International Journal of Mathematical Archive-5(8), 2014, 21-24
IJMA Available online through www.ijma.info ISSN 2229 – 5046

FIXED POINT THEOREMS IN GENERALIZED METRIC SPACES

D. P. Shukla*
Dept. of Mathematics/Computer Science, Govt. P.G. Science College, Rewa (M.P.), 486001, India.

(Received On: 14-07-14; Revised & Accepted On: 06-08-14)

ABSTRACT

In this paper we have established fixed point theorems in generalized metric space using self mapping and contraction mapping and the obtained result can be considered as an extention and generalization of some well known results of fixed point theorems in generalized metric spaces.

Key words: *Cauchy sequence, Fixed point, Generalized metric space.*

Mathematical Subject Classification: *47H10, 54H25.*

1. INTRODUCTION

The concept of generalized metric space introduced by Branciari [1] where the triangular inequality of a metric space has been replaced by a tetrahedral inequality, correspondingly, the Banach contraction principle was extended to the case of this generalized metric space. Under the situation, it is reasonable to consider if other important fixed point theorems can be obtained in such a space. Some works have already been done in this respect [3], [4] and [6].

In this paper we will show that unique fixed point under some general condition in a generalized metric space.

In this same way, we prove a fixed point theorem and common fixed point theorems for the mappings satisfying different types of contractive conditions in general metric spaces

If (X, d) is a complete metric space and $T: X \to X$ is a contraction mapping i.e. $d(Tx, Ty) \leq \alpha\, d(x, y)$, $0 < \alpha < 1$ for all $x, y \in X$ then the widely known Banach is contraction mapping principle tell that T has a unique fixed point in X.

2. PRELIMINARY NOTES

Let R^+ denote the set of all non-negative real numbers and N denote the set of all positive integers.

Definition 2.1: Let X be a non empty set and $d: X^2 \to R^+$ be a mapping such that for all $x, y \in X$ and for all distinct points $z, w \in X$ each of them different from x and y, one has the following:
(i) $d(x, y) = 0$ if and only if $x = y$,
(ii) $d(x, y) = d(y, x)$
(iii) $d(x, y) \leq d(x, z) + d(z, w) + d(w, y)$ (Tetrahedral inequality)

Then d is called a generalized metric and (X, d) is a generalized metric space (or shortly g.m.s)

Any metric space is a g.m.s. but the converse is not true [1]

We present an example to show that not every generalized metric on a set X is a metric on X.

Example 2.2 [2]: Let $X = \{t, 2t, 3t, 4t, 5t\}$ with $t > 0$ be a constant, and we defined $d: X \times X \to [0, \infty]$ by
(i) $d(x, x) = 0$ for all $x \in X$
(ii) $d(x, y) = d(y, x)$, for all $x, y \in X$
(iii) $d(t, 2t) = 3\gamma$
(iv) $d(t, 3t) = d(2t, 3t) = \gamma$,
(v) $d(t, 4t) = d(2t, 4t) = d(3t, 4t) = 2\gamma$,
(vi) $d(t, 5t) = d(2t, 5t) = d(3t, 5t) = d(4t, 5t) = 3/2\, \gamma$,

*Corresponding author: P V H S Sai Kumar**

where $\gamma > 0$ is a constant. Then let (X, d) be a generalized metric space but if is not a metric space, because

$$d(t, 2t) = 3\gamma > d(t, 3t) + d(3t, 2t) = 2\gamma,$$

Example 2.3 [5]: Let us define

$X = \{1/n: n = 1, 2.....\} \cup \{0, 2\}$,

$$d : X\times X\to R^{+}: d(x, y) \begin{cases} 0, \text{ for } x = y \\ 1/n, \text{ for } x \in \{0, 2\}, y = 1/n \\ 1/n, \text{ for } x = 1/n, y \in \{0, 2\} \\ 1, \text{ otherwise} \end{cases}$$

Then if is easy to see that (X, d) is a generalized metric space but (X, d) is not a standard metric space because it lacks the tringular property

$$1 = d(½, ⅓) > d(½, 0) + d(0, ⅓) = ½ + ⅓ = 5/6$$

Definition 2.4: A sequence $\{x_n\}$ in a generalized metric space (X, d) is said to be convergent with limit $x\in X$ if for every $\in > 0$ there exists $n_0 \in N$ such that $d(x_n, x) < \in$ for all $n > n_0$, it follows that if the sequence $\{x_n\}$ converges to x and x is the limit of $\{x_n\}$ we denote by $\lim x_n = x$ as $n \to\infty$.

