International Journal of Mathematical Archive-5(8), 2014, 92-108
Available online through www.ijma.info ISSN 2229 – 5046

# F – HARMONIC MEAN GRAPHS

**A. Durai Baskar and S. Arockiaraj***

*Department of Mathematics, Mepco Schlenk Engineering College*
*Mepco Engineering College (PO) – 626 005, Sivakasi, Tamil Nadu, India.*

*(Received On: 03-02-14; Revised & Accepted On: 28-02-14)*

***ABSTRACT***

*A function f is called a F –Harmonic mean labeling of a graph G(V,E) with p vertices and q edges if* $f:V(G)\to\{1,2,3,....,q+1\}$ *is injective and the induced function* $f^*$ *defined as* $f^*(uv)=\left\lfloor \frac{2f(u)f(v)}{f(u)+f(v)}\right\rfloor$, *for all* $uv\in E(G)$, *is bijective. A graph that admits an F – Harmonic mean labeling is called an F – Harmonic mean graph. In this paper, we have discussed the F – Harmonic meanness of a path* $P_n$, *the star graph* $S_n$, *union of a path and a cycle, the graph* $P_n\odot S_m$, *the graph* $C_n\odot S_1$, *the graph* $TW(P_n)$, *the graph* $(P_n;S_m)$, *the middle graph of a path, the graph* $T(n,k)$, *the H-graph, the graph* $S(P_n\odot S_1)$ *and the arbitrary subdivision of* $S_3$.

***Keywords:*** *F – Harmonic mean labelling, F – Harmonic mean graph.*

***2010 Mathematics Subject Classification:*** *05C78.*

## 1. INTRODUCTION

Throughout this paper, by a graph we mean a finite, undirected and simple graph. Let $G(V,E)$ be a graph with p vertices and q edges. For notations and terminology, we follow [4]. For a detailed survey on graph labeling, we refer [3].

Path on n vertices is denoted by $P_n$ and a cycle on n vertices is denoted by $C_n$. A star graph $S_n$ is the complete bipartite graph $K_{1,n}$. The graph $G\odot S_m$ is obtained from G by attaching m pendant vertices to each vertex of G. A Twig $TW(P_n),\ n\geq 3$ is a graph obtained from a path by attaching exactly two pendant vertices to each internal vertices of the path. If $v_1^{(i)},v_2^{(i)},v_3^{(i)},...,v_{m+1}^{(i)}$ and $u_1,u_2,...,u_n$ be the vertices of the star graph $S_m$ and the path $P_n$, then the graph $(P_n;S_m)$ is obtained from n copies of $S_m$ and the path $P_n$ by joining $u_i$ with the central vertex $v_1^{(i)}$ of the $i^{th}$ copy of $S_m$ by means of an edge, for $1\leq i\leq n$.

The middle graph M(G) of a graph G is the graph whose vertex set is $\{v:v\in V(G)\}\cup\{e:e\in E(G)\}$ and the edge set is $\{e_1e_2:e_1,e_2\in E(G)\ and\ e_1\ and\ e_2\ are\ adjacent\ edges\ of\ G\}\cup\{ve:v\in V(G),e\in E(G)\ and\ e\ is\ incident\ with\ v\}$ The graph Tadpoles $T(n,k)$ is obtained by identifying a vertex of the cycle $C_n$ to an end vertex of the path $P_k$. The H-graph is obtained from two paths $u_1,u_2,...,u_n\ and\ v_1,v_2,...,v_n$ of equal length by joining an edge $u_{\frac{n+1}{2}}v_{\frac{n+1}{2}}$ when n is odd and

*Corresponding author: S. Arockiaraj**
*Department of Mathematics, Mepco Schlenk Engineering College*
*Mepco Engineering College (PO) – 626 005, Sivakasi, Tamil Nadu, India.*

$u_{\frac{n+2}{2}} v_{\frac{n}{2}}$ when n is even. A subdivision of a graph G, denoted by S(G), is a graph obtained by subdividing each edge of G by a vertex. An arbitrary subdivision of a graph G is a graph obtained from G by a sequence of elementary subdivisions forming edges into paths through new vertices of degree 2.

The concept of geometric mean labeling was introduced and studied by A. Durai Baskar *et al*. in [1, 2]. Motivated by the works of so many authors in the area of graph labeling, we introduced a new type of labeling called F - Harmonic mean labeling.

A function f is called a F –Harmonic mean labeling of a graph G(V,E) with p vertices and q edges if $f : V(G) \to \{1,2,3,....,q+1\}$ is injective and the induced function $f^*$ defined as $f^*(uv) = \left\lfloor \frac{2f(u)f(v)}{f(u)+f(v)} \right\rfloor$, $for\ all\ uv \in E(G),$ is bijective. A graph that admits an F – Harmonic mean labeling is called an F – Harmonic mean graph.

In [5], S. S. Sandhya et al. defined the harmonic mean labeling as follows:

A graph $G = (V, E)$ with p vertices and q edges is called a Harmonic mean graph if it is possible to label the vertices $x \in V$ with distinct labels f(x) from $1,2,...,q+1$ in such a way that when each edge $e = uv$ is labelled with $f(uv) = \left\lceil \frac{2f(u)f(v)}{f(u)+f(v)} \right\rceil$ $or$ $\left\lfloor \frac{2f(u)f(v)}{f(u)+f(v)} \right\rfloor$ then the edge labels are distinct. In this case f is called Harmonic mean labelling of G.

In the above definition, the readers will get some confusion in finding the edge labels which edge is assigned by flooring function and which edge is assigned by ceiling function.

In [5], they have given the harmonic mean labeling of the graph $C_5 \cup P_4$ as in the figure 1.

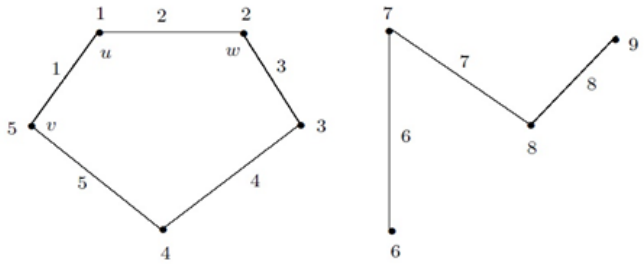


Figure 1

From the above figure, for the edge uv, they have used flooring function $\left\lfloor \frac{2f(u)f(v)}{f(u)+f(v)} \right\rfloor$ and for the edge vw, they have used ceiling function $\left\lceil \frac{2f(u)f(v)}{f(u)+f(v)} \right\rceil$ for fulfilling their requirement. To avoid the confusion of assigning the edge labels in their definition, we just consider the flooring function $\left\lfloor \frac{2f(u)f(v)}{f(u)+f(v)} \right\rfloor$ for our discussion. Based on our definition, the F - Harmonic mean labeling of the same graph $C_5 \cup P_4$ is given in figure 2.

