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GRAPHICAL PARTITION AND SOME GRAPH OF SPECIAL TYPES

Jayanta Kr. Choudhury*
Department of Mathematics,
Swadeshi Academy Junior College, Guwahati-781005, Assam, India.

Bichitra Kalita
Department of Computer Applications (M.C.A.),
Assam Engineering College, Guwahati-781013, Assam, India.

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ABSTRACT

A graphical partition of the even integer n is a partition of n of which each part of the partition is the degree of a vertex in a simple graph and the degree sum of the graph is n. In this paper, we study some theoretical investigations related to the existence of pan graph, sun graph, helm graph, barbell graph, and complete bipartite graph from the graphical parts of the numbers $2(n+1)$, $n(n+3)$, $6n$, $2(n^2-n+1)$ for $n \geq 3$ and $2n^2$ for $n \geq 2$ and an algorithm is developed to determine different forms of graphs of the graphical parts.

Key words: *Partition, Integer, Graphical partition, Algorithm.*

Mathematics Subject classification (MSC2010): *05C70, 05C85.*

1. INTRODUCTION

Partition of the vertices of graphs is a fundamental concept in graph theory. Partitions have a number of applications in graph theory. The partition of a graph is the partition of $2q$ as the sum of the degrees of the vertices $2q = \sum d_i$. A partition $\sum d_i$ of n into p parts is graphical if there is a graph G whose vertices have degrees d_i. If such a partition is graphical, then certainly every $d_i \leq p-1$ and n is even. These two conditions are not sufficient for a partition to be graphical. Because, firstly one can't say whether a given partition is graphical or not and secondly, how to construct a graph for a given graphical partition? The answer to the first was given by Erdös and Gallai [6] and another answer given independently by Havel [7]. Hakimi [8] forwarded a solution for second question. Moreover, some results about partition of even number which are graphical or non-graphical were forwarded by Erdös and Richamond [5]. They established two results for the bounds of graphical partition $g(n) = |G(n)|$ [$G(n)$ denote the set of graphical partitions of an even integer n]. But Rousseau and Ali [15] established the upper bound that $\lim_{n\to\infty} \frac{g(n)}{p(n)} \leq 0.25$ [$p(n)$ denotes the number of unrestricted partitions of n]. Finally, Pittel [14] shown that $\frac{g(n)}{p(n)} \to 0$ as $n \to \infty$, which is an open question originally posed by H. S. Wilf [11], known as Wilf's conjectures. J. M. Nolan *et al* [12], in their paper, discussed about the efficient counting and generating graphical partitions and use of them to count graphical partitions. G. H. Hardy and Ramunajan [10] discussed the circular method of the partition function. DeTemple *et al* [3] answered two questions in the study of partitions graphs by recently discovered examples and they provided an enumeration of the partition graphs on ten or fewer vertices. Moore [1] defines the distance partition of a finite graph and shows how this partition is equitable. He connects this partition to a number of fundamental ideas in graph theory. Deborah *et al* [4] discussed the problem of enumerating graphical forest partitions and they proved that $gf(2k) = p(0) + p(1) + p(2) + \cdots\cdots + p(k-1)$ where $gf(2k)$ is the number of graphical forest partitions of $2k$ and $p(j)$ is the ordinary partition functions which counts the number of integer partitions of j. B.Nath *et al* [13] investigated some properties of graphical partitions relating to the existence of binary tree, regular graphs having degree one and degree two and the complete graph from the graphical parts of the numbers $(2n+4)$, $(4n+4)$, $(n+1)(n+2)$, $(6n+2)$ for $n \geq 1$. They have also found some structures from the graphical partitions of some even numbers which satisfy the saturated hydro-carbon C_nH_{2n+2} of chemistry and finally they have developed an algorithm to find out the various forms of the graphs among all graphical parts. Although, many authors have tried to determine an efficient formula for $g(n)$, still it remains under investigations.

