International Journal of Mathematical Archive-5(8), 2014, 159-161
IJMA Available online through www.ijma.info ISSN 2229 – 5046

SUPERCYCLICITY OF PARANORMAL OPERATORS

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(Received On: 02-08-14; Revised & Accepted On: 26-08-14)

ABSTRACT

An operator T on a Banach space X is paranormal if for all $x \in X$, $\|Tx\|^2 \leq \|T^2x\| \|x\|$. In this note, we show that any paranormal operator T is not supercyclic; i.e., the projective orbit $\{\alpha T^n x : \alpha \in C,\ n \geq 0\}$ is not dense in X for all $x \in X$.

Key words and Phrases: *paranormal operators, supercyclicity.*

2010 Mathematics Subject Classification: *47A16.*

1. INTRODUCTION

Suppose that T is a bounded linear operator on a separable Banach space X. For $x \in X$ the orbit of x under T is

$$orb(T, x) = \{T^n x : n = 0,1,2,\cdots\}.$$

We recall that a vector x in X is called a hypercyclic vector for T if the set $orb(T, x)$ is dense in X. Also x is a supercyclic vector for T if the set of all scalar multiples of the elements of $orb(T, x)$ is dense in X. An operator T is called hypercyclic (supercyclic) if it has a hypercyclic (supercyclic) vector.

Rolewicz in 1969 has given an example of a hypercyclic operator on a Banach space [8]; but the study of hypercyclicity was really begun with Kitai's thesis in 1982 [6]. Clearly every hypercyclic operator is supercylcic but not vice versa. The backward shift on ℓ^2 is an example of a suprecyclic operator which is not hypercyclic. Two good sources on hypercyclicity and supercyclicity of operators are [2] and [4].

Recall that an operator T on a Hilbert space H is hyponormal if $T^*T - TT^* \geq 0$ which is equivalent to the fact that $\|Tx\| \geq \|T^*x\|$ for all $x \in H$. Hilden and Wallen in [5] have shown that normal operators are never supercyclic. The non-hypercyclicity of hyponormal operators has proved by Kitai [6]; later on, Bourdon showed that hyponormal operators are not supercylic [3]. An operator T on a Banach space X is paranormal if $\|Tx\|^2 \leq \|T^2x\| \|x\|$ for all $x \in X$. It is known that every hyponormal operator on a Hilbert space is paranormal. The non-hypercyclicty of paranormal operators have proved in Theorem 5.30 of [4]. In this paper, we observe that paranormal operators are not supercyclic.

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2. MAIN RESULTS

Theorem 1: No paranormal operator on a Banach space with dimension more that one is supercylcic.

Proof: Suppose that T is a paranormal operator on a Banach space X with $\dim X > 1$. Observe that λT is also paranormal for every scalar λ. Let $\varepsilon > 0$ and put $\lambda_\varepsilon = (1+\varepsilon) \| T \|^{-1}$. Assume that T is supercyclic. Then $\lambda_\varepsilon T$ is a paranormal, supercyclic operator. Hence there is a supercyclic vector x_ε for $\lambda_\varepsilon T$ such that $\| \lambda_\varepsilon T x_\varepsilon \| > \| x_\varepsilon \|$. Indeed, otherwise $\| \lambda_\varepsilon T_x \| \le \| x \|$ for all supercyclic vectors x. But the set of all supercyclic vectors for $\lambda_\varepsilon T$ is a dense subset of X (Theorem 1.12 of [2]), thus $\| \lambda_x T_x \| \le \| x \|$ for all $x \in X$. Consequently, $(1+\varepsilon) = \| \lambda_\varepsilon T \| \le 1$ which is absurd.

Now, the paranormality of $\lambda_\varepsilon T$ implies that

$$\| (\lambda_\varepsilon T)^2 x_\varepsilon \| \ge \frac{\| (\lambda_\varepsilon T) x_\varepsilon \|^2}{\| x_\varepsilon \|} > \| (\lambda_\varepsilon T) x_\varepsilon \|.$$

Therefore,

$$\| (\lambda_\varepsilon T)^3 x_\varepsilon \| \ge \frac{\| (\lambda_\varepsilon T)^2 x_\varepsilon \|^2}{\| (\lambda_\varepsilon T) x_\varepsilon \|} > \| (\lambda_\varepsilon T)^2 x_\varepsilon \|.$$

By continuing this process we get

$$\| (\lambda_\varepsilon T)^{n+1} x_\varepsilon \| > \| (\lambda_\varepsilon T)^n x_\varepsilon \|, \ \forall n \ge 0.$$

On the other hand, for $y \in X$ there is a sequence of scalars $(\alpha_i)_i$ and a sequence of integers $(n_i)_i$ so that

$$\alpha_i (\lambda_\varepsilon T)^{n_i} x_\varepsilon \to y$$

as $i \to \infty$. Since

$$| \alpha_i | \, \| (\lambda_\varepsilon T)^{n_i+1} x_\varepsilon \| > | \alpha_i | \, \| (\lambda_\varepsilon T)^{n_i} x_\varepsilon \|$$

for all i, we obtain

$$\| (\lambda_\varepsilon T) y \| \ge \| y \|.$$

When ε converges to zero we conclude that

$$\| T \| \, \| y \| \ge \| Ty \| \ge \| T \| \, \| y \|, \ \forall y \in X$$

which, in turn, implies that the operator $\dfrac{T}{\| T \|}$ is an supercyclic, isometric operator on X. But it is known that an isometry on a Banach space with dimension greater than one is not supercyclic [1] or [7].

Recall that an operator T on a Hilbert space H is quasi-hyponormal if $T^*(T^*T - TT^*)T \ge 0$ which is equivalent to $\| T^2 x \| \ge \| T^* T x \|$ for all $x \in H$. Every quasihyponormal operator T is paranormal; indeed, by the Cauchy-Scharwz inequality we have

$$\| Tx \|^2 = < Tx, Tx > \le \| T^* T x \| \, \| x \| \le \| T^2 x \| \, \| x \|$$

for all $x \in X$. Thus, we have the following result.

Corollary 1: No quasi-hyponormal operator is supercyclic**.**

ACKNOWLEDGEMENT

This research was in part supported by a grant from Shiraz University Research Council.

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Source of support: Shiraz University Research Council, Iran. Conflict of interest: None Declared