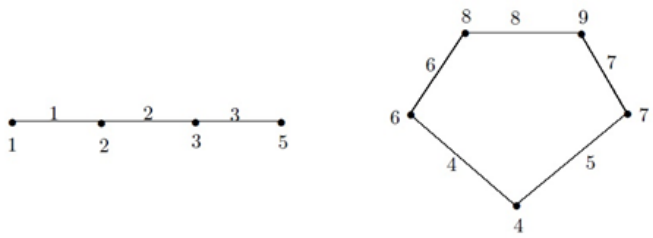


Figure 2

In this paper, we have discussed the F – Harmonic meanness of a path $P_n$, the star graph $S_n$ $for\ n \le 5$, the graph $P_n \odot S_m$ $for\ n \ge 1\ and\ m \le 3$, the graph $C_n \odot S_1$ $for\ n \ge 3,$ the graph $TW(P_n)$ $for\ n \ge 3,$ the graph $(P_n; S_m)$ $for\ n \ge 1\ and\ m \le 3,$ the middle graph of a path, the graph $T(n,k)$ $for\ n \ne 2,$ the H-graph, the graph $C_n \cup P_m$ $for\ n \ne 3\ and\ m \ne 2$, the graph $S(P_n \odot S_1)$ and the arbitrary subdivision of $S_3$.

## 2. MAIN RESULTS

**Theorem 2.1:** Every path $P_n$ is an F – Harmonic mean graph.

**Proof:** Let $v_1, v_2, ..., v_n$ be the vertices of the path $P_n$. We define $f : V(G) \to \{1,2,3,...,n\}$ as $f(v_i) = i,\ for\ 1 \le i \le n$ and the induced edge labeling is $f^*(v_i v_{i+1}) = i,\ for\ 1 \le i \le n-1.$ Hence f is an F – Harmonic mean labelling of the path $P_n$. Thus the path $P_n$ is an F – Harmonic mean graph.

An F – Harmonic mean labelling of $P_6$ is shown in figure 3.

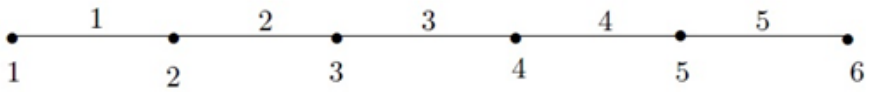


Figure 3

**Theorem 2.2:** union of any two trees is not an F – Harmonic mean graph.

**Proof:** Let G be the union of two trees S and T, then $|V(G)| = |V(S)| + |V(T)|$ *and* $|E(G)| = |E(S)| + |E(T)| = |V(S)| + |V(T)| - 2$. Since $|V(G)| > |E(G)| + 1$, a F – Harmonic mean labelling does not exist on V(G).

**Corollary 2.3:** Any forest is not an F – Harmonic mean graph.

**Proof:** By the above theorem 2.2, the result follows.

**Theorem 2.4:** The star graph $S_n$ is a F – Harmonic mean graph only when $n \le 5$.

**Proof:** The number of vertices and edges of $S_n$ are n+1 and n respectively. If f is a F – Harmonic mean labelling of $S_n$, then it is a bijective function from $V(S_n)$ *to* $\{1,2,...,n+1\}$. By the definition, 1 should be the label for a pendant vertex.

As we have to label 2 for an edge, the central vertex of $S_n$ may be labelled as either 2 or 3 or 4 or 5. 2 cannot be a label for the central vertex in case of $n \ge 3$, since two of the pendant vertices of $S_n$ are to be labelled as 3 and 4. When $n \ge 4$, 3 cannot be a label for the central vertex since two of the pendant vertices having the labels 4 and 5. When $n \ge 5$, 4 cannot be a label for the central vertex since two of the pendant vertices having the labels 5 and 6. When $n \ge 6$, by the above arguments, none of 2, 3, 4 are the labels of the central vertex. If the central vertex is labelled by 6 or more, then 2 cannot be an edge label. So 5 only the possible labels for the central vertex. If 5 is labelled in the central vertex, then at least two edges may have the same labels.

The F – Harmonic mean labelling of $S_n, n \le 5$ are as follows in figure 4.

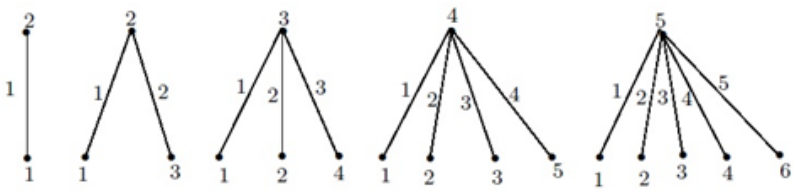

Figure 4

**Theorem 2.5:** If G is a graph with every vertices of degree at least 2, then G is not a F – Harmonic mean graph.

**Proof:** Let G be a graph with $\delta(G) \geq 2$. Let v be a vertex with degree $\delta(G)$ which is labelled by 1.

Let $v_1$ *and* $v_2$ be the vertices adjacent to v whose labels are m and n. By the definition of F – Harmonic mean labelling, the edge labels of $vv_1$ *and* $vv_2$ are one and the same as $\left\lfloor \frac{2m}{m+1} \right\rfloor$ *is equal to* $\left\lfloor \frac{2n}{n+1} \right\rfloor$. Hence the result follows.

**Corollary 2.6:** Every cycle $C_n$ is not an F – Harmonic mean graph, $n \geq 3$.

**Proof:** By the above theorem 2.5, the result follows.

**Corollary 2.7:** Every complete graph $K_n$ is not an F – Harmonic mean graph, for $n \geq 3$ .

**Proof:** By the above theorem 2.5, the result follows.

**Corollary 2.8:** The Harary graph $H_{m,n}$ is not an F – Harmonic mean graph, for any positive integers $m, n \geq 2$.

**Proof:** By the above theorem 2.5, the result follows.

**Theorem 2.9:** The graph $P_m \cup C_n$ is a F – Harmonic mean graph for $m \neq 2$ *and* $n \neq 3$.

**Proof:** Let $u_1, u_2, ..., u_m$ *and* $v_1, v_2, ..., v_n$ be the vertices of the path $P_m$ and the cycle $C_n$ respectively.

We define $f : V(P_m \cup C_n) \to \{1, 2, 3, ..., m+n\}$ as follows

$$f(u_i) = i, \ for\ 1 \leq i \leq m-1,$$

$$f(u_m) = m+1 \ \ and$$

$$f(v_i) = \begin{cases} m+n+2-2i & 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor \\ m & i = \left\lfloor \frac{n}{2} \right\rfloor + 1 \\ m-n-1+2i & \left\lfloor \frac{n}{2} \right\rfloor + 2 \leq i \leq n \end{cases} .$$

The induced edge labeling is obtained as follows

$$f^*(u_i u_{i+1}) = i, \ for\ 1 \leq i \leq m-1,$$

$$f^*(v_i v_{i+1}) = \begin{cases} m+n-2i & 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor \\ m & i = \left\lfloor \frac{n}{2} \right\rfloor + 1 \ and\ n\ is\ odd \\ m+1 & i = \left\lfloor \frac{n}{2} \right\rfloor + 1 \ and\ n\ is\ even \\ m-n-1+2i & \left\lfloor \frac{n}{2} \right\rfloor + 2 \leq i \leq n-1 \end{cases}$$

$$and\ \ f^*(v_n v_1) = m+n-1.$$

Hence f is an F – Harmonic mean labelling of $P_m \cup C_n$. Thus the graph $P_m \cup C_n$ is an F – Harmonic mean graph for $m \neq 2$ *and* $n \neq 3$.