Corresponding author: Jayanta Kr. Choudhury*

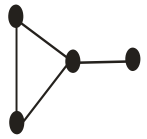
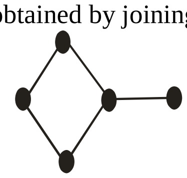
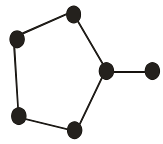
In this paper, we have discussed partitions of number n possessing graphical sequences. A graphical sequence means a sequence whose terms represent the degrees of the vertices in a simple graph. Again, the partition of a graph is the partition of $2q$ as the sum of the degrees of the vertices, $2q = \sum d_i$ (Hand seeking lemma). Thus a partition of $\sum d_i$ of the number n into p-parts is graphical if there exist a graph whose points have degrees d_i. Here we have studied some theoretical investigations related to the existence of pan graph, sun graph, helm graph, barbell graph, and complete bipartite graph from the graphical parts of the numbers $2(n+1), n(n+3), 6n, 2(n^2 - n + 1)$ for $n \geq 3$ and $2n^2$ for $n \geq 2$.

The paper is organized as follows**:**

The section 1 includes the introduction part containing the works of the other researchers. In section 2, preliminaries are discussed. In section 3, some theorems related to some special type of even numbers are stated and proved. In section 4, we present an algorithm to produce different structures of graphs of the graphical parts. Conclusion is included in section 5.

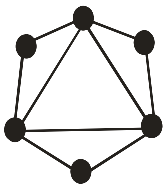
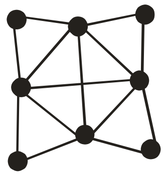
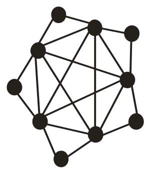
2. PRELIMINARIES

Pan Graph: An n-pan graph is the graph obtained by joining a cycle graph C_n to a singleton graph K_1 with a bridge.

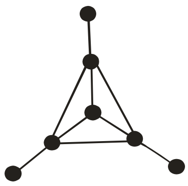
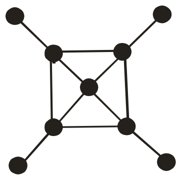
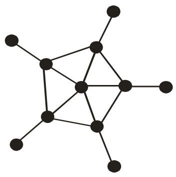
$n = 3$ $n = 4$ $n = 5$

Sun Graph: An n-sun graph is a graph on $2n$ vertices consisting of a complete graph K_n with an outer ring of n vertices, each of which is joined to both endpoints of the closest outer edge of the central core.

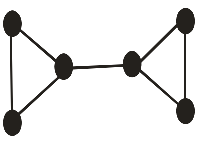
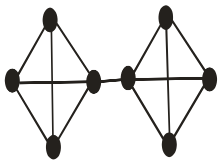
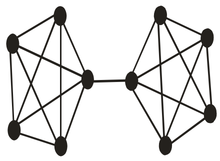
$n = 3$ $n = 4$ $n = 5$

Helm Graph: A Helm graph H_n is the graph obtained from an n-wheel graph by adjoining a pendant edge at each vertex of the cycle.

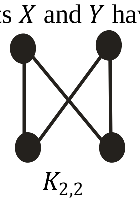
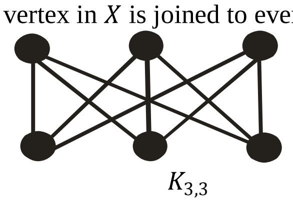
$n = 3$ $n = 4$ $n = 5$

Barbell Graph: An n-barbell graph is the simple graph obtained by connecting two copies of a complete graph K_n by a bridge.

$n = 3$ $n = 4$ $n = 5$

Complete bipartite graph: A complete bipartite graph $K_{m,m}$ is a simple bipartite graph G, with bipartition of its vertex set V into two sets X and Y having m vertices each, in which every vertex in X is joined to every vertex of Y.

