., Sharma, B. K., & Sahu, D. R.(1995). Semi compatibility and fixed points. Math. Japon, 42(1), 91-98.

[2] Ghaler, S. (1963). 2-metricsche raume ihre topologische structur. Math. Nachr, 26, 115-148.

[3] Fisher, B. & Murty, P.P. (1987). Related fixed point theorem for two pair of mappings on two metric space. Kyungpook Math. J., 37, 343-347.

[4] Iseki, K. (1975). Fixed point theorems in 2-metric spaces. Math. Sem. Notes. Kobe. Univ., 3, 133-136.

[5] Naidu, S. V. R & Prasad, J. R. (1986). Fixed point theorems in 2-metric spaces. Indian J. Pure Appl. Math., 17, 974-993.

[6] Popa, V. (2002). Fixed points for non-surjective expansion mappings satisfying an implicit relation. Bul. Stiint. Univ. Baia Mare Ser. B. Fasc. Mat. Inform, 18(1), 105-108.

[7] Saliga, L. M. (1996). Fixed point theorems for non self maps in d-complete topological Spaces. Int. J. Math. Math. Sci., 19, 103-110.

[8] Sharma, B. K., & Sahu, D. R. & Bounias, M. & Bonaly, A. (1995). Fixed points for non-surjective expansions mappings. Int. J. Math. Math. Sci., 21(2), 277-288.

[9] Singh, B. & Jain, S.(2005). Semi compatibility and fixed point theorems in fuzzy metric spaces using implicit relation. Int. Math. Math. Sci., (16), 2617-2629.

[10] Singh, B. & Jain, S.(2004). Semi compatibility and fixed point theorems in Menger spaces. Journal of the Chungcheong Mathematical Society, 17(1), 1-17.

[11] Singh, B. & Jain, S. (2005). Semi compatibility, compatibility and fixed point theorems in fuzzy metric spaces. Journal of the Chungcheong Mathematical Society, 18(1), 1-23.

[12] Singh, B. & Jain, S. (2005). Semi compatibility and fixed point theorems in an unbounded D metric space. Int. J. Math. Math. Sci., 5, 789-801.

[13] Sharma, D. (2009). Common fixed point theorems for compatible mappings. International Journal of Theoretical & Applied Sciences, 1(2), 79-82.

Source of support: Nil, Conflict of interest: None Declared