An F – Harmonic mean labelling of $P_6 \cup C_7$ is shown in figure 5.

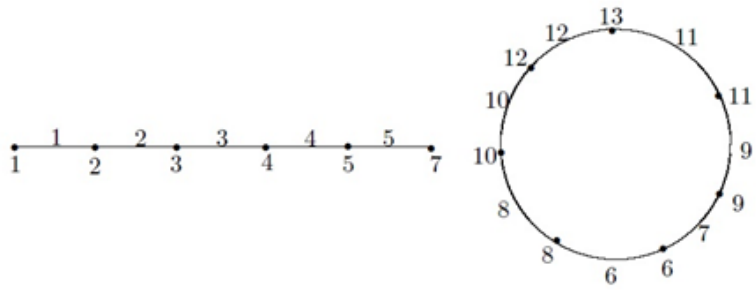


Figure 5

**Theorem 2.10:** ***For a T – graph*** $T_n$, $T_n \cup C_m$ ***is a F – Harmonic mean graph for*** $n \geq 4$ *and* $m \geq 3$.

**Proof:** Let $u_1, u_2, \ldots, u_{n-1}$ be the vertices of the path $P_{n-1}$ and $u_n$ be the pendant vertex identified with $u_2$. Let $v_1, v_2, \ldots, v_m$ be the vertices of the cycle $C_m$.

$$\therefore V(T_n \cup C_m) = V(C_m) \cup V(P_n) \cup \{u_n\} \text{ and}$$
$$E(T_n \cup C_m) = E(C_m) \cup E(P_n) \cup \{u_2 u_n\}.$$

We define $f : V(P_m \cup C_n) \to \{1, 2, 3, \ldots, m+n\}$ as follows

$$f(u_i) = i, \text{ for } 1 \leq i \leq n-2,$$
$$f(u_{n-1}) = n-1, f(u_n) = 1 \text{ and}$$
$$f(v_i) = \begin{cases} m+n+2-2i & 1 \leq i \leq \left\lfloor \frac{m}{2} \right\rfloor \\ n & i = \left\lfloor \frac{m}{2} \right\rfloor + 1 \\ n-m-1+2i & \left\lfloor \frac{m}{2} \right\rfloor + 2 \leq i \leq m \end{cases}.$$

The induced edge labeling is obtained as follows

$$f^*(u_i u_{i+1}) = i+1, \text{ fo } 1 \leq i \leq n-2, f^*(u_2 u_n) = 1,$$
$$f^*(v_i v_{i+1}) = \begin{cases} m+n-2i & 1 \leq i \leq \left\lfloor \frac{m}{2} \right\rfloor \\ n & i = \left\lfloor \frac{m}{2} \right\rfloor + 1 \text{ and } m \text{ is odd} \\ n+1 & i = \left\lfloor \frac{m}{2} \right\rfloor + 1 \text{ and } m \text{ is even} \\ n-m-1+2i & \left\lfloor \frac{m}{2} \right\rfloor + 2 \leq i \leq m-1 \end{cases}$$
$$\text{and } f^*(v_1 v_m) = m+n-1.$$

Hence f is an F – Harmonic mean labelling of $T_n \cup C_m$. Thus the graph $T_n \cup C_m$ is an F – Harmonic mean graph for $n \geq 4$ *and* $m \geq 3$.

An F – Harmonic mean labelling of $T_7 \cup C_6$ is shown in figure 6.

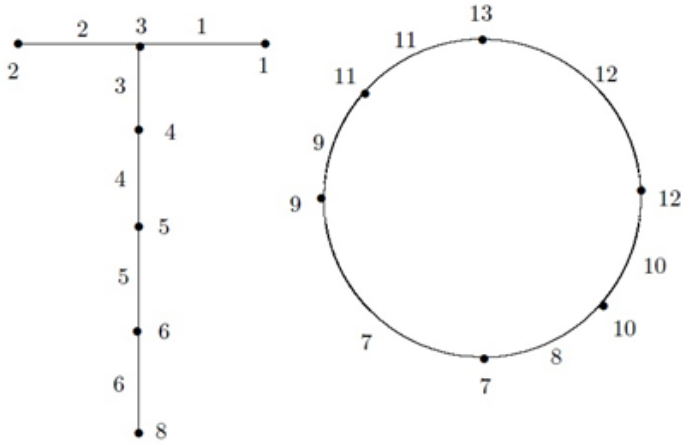


Figure 6

**Theorem 2.11:** The graph $P_n \odot S_m$ is a F – Harmonic mean graph for $n \geq 1$ *and* $m \leq 3$.

**Proof:** Let $u_1, u_2, \ldots, u_n$ be the vertices of the path $P_n$ and $v_1^{(i)}, v_2^{(i)}, \ldots, v_m^{(i)}$ be the pendant vertices at each vertex $u_i$ of the path $P_n$ , for $1 \leq i \leq n$.

**Case-(i):** $m = 1$

We define $f : V(P_n \odot S_1) \to \{1, 2, 3, \ldots, 2n\}$ as follows

$$f(u_i) = 2i, \text{ for } 1 \leq i \leq n \text{ and } f(v_1^{(i)}) = 2i - 1, \text{ for } 1 \leq i \leq n.$$

The induced edge labeling is obtained as follows

$$f^*(u_i u_{i+1}) = 2i, \text{ for } 1 \leq i \leq n-1 \text{ and } f^*(v_1^{(i)} u_i) = 2i - 1, \text{ for } 1 \leq i \leq n.$$

**Case-(ii):** $m = 2$

We define $f : V(P_n \odot S_2) \to \{1, 2, 3, \ldots, 3n\}$ as follows

$$f(u_1) = 3, \ f(u_i) = 3i - 1, \text{ for } 2 \leq i \leq n, \ f(v_1^{(i)}) = 3i - 2, \text{ for } 1 \leq i \leq n,$$
$$f(v_2^{(1)}) = 2 \text{ and } f(v_2^{(i)}) = 3i, \text{ for } 2 \leq i \leq n.$$

The induced edge labeling is obtained as follows

$$f^*(u_i u_{i+1}) = 3i, \text{ for } 1 \leq i \leq n-1, \ f^*(v_1^{(i)} u_i) = 3i - 2, \text{ for } 1 \leq i \leq n \text{ and}$$
$$f^*(v_2^{(i)} u_i) = 3i - 1, \text{ for } 1 \leq i \leq n.$$

**Case-(iii):** $m = 3$

We define $f : V(P_n \odot S_3) \to \{1, 2, 3, \ldots, 4n\}$ as follows

$$f(u_i) = 4i - 1, \text{ for } 1 \leq i \leq n, \ f(v_1^{(i)}) = 4i - 3, \text{ for } 1 \leq i \leq n,$$
$$f(v_2^{(i)}) = 4i - 2, \text{ for } 1 \leq i \leq n \text{ and } f(v_3^{(i)}) = 4i, \text{ for } 1 \leq i \leq n.$$