$K_{2,2}$ $K_{3,3}$

3. THEORETICAL INVESTIGATION

Theorem 3.1: In the graphical partitions of the number $a_n = 2(n+1)$ for $n \geq 3$, there always exist an n-pan graph for $n \geq 3$.

Proof: We shall proof the theorem by method of mathematical induction. We know that the number of vertices in the n-pan graph for $n \geq 3$ is $(n+1)$ in which $(n-1)$ vertices of degree two each, one vertex of degree one and one vertex of degree three.

For $n = 3$, $a_3 = 8 = (1+3) + (2+2)$. Then this partition represents a graph having two vertices of degree two each, one vertex of degree one and one vertex of degree three as shown in Figure-1. Clearly, this graph is an n-pan graph for $n = 3$.

Thus the result is true for $n = 3$.

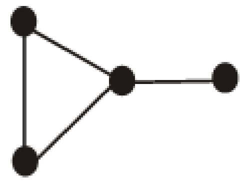


Figure-1

For $n = 4$, $a_4 = 10 = (1+3) + (2+2+2)$, which represents a graph having three vertices of degree two each, one vertex of degree one and one vertex of degree three as shown in Figure-2. The graph in Figure-2 is an n-pan graph for $n = 4$. Thus the result is true for $n = 4$.

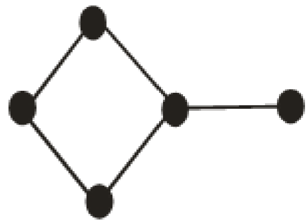


Figure-2

Let us assume that the result is true for $n = k$, that is, in the graphical partitions of the number $a_k = 2(k+1)$, there always exists a k-pan graph in which $(k-1)$ vertices having degree two each, one vertex of degree one and one vertex of degree three. Then we have

$$a_k = (1+3) + \{2+2+2 \cdots\cdots to\ (k-1)\ terms\}$$

Now, if we introduce a new vertex in any one edge lying in the cycle C_k in the k-pan graph as shown in the Figure-3, then the resulting graph will be a pan graph, having k vertices of degree two each, one vertex of degree one and one vertex of degree three.

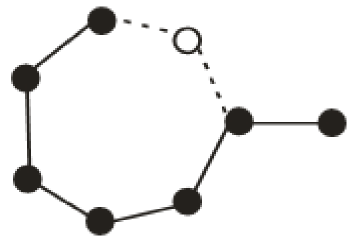


Figure-3

Then the graphical partitions of this graph will be

$(1+3) + [\{2+2+2+\cdots\cdots to\ (k-1)\ terms\} + 2]$

$= (1+3)\{2+2+2+\cdots\cdots to\ k\ terms\}$

$= (1+3) + 2k$

$= 2k + 4$

$= 2(k+2)$

$= 2\{(k+1)+1\}$

$= a_{k+1}$

Thus the result is true for $n = k+1$, whenever it is true for $n = k$. Hence completes the proof.

Theorem 3.2: In the graphical partitions of the number $a_n = n(n+3)$ for $n \geq 3$ there always exists an n-sun graph for $n \geq 3$.

Proof: We shall proof the theorem by method of mathematical induction. We know that the number of vertices in n-sun graph for $n \geq 3$ is $2n$, out of which n vertices having degree two each and n vertices are of degree $(n+1)$ each.

For $n = 3$, $a_3 = 18 = (2+2+2) + (4+4+4)$. Then this partition represents a graph having three vertices of degree two each and three vertices of degree four each as shown in Figure-4. Clearly, this graph is an n-sun graph for $n = 3$. Thus the result is true for $n = 3$.

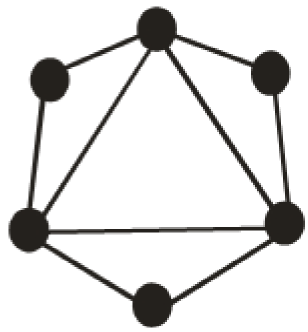


Figure-4

For $n = 4$, $a_4 = 28 = (2+2+2+2) + (5+5+5+5)$
$= (2+2+2+2) + \{(4+1) + (4+1) + (4+1) + (4+1)\}$.