Definition 2.5: A sequence $\{x_n\}$ is a generalized metric space (X, d) is said to be Cauchy sequence if for every $\in > 0$, there exists $n_0 \in N$ such that $d(x_n, x_m) < \in$ for all $n, m \geq n_0$.

Definition 2.6: A generalized metric space (X, d) is said to be complete if every Cauchy sequence is convergent in X.

Proposition 2.7: If a sequence is convergent is g.m.s. then it is a Cauchy sequence.

Proposition 2.8: Limit of a sequence in a g.m.s., if exists, is unique.

Proposition 2.9: A sequence $\{x_n\}$ in a g.m.s. converges to x then every subsequence of $\{x_n\}$ also converges to the same limit x.

3. MAIN RESULTS

Theorem 3.1: Let (X, d) be a complete generalized metric space and $T: X\to X$ satisfy the condition

$$d(Tx, Ty) \leq \alpha [d(x, Tx) + d(y, Tx) + d(y, Ty) + d(x, y)] \qquad \text{(A)}$$

for all $x, y \in X$ where $0 < \alpha < ¼$, then, T has a unique fixed point in X.

Proof: Let x_0 be any arbitrary point in X; if x_0 is not fixed point of T, Define sequence $\{x_n\}$ we can choose $x_1 \in X$ such that

$Tx_0 = x_1$
$Tx_1 = x_2$
$Tx_{n+1} = x_n$
$Tx_n = x_{n+1}$
if x_n is not a fixed point of T then

Now by (A), we have

$$d(x_1, x_2) = d(Tx_0, Tx_1) \leq \alpha [d(x_0, Tx_0) + d(x_1, Tx_0) + d(x_1,Tx_1) + d(x_0,x_1)]$$
$$d(x_1, x_2) \leq \alpha [d(x_0, x_1) + d(x_1, x_1) + d(x_1,x_2) + d(x_0, x_1)]$$
$$d(x_1, x_2) - \alpha d(x_1, x_2) \leq \alpha d(x_0, x_1) + \alpha d(x_0,x_1)$$
$$(1-\alpha) d(x_1, x_2) \leq 2\alpha\, d(x_0, x_1)$$
$$d(x_1, x_2) \leq \frac{2\alpha}{1-\alpha} d(x_0, x_1)$$

where $r = \dfrac{2\alpha}{1-\alpha} < 1$

$$r < 1$$
$$d(x_1, x_2) \leq rd(x_0, x_1) \qquad (1)$$
$$d(x_2, x_3) \leq rd(x_1, x_2)$$
$$\leq r[rd(x_0, x_1)]$$
$$d(x_2, x_3) \leq r^2 d(x_0, x_1)$$
$$d(x_3, x_4) \leq r^3 d(x_0, x_1)$$

In general, for any positive integer n, we have

$d(x_n, x_{n+1}) \le rd(x_{n-1}, x_n)$

$d(x_n, x_{n+1}) \le r^n d(x_0, x_1)$ (2)

We claim that the sequence $\{x_n\}$ is a Cauchy sequence in X

For m > n, we have

$d(x_n, x_m) \le d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_m)$

$d(x_n, x_m) \le r^n d(x_0, x_1) + r^{n+1} d(x_0, x_1) + d(x_{n+2}, x_m)$

$d(x_n, x_m) = (r^n + r^{n+1}) d(x_0, x_1) + d(x_{n+2}, x_m)$

. . . .

. . . .

$d(x_n, x_m) \le (r^n + r^{n+1} + ...+ r^{m-2}) d(x_0, x_1) + d(x_{m-1}, x_m)$

$d(x_n, x_m) \le (r^n + r^{n+1} + ...+ r^{m-2} + r^{m-1}) d(x_0, x_1)$

$$d(x_n, x_m) \le \frac{r^n}{1-r} d(x_0, x_1) \tag{3}$$

As n, m $\to\infty$ in equation (3), we have $d(x_n, x_m) \to 0$

Thus $\{x_n\}$ is a Cauchy sequence in the complete generalized metric space, there exists a point $u \in X$ such that $\{x_n\} \to u$.