*© 2014, IJMA. All Rights Reserved* 

The induced edge labeling is obtained as follows

$$f^*(u_iu_{i+1}) = 4i,\ for\ 1 \le i \le n-1,\ f^*(v_1^{(i)}u_i) = 4i-3,\ for\ 1 \le i \le n,$$
$$f^*(v_2^{(i)}u_i) = 4i-2,\ for\ 1 \le i \le n\ and\ f^*(v_3^{(i)}u_i) = 4i-1,\ for\ 1 \le i \le n.$$

Hence f is an F – Harmonic mean labelling of $P_n \odot S_m$. Thus the graph $P_n \odot S_m$ is an F – Harmonic mean graph for $n \ge 1\ and\ m \le 3.$

The F – Harmonic mean labelling of $P_5 \odot S_1, P_6 \odot S_2\ and\ P_4 \odot S_3$ is shown in figure 7.

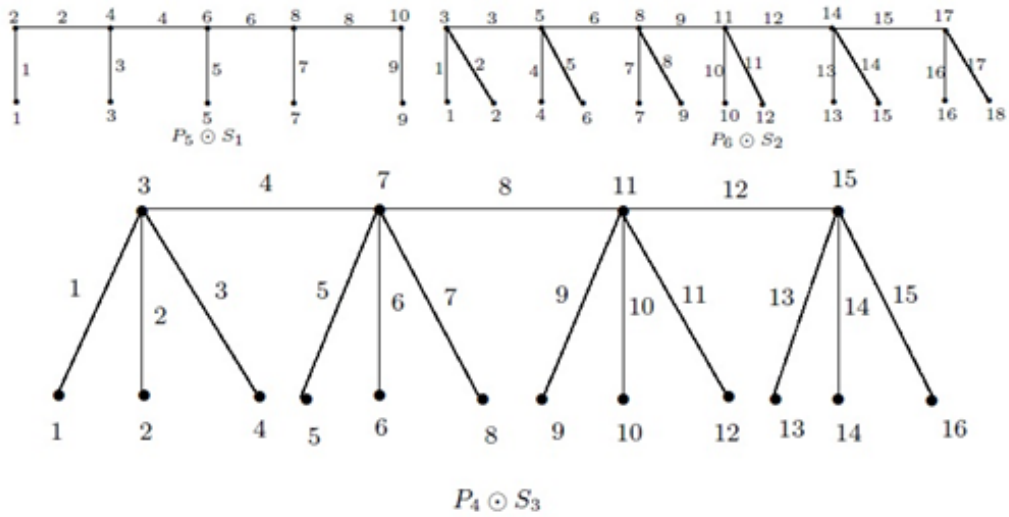


Figure 7

**Theorem 2.12:** $C_n \odot K_1$ is a F – Harmonic mean graph for $n \ge 3$.

**Proof:** Let $v_1, v_2, \ldots, v_n$ be the vertices of the cycle $C_n$ and let $u_i$ be the pendant vertices attached at each $v_i,\ for\ 1 \le i \le n.$

We define $f : V(C_n \odot K_1) \to \{1,2,3,\ldots,2n+1\}$ as follows

$$f(v_i) = \begin{cases} 2 & i = 1 \\ 2i+1 & 2 \le i \le n \end{cases}\ and\ \ f(u_i) = \begin{cases} 1 & i = 1 \\ 2i & 2 \le i \le n \end{cases}.$$

The induced edge labeling is obtained as follows

$$f^*(v_iv_{i+1}) = \begin{cases} 2 & i = 1 \\ 2i+1 & 2 \le i \le n-1 \end{cases},\ f^*(v_nv_1) = 3\ and$$
$$f^*(u_iv_i) = \begin{cases} 1 & i = 1 \\ 2i & 2 \le i \le n \end{cases}.$$

Hence f is an F – Harmonic mean labelling of $C_n \odot K_1$. Thus the graph $C_n \odot K_1$ is an F – Harmonic mean graph for $n \ge 3.$

An F – Harmonic mean labelling of $C_9 \odot K_1$ is shown in figure 8.

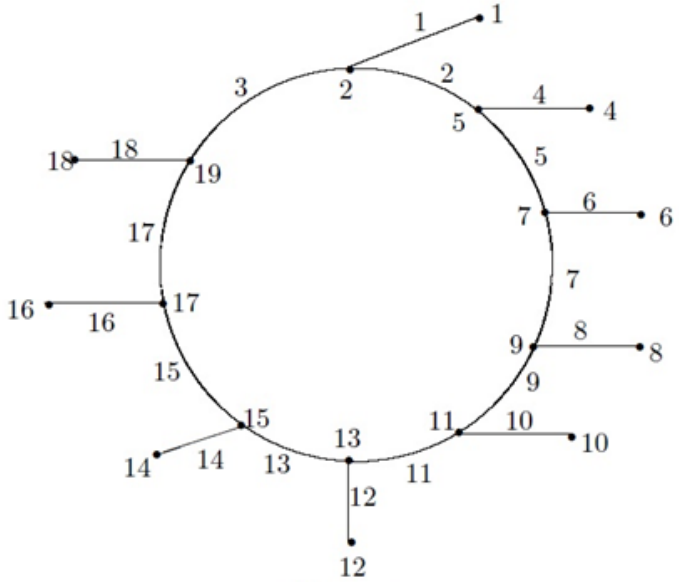


Figure 8

**Theorem 2.13:** The graph $TW(P_n)$ is a F – Harmonic mean graph for $n \geq 3$.

**Proof:** Let $u_1, u_2, ..., u_n$ be the vertices of the path $P_n$ and $v_1^{(i)}, v_2^{(i)}$ be the pendant vertices at each vertex $u_i$, $for\, 1 \leq i \leq n.$

$$\therefore V\left(TW\left(P_n\right)\right) = V(P_n) \cup \left\{v_1^{(i)}, v_2^{(i)}\ ;\ 2 \leq i \leq n-1\right\}\ and$$

$$E\left(TW\left(P_n\right)\right) = E(P_n) \cup \left\{u_i v_1^{(i)}, u_i v_2^{(i)}\ ;\ 2 \leq i \leq n-1\right\}.$$

We define $f : V\left(TW\left(P_n\right)\right) \to \{1, 2, 3, ..., 3n-4\}$ as follows

$$f(u_1) = 1,\ f(u_i) = 3i-3,\ fo\ \ r 2 \leq i \leq n-2,$$

$$f(u_{n-1}) = 3n-5, f(u_n) = 3n-4,\ f(v_1^{(i)}) = 3i-4,\ for\ 2 \leq i \leq n-1,$$

$$f(v_2^{(i)}) = 3i-2,\ for\ 2 \leq i \leq n-2\ and\ \ f(v_2^{(n-1)}) = 3n-6.$$

The induced edge labeling is obtained as follows

$$f^*(u_i u_{i+1}) = 3i-2,\ for\ 1 \leq i \leq n-1, f^*(u_i v_1^{(i)}) = 3i-4,\ for\ 2 \leq i \leq n-1$$

$$and\ \ f^*(u_i v_2^{(i)}) = 3i-3,\ for\ 2 \leq i \leq n-1.$$

Hence f is an F – Harmonic mean labelling of $TW(P_n)$. Thus the graph $TW(P_n)$ is an F – Harmonic mean graph for $n \geq 3$.