Then this partition represents a graph having four vertices of degree two each and four vertices of degree five each as shown in Figure-5. The graph in figure-5 is an n-sun graph for $n = 4$. So, the result is true for $n = 4$.

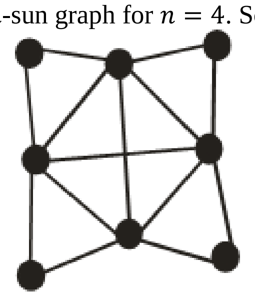


Figure-5

Let us assume that the result is true for $n = m$, that is, in the graphical partitions of the number $a_m = m(m+3)$, there always exists a m-sun graph in which number of vertices are $2m$, out of which m vertices having degree two each and m vertices of degree $(m+1)$ each. Then we have

$$a_m = (2+2+2+\cdots\cdots to\ m\ terms) + \{(m+1) + (m+1) + (m+1) + \cdots\cdots to\ m\ terms\}$$

Now, if we introduce two new vertices in such a way that one of them considered in any one edge lying in the cycle of the inner complete graph K_m and connect it with other vertices of the complete graph K_m and with the other new vertex as shown in the Figure-6, then the resulting graph will be a sun graph, having $(m+1)$ vertices of degree two each and $(m+1)$ vertices of degree $\{(m+1)+1\} = (m+2)$ each.

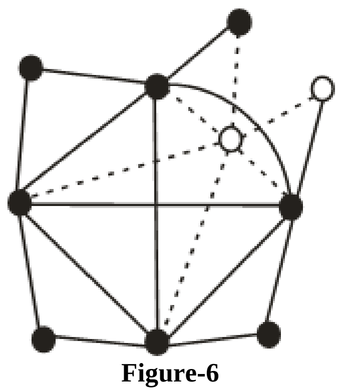


Figure-6

Then the graphical partitions of this graph will be

$\{2 + 2 + 2 + \cdots\cdots to\ (m + 1)\ terms\} + \{(m + 2) + (m + 2) + (m + 2) + \cdots\cdots to\ (m + 1)\ terms\}$

$= 2(m + 1) + (m + 2)(m + 1)$

$= (m + 1)(2 + m + 2)$

$= (m + 1)(m + 4)$

$= (m + 1)\{(m + 1) + 3\}$

$= a_{m+1}$

Thus the result is true for $n = m + 1$, whenever it is true for $n = m$. Hence completes the proof.

Theorem 3.3: For $n \geq 3$, the graphical partitions of the number $a_n = 6n$ always contain a helm graph H_n.

Proof: We shall proof the theorem by method of mathematical induction. We know that there are $(2n + 1)$ number of vertices in the helm graph H_n for $n \geq 3$, among which n vertices having degree one each, n vertices of degree four each and one vertex of degree n.

For $n = 3$, $a_3 = 18 = (1 + 1 + 1) + (4 + 4 + 4) + 3$. Then this partition represents a graph in which three vertices having degree one each, three vertices of degree four each and one vertex of degree three as shown in Figure-7. Clearly, this graph is a helm graph H_n.Thus the result is true for $n = 3$.

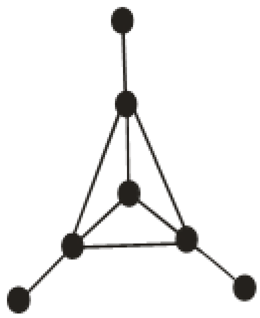


Figure-7

For $n = 4$, $a_4 = 24 = (1 + 1 + 1 + 1) + (4 + 4 + 4 + 4) + 4$.

Then this partitions represents a graph having four vertices of degree one each, four vertices of degree four each and one vertex of degree four as shown in Figure-8. This graph is a helm graph H_n and thus the result is true for $n = 4$.

