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A REMARK ON THE MULTIPLIER OF FUNCTION SPACES

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ABSTRACT

Let E be a reflexive Banach space of analytic functions. It is shown that the inclusion $M(E)[f] \subseteq [f]\ (f \in E)$ holds for certain E, whenever $M(E)$ is the set of all multipliers of E and $[f]$ denotes the closure in E of the polynomial multiples of f.

Key words and phrases: *Banach space of analytic functions, multiplier, polynomial.*

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1. INTRODUCTION

Let G be a bounded domain in the complex plane C. Suppose that E is a reflexive Banach space consisting of functions that are analytic on G such that E contains the polynomials as a dense subset and for each $\lambda \in G$, the functional $e(\lambda): E \to C$ of evaluation at λ given by $e(\lambda)(f) = \langle f, e(\lambda)\rangle = f(\lambda)$ is bounded, and if $\in E$ then $z f \in E$.

Note that the last condition allows us to define $M_z : E \to E$ by $M_z f = z f$. It is easy to see that M_z is actually a bounded operator on E. The operator M_z and many of its properties have been studied in [1], [3], [5-8], [10] and [12]. We just give an example to illustrate the existence of such spaces. For $-\infty < \alpha < \infty$ let D_α consist of all functions f analytic in unit disc D with Taylor series $f(z) = \sum_{n=0}^{\infty} \hat{f}(n) z^n$ such that

$$\| f \|_\alpha^2 = \sum_{n=0}^{\infty} (n+1)^\alpha \, |\hat{f}(n)|^2 < \infty.$$

For good sources on D_α see [2] and [11].

A Caratheodory region is an open connected subset of C whose boundary equals to its outer boundary. It is easy to see that G is a Caratheodory region if and only if G is the interior of the polynomially convex hull of $\overline{G}$. In this case, Farrell-Rubel-Shields Theorem holds [4, Theorem 5.1, p. 151]. If $f \in H^\infty(G)$ then there exists a sequence of polynomials $(p_n)_n$ such that $\sup_n \| p_n \|_G < c$ for a constant c and $p_n(z) \to f(z)$ for all $z \in G$.

A complex-valued function φ on G is called a multiplier of E if $\varphi E \subset E$. In general each multiplier φ of E determines a multiplication operator $M_\varphi f = \varphi f\ (f \in E)$. Also, $M_\varphi^* e(\lambda) = \varphi(\lambda) e(\lambda)\ (\lambda \in G)$. The set of all multipliers is denoted by $M(E)$. It is well known that $M(E) \subseteq E \cap H^\infty(G)$, whenever $H^\infty(G)$ denotes the space of bounded analytic functions in G, with the supermum norm. Also, $[f]$ denotes the closure in E of the polynomial multiples of $f\ (f \in E)$.

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2. MAIN RESULTS

In general, it is an open question that for which Banach spaces of analytic functions $E, M(E)[f] \subseteq [f]$ for all $f \in E$ (Question 2 of [2]). Clearly, this is equivalent to $M(E)f \subseteq [f](f \in E)$. Now, we bring the following result.

Theorem 1: Suppose that E is a reflexive Banach space of analytic functions on a Caratheodory region G such that $M(E)$ is closed in $H^\infty(G)$ then $M(E) = H^\infty(G)$ and $M(E)[f] \subseteq [f]$ for every $f \in E$.

Proof: Define the mapping $\varphi \mapsto M_\varphi$ from $M(E)$ with the supermum norm into $B(E)$, the set of all bounded operators on E . By the closed graph theorem, this map is continuous. In fact, if $\varphi_n \to 0$ and $M_{\varphi n} \to A$ then $\varphi_n(\lambda) f(\lambda) \to (Af)(\lambda)$ for every $f \in E$ and for all λ in G ; so $Af = 0$. Therefore, there exists a constant c such that $\| M_p \| \le c \| p \|_G$ for every polynomial p . Let $\varphi \in H^\infty(G)$. By the Farrell-Rubel-Shields Theorem there exists a sequence of polynomials $(p_n)_n$ such that $\sup_n \| p_n \|_G < \infty$ and $p_n(\lambda) \to \phi(\lambda)$ $(\lambda \in G)$. It follows that $\sup_n \| M_{p_n} \| < \infty$. But ball $B(E)$ is WOT compact; so by passing to a subsequence if necessary we may assume that M_{p_n} converges in WOT to some operator B . Therefore,

$$(Bf)(\lambda) = \lim_n \langle p_n f, e(\lambda) \rangle = \lim_n p_n(\lambda) f(\lambda) = \varphi(\lambda) f(\lambda), \ (\lambda \in G)$$

that is, $B = M_\varphi$ and $\varphi \in M(E)$. Now, since $M(E) \subseteq H^\infty(G)$ the equality holds. On the other hand, $\| p_n f \|_E \le c \| p_n \|_G \| f \|_E < \infty$. Hence $p_n f \to \varphi f$ weakly. It follows that $\varphi f \in [f]$ for every $f \in E$ and the proof is complete.

Corollary 1: Let $L_a^p(G)$ $(1 < p \le 2)$ be the Bargman space on a Caratheodory region G. It is clear that $L_a^p(G)$ satisfies all hypotheses of Theorem 1. So $H^\infty(G)[f] \subseteq [f]$ for every $f \in L_a^p(G)$.

The Caratheodory condition on G is not necessary for $M(E)[f] \subseteq [f]$, $f \in E$. Indeed, the following theorem holds.

Theorem 2: Suppose that E is a reflexive Banach space of analytic functions on $G = \Omega - K$ where Ω is a Caratheodory region and K is a compact subset of Ω . If $M(E)$ is closed in $H^\infty(G)$ then $M(E) = H^\infty(G) \cap E$ and $M(E)[f] \subseteq [f]$, for every $f \in E$.

Proof: By the proof of Theorem 1, there exists a constant c such that $\| M_p \| \le c \| p \|_G$ for every polynomial p . Also, clearly $M(E) \subseteq H^\infty(G) \cap E$. Now, let $\varphi \in H^\infty(G) \cap E$. So there exists a sequence $(p_n)_n$ of polynomials converging to φ in E . Thus, $(p_n)_n$ converges uniformly on compact subsets of G . Choose the oriented line intervals $\gamma_1, \ldots, \gamma_N$ in G [9, 13.5 Theorem, p. 254] such that

$$| p_m(z) - p_n(z) | \le \sum_{j=1}^{n} \frac{1}{2\pi} \int_{\gamma j} \left| \frac{p_m(\lambda) - p_n(\lambda)}{z - \lambda} \right| | d\lambda |, \ (z \in K).$$

Since $(p_n)_n$ converges uniformly on $\gamma_1, \ldots, \gamma_N$, it is uniformly Cauchy on K . It follows that $(p_n)_n$ is uniformly convergent on compact subsets of Ω . Let ψ be the limit of $(p_n)_n$. In fact, ψ is the extension of φ on Ω . By the maximum modulus theorem $\| \psi \|_\Omega = \| \varphi \|_G < \infty$, and thanks to the Farrell-Rubel-Shields Theorem there exists a sequence of polynomials $(q_n)_n$ such that $q_n(z) \to \varphi(z)$ and $\sup_n \| q_n \|_G < \infty$. Now, as the proof of Theorem 1, $\varphi f \in M(E)$ and $\varphi f \in [f]$ for all $f \in E$.

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