An F – Harmonic mean labelling of $TW(P_6)$ is shown in figure 9.

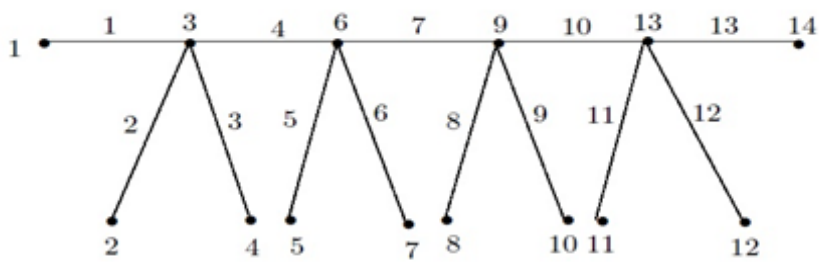


Figure 9

**Theorem 2.14:** The graph $[P_n;S_m]$ is a F – Harmonic mean graph for $n \geq 1$ *and* $m \leq 2$.

**Proof:** Let $u_1,u_2,...,u_n$ be the vertices of the path $P_n$ and $v_1^{(i)},v_2^{(i)},...,v_{m+1}^{(i)}$ be the vertices of the star graph $S_m$ such that $v_1^{(i)}$ is the central vertex of $S_m$, for $1 \leq i \leq n$.

**Case-(i):** $m = 1$

We define $f : V[P_n;S_1] \to \{1,2,3,...,3n\}$ as follows

$$f(u_i) = \begin{cases} 3 & i = 1 \\ 3i-1 & 2 \leq i \leq n \end{cases}, \quad f(v_1^{(i)}) = \begin{cases} 2 & i = 1 \\ 3i & 2 \leq i \leq n \end{cases}$$

$$and \;\; f(v_2^{(i)}) = 3i-2, \; for \; 1 \leq i \leq n.$$

The induced edge labeling is obtained as follows

$$f^*(u_iu_{i+1}) = 3i, \; for \; 1 \leq i \leq n-1, \; f^*(v_1^{(i)}u_i) = 3i-1, \; for \; 1 \leq i \leq n$$

$$and \;\; f^*(v_2^{(i)}u_i) = 3i-2, \; for \; 1 \leq i \leq n.$$

**Case-(ii):** $m = 2$

We define $f : V[P_n;S_2] \to \{1,2,3,...,4n\}$ as follows

$$f(u_i) = \begin{cases} 4i & 1 \leq i \leq n \; and \; i \; is \; odd \\ 4i-2 & 1 \leq i \leq n \; and \; i \; is \; even \end{cases},$$

$$f(v_1^{(i)}) = 4i-1, \; for \; 1 \leq i \leq n, \; f(v_2^{(i)}) = 4i-3, \; for \; 1 \leq i \leq n$$

$$and \;\; f(v_3^{(i)}) = \begin{cases} 4i-2 & 1 \leq i \leq n \; and \; i \; is \; odd \\ 4i & 1 \leq i \leq n \; and \; i \; is \; even \end{cases}.$$

The induced edge labeling is obtained as follows

$$f^*(u_iu_{i+1}) = 4i, \; for \; 1 \leq i \leq n-1,$$

$$f^*(v_1^{(i)}u_i) = \begin{cases} 4i-1 & 1 \leq i \leq n \; and \; i \; is \; odd \\ 4i-2 & 1 \leq i \leq n \; and \; i \; is \; even \end{cases},$$

$$f^*(v_1^{(i)}v_2^{(i)}) = 4i-3, \; for \; 1 \leq i \leq n$$

$$and \;\; f^*(v_1^{(i)}u_i) = \begin{cases} 4i-2 & 1 \leq i \leq n \; and \; i \; is \; odd \\ 4i-1 & 1 \leq i \leq n \; and \; i \; is \; even \end{cases}.$$

Hence f is an F – Harmonic mean labelling of $[P_n;S_m]$. Thus the graph $[P_n;S_m]$ is an F – Harmonic mean graph for $n \geq 1$ *and* $m \leq 2$.

The F – Harmonic mean labelling of $[P_5;S_1]$ *and* $[P_6;S_2]$ is shown in figure 10.

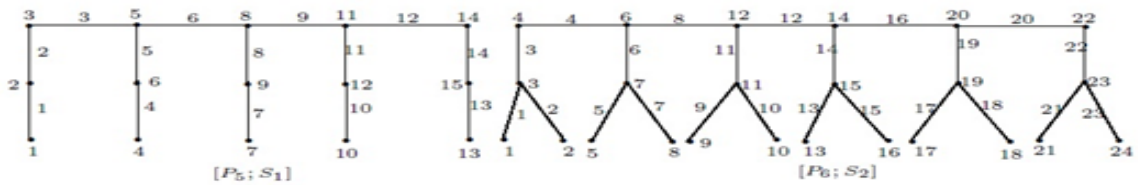


Figure 10

**Theorem 2.15:** The middle graph of a path is a F – Harmonic mean graph.

**Proof:** Let $V(P_n)=\{v_1,v_2,...,v_n\}$ *and* $E(P_n)=\{e_i=v_iv_{i+1}\ ;\ 1\le i\le n-1\}$ be the vertex set and edge set of the path $P_n$. Then

$$V(M(P_n))=\{v_1,v_2,...,v_n,e_1,e_2,...,e_{n-1}\}$$
$$and\ E(M(P_n))=\{v_ie_i,e_iv_{i+1}\ ;\ 1\le i\le n-1\}\cup\{e_ie_{i+1}\ ;\ 1\le i\le n-2\}.$$

We define $f:V(M(P_n))\to\{1,2,3,...,3n-3\}$ as follows

$$f(v_i)=3i-1,\ fo\ \ 1\le i\le n-1,\ f(v_n)=1$$
$$and\ \ f(e_i)=3i,\ for\ 1\le i\le n-1.$$

The induced edge labeling is obtained as follows

$$f^*(v_ie_i)=3i-1,\ for\ 1\le i\le n-1,\ f^*(e_iv_{i+1})=3i,\ for\ 1\le i\le n-2$$
$$f^*(e_{n-1}v_n)=1\ and\ f^*(e_ie_{i+1})=3i+1,\ for\ 1\le i\le n-2.$$

Hence f is an F – Harmonic mean labelling of $M(P_n)$. Thus the middle graph of the path is an F – Harmonic mean graph.

An F – Harmonic mean labelling of $M(P_7)$ is shown in figure 11.