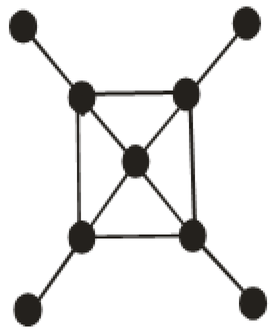


Figure-8

Let us assume that the result is true for $n = k$, that is, in the graphical partitions of the number $a_k = 6k$, there always exists a helm graph H_k in which k vertices having degree one each, k vertices of degree four each and one vertex of degree k. Then we have

$$a_k = (1 + 1 + 1 + \cdots\cdots to\ k\ terms) + (4 + 4 + 4 + \cdots\cdots to\ k\ terms) + k$$

Now, if we introduce a new vertex in any one edge of the wheel W_k of the helm graph H_k and adjoining a pendant edge to this new vertex as shown in the Figure-9, then the resulting graph will be a helm graph, having $(k + 1)$ vertices of degree one each, $(k + 1)$ vertices of degree four each and one vertex of degree $(k + 1)$.

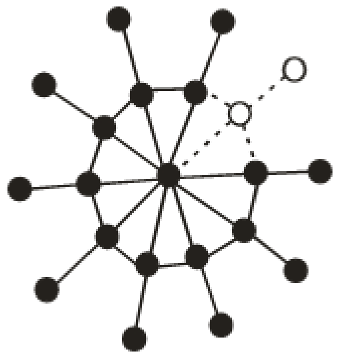


Figure-9

Then the graphical partitions of this graph will be

$\{1 + 1 + \cdots\cdots to\ (k+1)\ terms\} + \{4 + 4 + \cdots\cdots to\ (k+1)\ terms\} + (k+1)$

$= 1(k+1) + 4(k+1) + (k+1)$

$= (k+1)(1+4+1)$

$= 6(k+1)$

$= a_{k+1}$

Thus the result is true for $n = k+1$, whenever it is true for $n = k$. Hence completes the proof.

Theorem 3.4: In the graphical partitions of the number $a_n = 2(n^2 - n + 1)$ for $n \geq 3$, there always exists an n-Barbell graph for $n \geq 3$.

Proof: We shall proof the theorem by method of mathematical induction. We know that the n-Barbell graph for $n \geq 3$ consists of two copies of a complete graph K_n connected by a bridge, having $2n$ vertices altogether, where $2(n-1)$ vertices are of degree $(n-1)$ each and two vertices of degree n each for $n \geq 3$.

For $n = 3$, we have $a_3 = 14 = (2+2+2+2) + (3+3)$. Then this partition represents a graph having four vertices of degree two each and two vertices of degree three each as shown in Figure-10. Clearly, this graph is an n-Barbell graph for $n = 3$. Thus the result is true for $n = 3$.

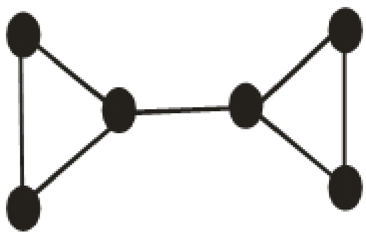


Figure-10

For $n = 4$, $a_4 = 26 = (3+3+3+3+3+3) + (4+4)$. Then this partitions represents a graph having six vertices of degree three each and two vertices of degree four each as shown in Figure-11, which is an n-Barbell graph for $n = 4$. So, the result is true for $n = 4$.