Now

$d(u, Tu) \le d(u, x_n) + d(x_n, x_{n+1}) + d(x_{n+1}, Tu)$

$d(u, Tu) \le d(u, x_n) + d(x_n, x_{n+1}) + d(Tx_n, Tu)$

By (A), we have

$d(u, Tu) \le d(u, x_n) + d(x_n, x_{n+1}) + \alpha[d(x_n, Tx_n) + d(u, Tx_n) + d(u, Tu) + d(x_n, u)]$

$d(u, Tu) \le d(u, x_n) + d(x_n, x_{n+1}) + \alpha d(x_n, Tx_n) + \alpha d(u, Tx_n) + \alpha d(u, Tu) + \alpha d(x_n, u)$

$d(u, Tu) \le d(u, x_n) + d(x_n, x_{n+1}) + \alpha d(x_n, x_{n+1}) + \alpha d(u, xn_{+1}) + \alpha d(u, Tu) + \alpha d(x_n, u)$

$(1-\alpha) d(u, Tu) \le d(u, x_n) + d(x_n, x_{n+1}) + \alpha d(x_n, x_{n+1}) + \alpha d(u, x_{n+1}) + \alpha d(x_n, u)$

$(1-\alpha) d(u, Tu) \to 0$ as $n\to\infty$

Thus $d(u, Tu) \to 0$

$d(Tu, u) \to 0$

$Tu = u$

Hence u is fixed point of T in X

Uniqueness of u

In order to prove that u is the unique fixed point of T.

Let u, v be two fixed points of T, then

$d(u, v) = d(Tu, Tv)$

By (A), we have

$d(u,v) \le \alpha[d(u,Tu) + d(v, Tu) + d(v, Tv) + d(u, v)]$

$d(u,v) \le \alpha[d(u,u) + d(v, u) + d(v, v) + d(u, v)]$

$d(u,v) \le 2\alpha(u,v)$

$\Rightarrow u = v$ as $0 < \alpha < ¼$

Hence uniqueness of fixed point follows

Theorem 3.2: Let (X, d) be a complete generalized metric space and T: $X\to X$ satisfying the condition (A)

$(Tx, Ty) \le \alpha[d(x,Tx) + d(y, Tx) + d(y, Ty) + d(x, y)]$

for all $x, y \in X$ where $0 < \alpha < ¼$, $\{T^n(x)\}$ has a subsequence $\{T^{n_k}(x)\}$ with $\lim_{k\to\infty}\{T^{n_k}(x)\} = u \in X$. Then u is the unique fixed point of T in X.

Proof: we have

$$d(u, Tu) \le d(u, T^{n_k}(x)) + d(T^{n_k}(x), T^{n_k+1}(x)) + d(T^{n_k+1}(x), Tu) \tag{4}$$

Now

$$d(T^{n_k+1}(x), Tu) = d(T(T^{n_k}(x), Tu)$$
$$\leq \alpha[d T^{n_k}(x), T^{n_k+1}(x)) + d(u, T^{n_k+1}(x)) + d(u, Tu) + d(T^{n_k}(x), u)]$$

Then from (4),

$$d(u, Tu) \leq d(u, T^{n_k}(x)) + d(T^{n_k}(x), T^{n_k+1}(x)) + \alpha[d(T^{n_k}(x)), T^{n_k+1}(x)) + d(u, T^{n_k+1}(x)) + d(u,Tu) + d(T^{n_k}(x))\ u)]$$

Taking limit as $k \to \infty$ on both sides of the inequality, we get
$\Rightarrow d(u, Tu) = 0$

So Tu = u and uniqueness follows very immediate.

ACKNOWLEDGMENTS

The author is thankful to the P. Das and L.K. Day for his valuable suggestions which improved the presentation of the paper.

REFERENCES

1. Branciari, A: "A fixed point theorem of Banach-caccioppoli type as a class of generalized metric spaces", Publ. Math. Debrecen, 57: 1-2 (2000) 31-37.
2. Chen Chi-Miag: "Common fixed point theorem in complete Generalized metric spaces" J. of Applied Mathematics (2012) 01-14.
3. DAS Pratulananda: "A fixed point theorem on a class of generalized metric spaces, Korean. J. Math. Sc. 9: 1 (2000), 29-33.
4. DAS P. and DAY L.K.: "A fixed point theorem in a generalized metric space", Soochow Journal of Mathematics 33 (2007) 33-39.
5. DAS P and DAY L.K.: "Fixed point of contractive mappings in generalized metric space", Math Slovaca 59 No. 4 (2009) 499-504.
6. Lahiri, B.K. and DAS. P: "Fixed point of a Ljubomir ciric quasi-contraction mapping in a generalized metric space", Publ. Maths Debrecen 61: 3-4 (2002) 589-594.

Source of support: Nil, Conflict of interest: None Declared