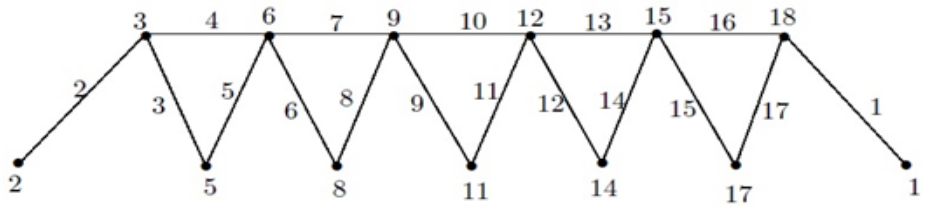


Figure 11

**Theorem 2.16:** Tadpoles $T(n,k)$ is a F – Harmonic mean graph for $n\ne 2.$

**Proof:** Let $u_1,u_2,...,u_n$ *and* $v_1,v_2,...,v_k$ be the vertices of the path $P_n$ and the cycle $C_k$ respectively.

Let $T(n,k)$ be the graph obtained by identifying the end vertex $u_n$ of the path $P_n$ to the vertex $v_1$ of the cycle $C_k$ .

We define $f:V(T(n,k))\to\{1,2,3,...,n+k\}$ as follows

$$f(u_i)=i,\ for\ 1\le i\le n\ and$$

$$f(v_i)=\begin{cases} n-2+2i & 2\le i\le\left\lfloor\frac{k}{2}\right\rfloor+1 \\ n-3+2i & i=\left\lfloor\frac{k}{2}\right\rfloor+2\ and\ k\ is\ odd \\ n-5+2i & i=\left\lfloor\frac{k}{2}\right\rfloor+2\ and\ k\ is\ even \\ n+2k+3-2i & \left\lfloor\frac{k}{2}\right\rfloor+3\le i\le k \end{cases}.$$

The induced edge labeling is obtained as follows

$$f^*(u_i u_{i+1}) = i, \ for \ 1 \le i \le n-1,$$

$$f^*(v_i v_{i+1}) = \begin{cases} n-2+2i & 1 \le i \le \left\lfloor \frac{k}{2} \right\rfloor \\ n-2+2i & i = \left\lfloor \frac{k}{2} \right\rfloor + 1 \ and \ k \ is \ odd \\ n+2k+1-2i & i = \left\lfloor \frac{k}{2} \right\rfloor + 1 \ and \ k \ is \ even \\ n+2k+1-2i & \left\lfloor \frac{k}{2} \right\rfloor + 2 \le i \le k-1 \end{cases}$$

$$and \ \ f^*(v_1 v_k) = n+1.$$

Hence f is an F – Harmonic mean labelling of $T(n,k)$. Thus the graph $T(n,k)$ is an F – Harmonic mean graph for $n \ne 2.$.

An F – Harmonic mean labelling of $T(6,8)$ is shown in figure 12.

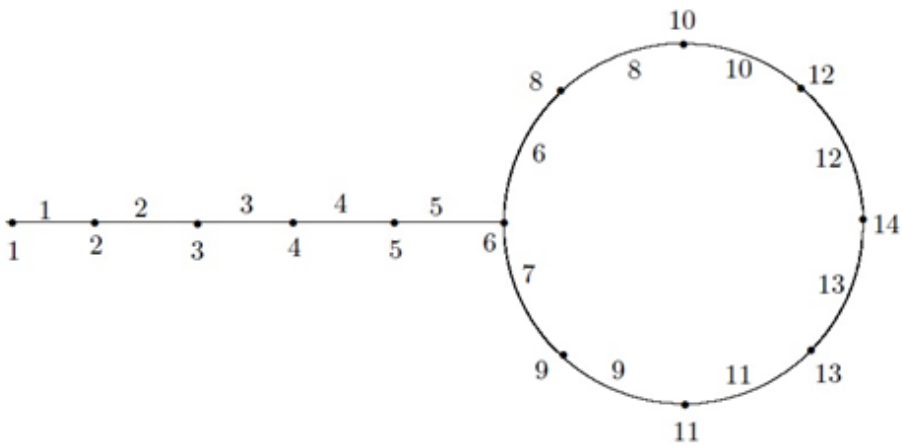


Figure 12

**Theorem 2.17:** Any H – graph G is a F – Harmonic mean graph.

**Proof:** Let $u_1, u_2, ..., u_n \ and \ v_1, v_2, ..., v_n$ be the vertices on the paths of length n in G.

**Case-(i):** n is odd and $n \ge 5.$

We define $f : V(G) \to \{1,2,3,...,2n\}$ as follows

$$f(u_i) = \begin{cases} i & 1 \le i \le m = \left\lfloor \frac{4n(n+1)}{5n+1} \right\rfloor \\ f(u_{i-1}) + 3 & m+1 \le i \le n \end{cases}$$

$$and \quad f(v_i) = \begin{cases} m+2 & i=1 \\ f(v_{i-1})+3 & 2 \le i \le n-m+1 \\ f(v_{i-1})+2 & n-m+2 \le i \le \left\lfloor \frac{n}{2} \right\rfloor \\ 2n & i = \left\lfloor \frac{n}{2} \right\rfloor + 1 \\ f(v_{i-1})-2 & \left\lfloor \frac{n}{2} \right\rfloor + 2 \le i \le m \\ f(v_{i-1})-3 & m+1 \le i \le n \end{cases}.$$

The induced edge labeling is obtained as follows

$$f^*(u_i u_{i+1}) = \begin{cases} i & 1 \le i \le m-1 \\ f^*(u_{i-1}u_i)+2 & i=m \\ f^*(u_{i-1}u_i)+3 & m+1 \le i \le n-1 \end{cases},$$

$$f^*(v_i v_{i+1}) = \begin{cases} m+3 & i=1 \\ f^*(v_{i-1}v_i)+3 & 2 \le i \le n-m \\ f^*(v_{i-1}v_i)+2 & n-m+1 \le i \le \left\lfloor \frac{n}{2} \right\rfloor \\ 2n-2 & i = \left\lfloor \frac{n}{2} \right\rfloor + 1 \\ f^*(v_{i-1}v_i)-2 & \left\lfloor \frac{n}{2} \right\rfloor + 2 \le i \le m \\ f^*(v_{i-1}v_i)-3 & m+1 \le i \le n-1 \end{cases}$$

$$and \quad f^*\left( u_{\frac{n+1}{2}} v_{\frac{n+1}{2}} \right) = m.$$

For n = 3, the F – Harmonic mean labelling of H – graph is shown in figure 13

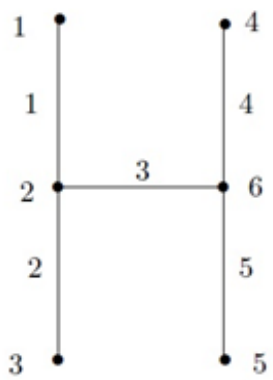


Figure 13

**Case-(i):** n is even and $n \geq 6$.

We define $f : V(G) \to \{1,2,3,...,2n\}$ as follows

$$f(u_i) = \begin{cases} i & 1 \leq i \leq m = \left\lfloor \dfrac{4n(n+1)}{5n+1} \right\rfloor \\ f(u_{i-1}) + 3 & m+1 \leq i \leq n \end{cases}$$

$$and \quad f(v_i) = \begin{cases} m+2 & i = 1 \\ f(v_{i-1}) + 3 & 2 \leq i \leq n-m+1 \\ f(v_{i-1}) + 2 & n-m+2 \leq i \leq \left\lfloor \dfrac{n}{2} \right\rfloor \\ 2n & i = \left\lfloor \dfrac{n}{2} \right\rfloor + 1 \\ f(v_{i-1}) - 2 & \left\lfloor \dfrac{n}{2} \right\rfloor + 2 \leq i \leq m \\ f(v_{i-1}) - 3 & m+1 \leq i \leq n \end{cases}.$$