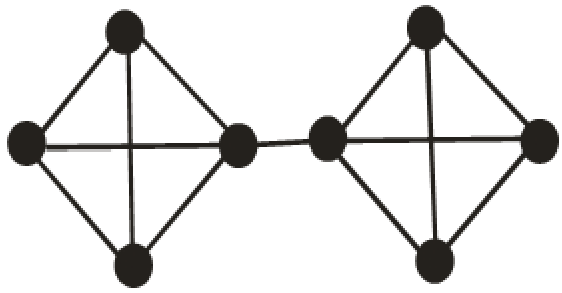


Figure-11

Let us assume that the result is true for $n = m$, that is, in the graphical partitions of the number $a_m = 2(m^2 - m + 1)$, there always exists an m-Barbell graph in which $2(m-1)$ vertices having degree $(m-1)$ each and two vertices of degree m each. Then we have

$$a_m = \{(m-1) + (m-1) + (m-1) + \cdots\cdots to\ 2(m-1)\ terms\} + (m+m)$$

Now, if we introduce two new vertices, each in any one edge of the cycle C_m of two copies of the complete graph K_m and connecting them to other vertices of the respective copies of K_m as shown in the Figure-12, then the resulting graph will be a Barbell graph, having $2m$ vertices of degree m each and two vertices of degree $(m+1)$ each.

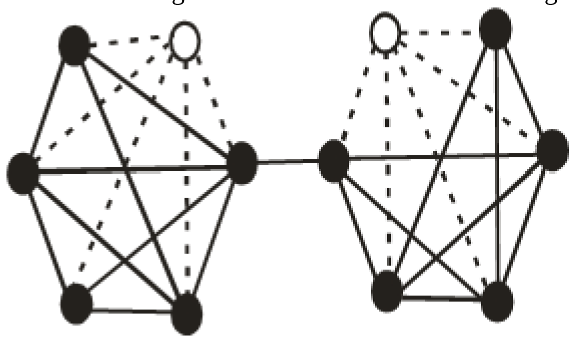


Figure-12

Then the graphical partitions of this graph will be
$\{m+m+m+\cdots\cdots to\ 2m\ terms\}+\{(m+1)+(m+1)\}$
$= 2m^2+2(m+1)$
$= 2m^2+2m+2$
$= 2(m^2+m+1)$
$= 2\{(m^2+2m+1)-m-1+1\}$
$= 2\{(m+1)^2-(m+1)+1\}$
$= a_{m+1}$

Thus the result is true for $n=m+1$, whenever it is true for $n=m$. Hence completes the proof.

Theorem 3.5: The graphical partitions of the number $a_n=2n^2$ for $n\geq 2$, there always contains a complete bipartite graph $K_{n,n}$ for $n\geq 2$.

Proof: We shall proof the theorem by method of mathematical induction. We know that the complete bipartite graph $K_{n,n}$ for $n\geq 2$, having $2n$ vertices is a regular graph of degree n.

For $n=2$, $a_2=8=2\cdot 2^2=(2+2)+(2+2)$. Then this partition represents a graph as shown in Figure-13. Clearly, this graph is a complete bipartite graph $K_{2,2}$ as well as regular having $2\times 2=4$ vertices of degree two each. Thus the result is true for $n=2$.

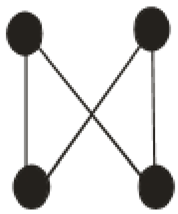


Figure-13

For $n=3$, $a_3=18=2\cdot 3^2=(3+3+3)+(3+3+3)$. Then this partition represents a graph as shown in Figure-14, which is a complete bipartite graph $K_{3,3}$ as well as regular having $2\times 3=6$ vertices of degree three each. So, the result is true for $n=3$.

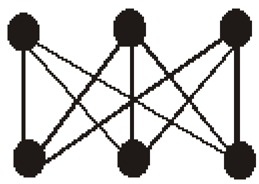


Figure-14

Let us assume that the result is true for $n=m$, that is, the graphical partitions of the number $a_m=2m^2$ always contain a complete bipartite graph $K_{m,m}$ which is regular having $2m$ vertices of degree m each. The wne have
$a_m = 2\cdot m^2 = m^2+m^2$
$= (m+m+m+\cdots\cdots to\ m\ terms)+(m+m+m+\cdots\cdots to\ m\ terms)$

Now, if we introduce two new vertices by joining each of them with $2m$ vertices, then the resulting graph will also be a complete bipartite graph which is also regular of degree $(m + 1)$ having $2(m + 1)$ vertices as shown in Figure-15.