The induced edge labeling is obtained as follows

$$f^*(u_i u_{i+1}) = \begin{cases} i & 1 \leq i \leq m-1 \\ f^*(u_{i-1}u_i) + 2 & i = m \\ f^*(u_{i-1}u_i) + 3 & m+1 \leq i \leq n-1 \end{cases},$$

$$f^*(v_i v_{i+1}) = \begin{cases} m+3 & i = 1 \\ f^*(v_{i-1}v_i) + 3 & 2 \leq i \leq n-m \\ f^*(v_{i-1}v_i) + 2 & n-m+1 \leq i \leq \left\lfloor \dfrac{n}{2} \right\rfloor - 1 \\ 2n-1 & i = \left\lfloor \dfrac{n}{2} \right\rfloor \\ f^*(v_{i-1}v_i) - 2 & \left\lfloor \dfrac{n}{2} \right\rfloor + 1 \leq i \leq m \\ f^*(v_{i-1}v_i) - 3 & m+1 \leq i \leq n-1 \end{cases}$$

$$and \quad f^*\left( u_{\frac{n+2}{2}} v_{\frac{n}{2}} \right) = m.$$

For n = 4, the F – Harmonic mean labelling of H – graph is shown in figure 14

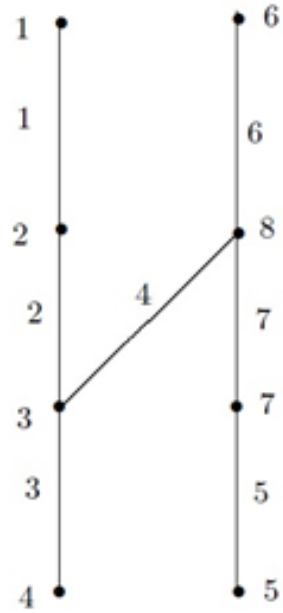


Figure 14

Hence f is an F – Harmonic mean labelling of the H – graph G. Thus the H – graph is an F – Harmonic mean graph.

The F – Harmonic mean labelling of H – graph with 9 vertices and 8 vertices are shown in figure 15.

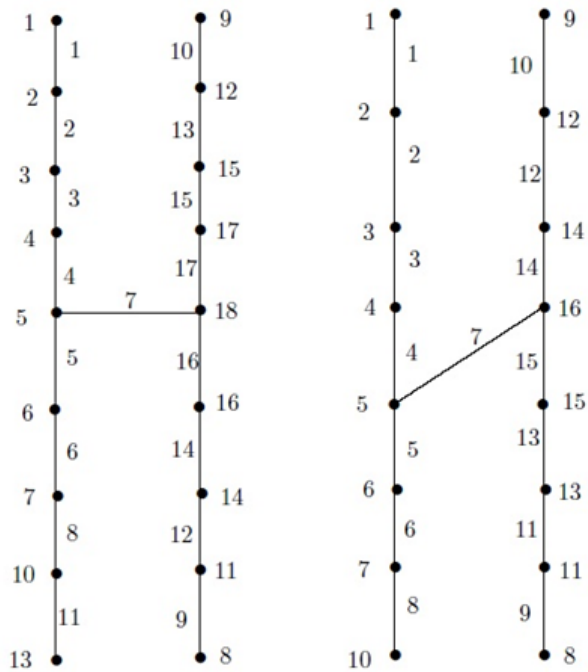


Figure 15

**Theorem 2.18:** $S\left(P_n \odot K_1\right)$ is a F – Harmonic mean graph.

**Proof:** Let $V\left(P_n \odot K_1\right)=\left\{u_i, v_i ; 1 \le i \le n\right\}$ .Let $x_i$ be the vertex which divides the edge $u_i v_i$, $for\ 1 \le i \le n$ and $y_i$ be the vertex which divides the edge $u_i v_{i+1}$, $for\ 1 \le i \le n-1.$

Then

$$\therefore V\left(S\left(P_n \odot K_1\right)\right)=\left\{u_i, v_i, x_i, y_j\ ;\ 1 \le i \le n,\ 1 \le j \le n-1\right\} and$$

$$E\left(S\left(P_n \odot K_1\right)\right)=\left\{u_i x_i, v_i x_i ;\ 1 \le i \le n\right\} \cup \left\{u_i y_i, y_i u_{i+1} ;\ 1 \le i \le n-1\right\}.$$

We define $f : V\left(S\left(P_n \odot k_1\right)\right) \to \{1,2,3,\ldots,4n-1\}$ as follows

$$f(u_i) = 4i-1,\ for\ 1 \le i \le n,\ f(y_i) = 4i+1,\ for\ 1 \le i \le n-1,$$

$$f(x_i) = 4i-2,\ for\ 1 \le i \le n\ and\ \ f(v_i) = \begin{cases} 1 & i=1 \\ 4i-4 & 2 \le i \le n \end{cases}.$$

The induced edge labeling is obtained as follows

$$f^*(u_i y_i) = 4i-1,\ for\ 1 \le i \le n-1,\ f^*(y_i u_{i+1}) = 4i+1,\ for\ 1 \le i \le n-1,$$

$$f^*(u_i x_i) = 4i-2,\ for\ 1 \le i \le n\ and\ \ f^*(v_i x_i) = \begin{cases} 1 & i=1 \\ 4i-4 & 2 \le i \le n \end{cases}.$$

Hence f is an F – Harmonic mean labelling of $S\left(P_n \odot K_1\right)$. Thus the graph $S\left(P_n \odot K_1\right)$ is an F – Harmonic mean graph.

An F – Harmonic mean labelling of $S\left(P_5 \odot K_1\right)$ is shown in figure 16.