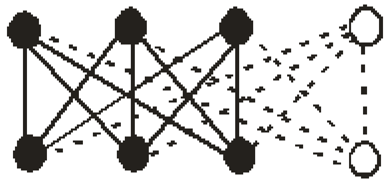


Figure-15

Then the graphical partitions of this graph will be
$\{(m + 1) + (m + 1) + \cdots\cdots to\ (m + 1)\ terms\} + \{(m + 1) + (m + 1) + \cdots\cdots to\ (m + 1)\ terms\}$
$= (m + 1)(m + 1) + (m + 1)(m + 1)$
$= (m + 1)^2 + (m + 1)^2$
$= 2(m + 1)^2$
$= a_{m+1}$

Thus the result is true for $n = m + 1$, whenever it is true for $n = m$. Hence completes the proof.

4. ALGORITHM

Input: Let $2(n + 1)$ / $n(n + 3)$ / $6n$ / $2(n^2 - n + 1)$ for $n \geq 3$ and $2n^2$ for $n \geq 2$ be different types of even numbers.

Output: To determine different forms of graphs produced from the graphical parts $g(2(n + 1))$ / $g(n(n + 3))$ / $g(6n)$ / $g(2(n^2 - n + 1))$ / $g(2n^2)$ of the partitions $p(2(n + 1))$ / $p(n(n + 3))$ / $p(6n)$ / $p(2(n^2 - n + 1))$ / $p(2n^2)$.

The following steps are considered.

Step 1: Study the graphical parts of $g(2(n + 1))$ / $g(n(n + 3))$ / $g(6n)$ / $g(2(n^2 - n + 1))$ of the partitions $p(2(n + 1))$ / $p(n(n + 3))$ / $p(6n)$ / $p(2(n^2 - n + 1))$ for $n \geq 3$ and $g(2n^2)$ of the partition $p(2n^2)$ for $n \geq 2$.

Step 2: If $\sum d(v_i) = 2e$, where $d(v_i)$ is the degree of v_i of the graphical parts $g(2(n + 1))$ for $n \geq 3$ and $e \geq 4$, then go to Step 3; otherwise go to Step 4.

Step 3: The graphical part is a pan graph.

Step 4: If $\sum d(v_i) = 2e$, where $d(v_i)$ is the degree of v_i of the graphical parts $g(n(n + 3))$ for $n \geq 3$ and $e = \frac{1}{2}(r + 2)(r + 5)$ $r \geq 1$ then go to Step 5; otherwise go to Step 6.

Step 5: The graphical part is a sun graph.

Step 6: If $\sum d(v_i) = 2e$, where $d(v_i)$ is the degree of v_i of the graphical parts $g(6n)$ for $n \geq 3$ and $e = 3n$, then go to Step 7; otherwise go to Step 8.

Step 7: The graphical part is a helm graph.

Step 8: If $\sum d(v_i) = 2e$, where $d(v_i)$ is the degree of v_i of the graphical parts $g(n^2 - n + 1)$ for $n \geq 3$ and $e = r^2 + 3r + 3$, $r \geq 1$ then go to Step 9; otherwise go to Step10.

Step 9: The graphical part is a barbell graph.

Step 10: If $\sum d(v_i) = 2e$, where $d(v_i)$ is the degree of v_i of the graphical parts $g(2n^2)$ for $n \geq 2$ and $e = n^2$ then go to Step 11 and Step 12.

Step 11: The graphical part is a complete bipartite graph.

Step 12: Stop

5. CONCLUSION

In our investigation, the graphical partitions of numbers $2(n+1)$, $n(n+3)$, $6n$, $2(n^2-n+1)$ for $n \geq 3$ and $2n^2$ for $n \geq 2$ have been studied with various properties. Further, an algorithm has been forwarded to produce different structures of graphs of the graphical parts.

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