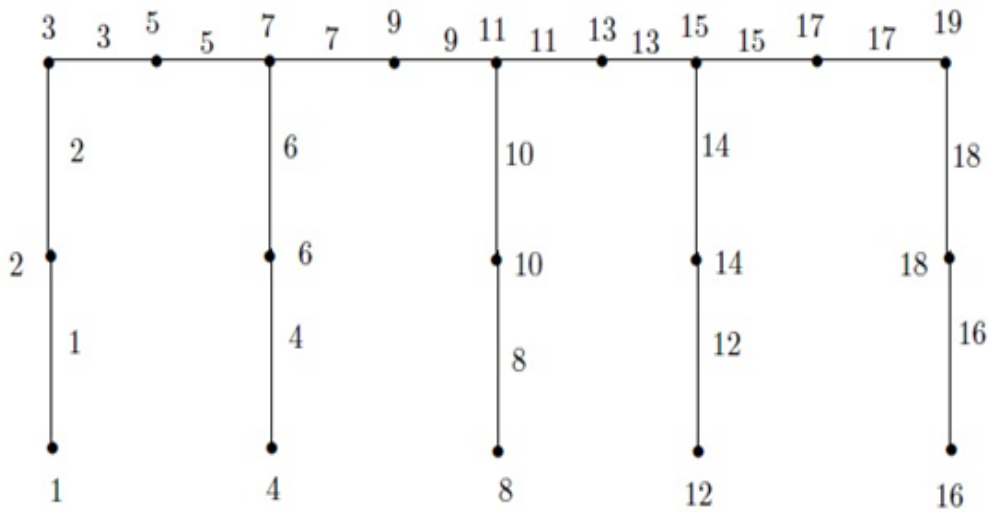


Figure 16

**Theorem 2.19:** Arbitrary subdivision of $S_3$ is a F – Harmonic mean graph.

**Proof:** Let G be the graph of arbitrary subdivision of $S_3$ .

Let $v_0, v_1, v_2\ and\ v_3$ be the vertices of G in which $v_0$ is the central vertex and $v_1, v_2\ and\ v_3$ are the pendant vertices of $S_3$.

Let the edges $v_0v_1, v_0v_2\ and\ v_0v_3\ of\ S_3\ be\ subdivided\ by\ p_1, p_2\ and\ p_3$ number of vertices respectively.

Let $v_0, v_1^{(1)}, v_2^{(1)}, v_3^{(1)}, \ldots, v_{p_1+1}^{(1)} (= v_1), v_0, v_1^{(2)}, v_2^{(2)}, v_3^{(2)}, \ldots, v_{p_2+1}^{(2)} (= v_2)$ and $v_0, v_1^{(3)}, v_2^{(3)}, v_3^{(3)}, \ldots, v_{p_3+1}^{(3)} (= v_3)$ be the vertices of G and $v_0 = v_0^{(i)},\ for\ 1 \le i \le 3$ .

Let $e_j^{(i)} = v_{j-1}^{(i)} v_j^{(i)},\ 1 \le j \le p_i + 1\ and\ 1 \le i \le 3$ be the edges of G and it has $p_1 + p_2 + p_3 + 4$ vertices and $p_1 + p_2 + p_3 + 3$ edges with $p_1 \le p_2 \le p_3$ .

We define $f : V(G) \to \{1,2,3,\ldots, p_1 + p_2 + p_3 + 4\}$ as follows

$$f(v_0) = p_1 + p_2 + 3,$$

$$f(v_j^{(1)}) = p_1 + p_2 + 4 - 2j, \quad fo \quad 1r \le j \le p_1 + 1,$$

$$f(v_j^{(2)}) = \begin{cases} p_1 + p_2 + 3 - 2j & 1 \le j \le p_1 + 1 \\ p_2 + 2 - j & p_1 + 2 \le j \le p_2 + 1 \end{cases}$$

$$and \;\; f(v_j^{(3)}) = p_1 + p_2 + 3 + j, \quad for \; 1 \le j \le p_3 + 1 \; .$$

The induced edge labeling is obtained as follows

$$f^*\left(v_j^{(1)} v_{j+1}^{(1)}\right) = p_1 + p_2 + 2 - 2j, \quad fo \quad 1r \le j \le p_1,$$

$$f^*\left(v_j^{(2)} v_{j+1}^{(2)}\right) = \begin{cases} p_1 + p_2 + 1 - 2j & 1 \le j \le p_1 \\ p_2 + 1 - j & p_1 + 1 \le j \le p_2 \end{cases},$$

$$f^*\left(v_j^{(3)} v_{j+1}^{(3)}\right) = p_1 + p_2 + 3 + j, \quad fo \quad 1r \le j \le p_3,$$

$$f^*\left(v_0 v_1^{(1)}\right) = p_1 + p_2 + 2,$$

$$f^*\left(v_0 v_1^{(2)}\right) = p_1 + p_2 + 1$$

$$and \;\; f^*\left(v_0 v_1^{(3)}\right) = p_1 + p_2 + 3.$$

Hence, f is an F –Harmonic mean labelling of G. Thus the arbitrary subdivision of $S_3$ is an F – Harmonic mean graph.

An F – Harmonic mean labelling of G with $p_1 = 3, p_2 = 5 \; and \; p_3 = 7$ is shown in figure 17.

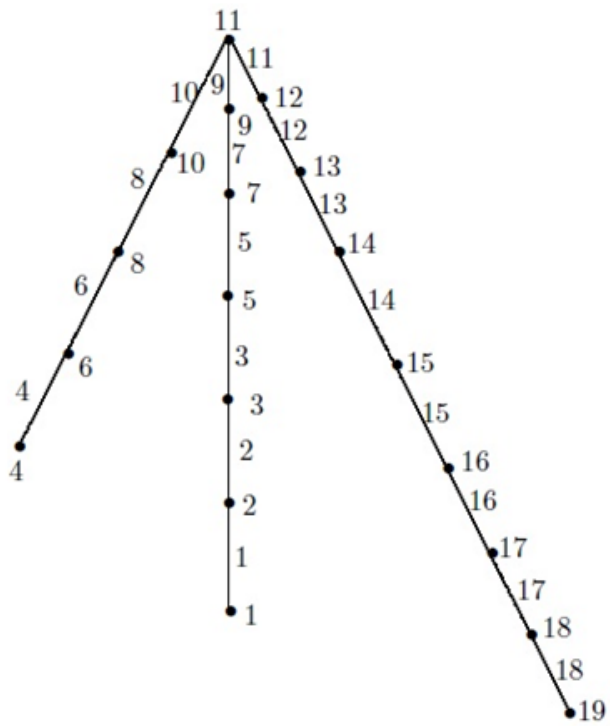


Figure 17

## REFERENCES

[1] A. Durai Baskar, S. Arockiaraj and B. Rajendran, F - Geometric mean labeling of some chain graphs and thorn graphs, Kragujevac Journal of Mathematics, Volume 37(1) (2013), Pages 163–186.
[2] A. Durai Baskar, S. Arockiaraj and B. Rajendran, Geometric meanness of graphs obtained from paths, Utilitas Mathematica (To appear).
[3] J. A. Gallian, A dynamic survey of graph labeling, The Electronic Journal of Combinatorics, 17(2011).
[4] F. Harary, Graph theory, Addison Wesely, Reading Mass., 1972.
[5] S. S. Sandhya, S. Somasundaram and R. Ponraj, Some Results on Harmonic Mean Graphs, Int. J. Contemp. Math. Sciences, Vol. 7, 2012, no. 4, 197 – 208.

**Source of support: Nil, Conflict of interest: None Declared**

***[Copy right © 2014. This is an Open Access article distributed under the terms of the International Journal of Mathematical Archive (IJMA), which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.]***