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NUMERICAL APPROACH TO FIND THE DOMINANT EIGENVALUE AND ITS CORRESPONDING EIGENVECTORS OF CERTAIN TWO-PARAMETER MATRIX EIGENVALUE PROBLEM

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ABSTRACT

This paper is concerned about the numerical study of two-parameter right definite eigenvalue problems in matrix form, which resulted in a discretization of a two-parameter Sturm–Liouville problem. The coupled spectral parameters of two-parameter eigenvalue problems will be separated using the Kronecker product. The dominant eigen-pair and its corresponding eigenvectors of the generalized eigenvalue problem of one parameter will be calculated with the help of Power Method. Finally, a numerical example is presented to illustrate the Method.

Key words: *Multiparameter eigenvalue problem, Tensor Product, kronecker product, power method.*

Subject classification: *65F15, 65N25, 65Fxx (2010).*

1. INTRODUCTION

The Multiparameter Eigenvalue problems (MEPs) is the more generalization concept of classical one-parameter standard eigenvalue problems $Ax = \lambda x$ and the generalized eigenvalue problems $Ax = \lambda Bx$ where the problem is to find a k-tuple values $\lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4 \dots \dots \dots \dots \lambda_k) \epsilon\, C^k$ and non-zero vector $x_i \epsilon\, C^{n_i}$ for $i = 1, 2, \dots \dots k$ such that

$$\left(A_i - \sum_{j=1}^{k} \lambda_j B_{ij}\right) x_i = 0, \quad i = 1, 2, \dots \dots k \tag{1.1}$$

where $\lambda_i \in R^i$; $i = 1,2,3, \dots \dots \dots k.$ are spectral parameters and $A_{i,} B_{ij}$; $j = 1,2,3 \dots \dots \dots \dots \dots k$ are self-adjoint, bounded linear operators that act in Hilbert Spaces H_i, $x_i \in H_i$. The k-tuple $\lambda\, \epsilon\, C^k$ is called an eigenvalue and the tensor product $x = x_1 \otimes x_2 \otimes x_3 \dots \dots \dots . \otimes x_k$ is the corresponding (right) eigenvector. Similarly left eigenvector can be defined. All authors impose some "definiteness condition" on the matrix operators B_{ij} .

It has long been known that the method of separation of variables technique can be applied to the Helmholtz equation, as a result MEPs has been formed. This type of problem originates largely in the classical analysis. In particular, they appear in mathematical physics when method of separation of variables is used to solve of boundary-value problems for partial differential equations. MEPs have been studied extensively by Atkinson [1] from the viewpoint of determinantal operators on the tensor product space.

The main motivation for investigating the problems (1.1) is the numerical study of Multiparameter Strum-Liouville eigenvalue problems [5]

$$-\frac{d}{dx_i}\left\{p_i(x_i)\frac{d}{dx_i} y_i(x_i)\right\} + q_i(x_i) y_i(x_i) = \sum_{j=1}^{k} \lambda_i a_{ij}\, (x_i) y_i(x_i), \ a_i \le x_i \le b_i \quad i = 1,2,3, \tag{1.2}$$

This form occurs when the separation constants cannot be decoupled. These equations can be discretised to give MEPs the matrix form (1.1).

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It has been observed that one-parameter problems have grained much more development both theoretically and numerically in comparison to Multiparameter case. Moreover, numerical study on MEPs, particularly, two-parameter problems are also well investigated by many authors in terms of differential equations [3]. But only a few authors has studied two-parameter eigenvalue problem in terms of matrix equations [2, 7]. This paper is concerned about the numerical treatment to find the dominant eigenvalue and the corresponding eigenvectors of two-parameter problem in light of Power Method. The rest of the paper is organized as follows: section 2 contains a brief description of origin of two-parameter problem in matrix form. Section 3 contains some basic preliminaries of Kronecker product and Power Method. Section 4 contains a brief explanation of separation of spectral parameter by Kronecker Product Method. In section 5 a numerical example is given and Power method is used to find approximate eigen pair and corresponding eigenvector. Finally, in section 6 a conclusion on the performance of the Knonecker Product Method is given.

2. TWO-PARAMETER PROBLEM

The two-parameter matrix eigenvalue problem is a particular case of problem (1) for $i, j = 1, 2$. It can be written as a system of two homogeneous linear equations given by

$$(A_1 - \lambda B_{11} - \mu B_{12})x = 0 \tag{2.1}$$

$$(A_2 - \lambda B_{21} - \mu B_{22}) y = 0 \tag{2.2}$$

where, A_i, B_{ij} , $i, j = 1,2.$ are square $n_1 \times n_2$ matrices.

An eigenvalue is defined to be a pair (λ_1, λ_2) of complex numbers for which there exists a non-zero decomposable element $u = u_1 \otimes u_2$. We also require a definiteness condition

$$D(u_1, u_2) = \begin{vmatrix} (B_{11}u_1, u_1) & (B_{12}u_1, u_1) \\ (B_{21}u_2, u_2) & (B_{22}u_2, u_2) \end{vmatrix} \geq 0 \tag{2.3}$$

We call problem (2.1) and (2.2) right definite if the condition (2.3) holds.

This type of problems generally arises in separation of variables technique. If we consider the Helmholtz equation $\Delta u + k^2 u = 0$ (where Δ is the Laplace operator for two-dimension) in R^2 that arises in the modeling of the vibration of a fixed membrane and perform separation of variables on an elliptic domain we obtain the Mathieu and modified Mathieu equations

$$y_1''(x_1) + (2\lambda\, cosh 2x_1 - \mu) y_1(x_1) = 0 \tag{2.3}$$

$$y_2''(x_2) + (2\lambda\, cos 2x_2 - \mu) y_2(x_2) = 0 \tag{2.4}$$

To solve (2.3) and (2.4) simultaneously a two-parameter eigenvalue problem has been formed. Here λ is the eigenvalue corresponding to the physical value k^2, whereas μ is certain artificial eigenvalue as it arises as one of the separation parameters.

Two parameter eigenvalue problems in matrix form are found in various engineering applications also. In particular, when calculating the electrical properties of a material from measurements or inter-digital dielectrometry sensors, the properties of a material that has two layers will be the eigenvalues from the corresponding two-parameter eigenvalue problem [8]. In electrical engineering, Multiparameter eigenvalue problems formulation has been seen in Power flow equation [6]. In this formulation, both eigenvalues and eigenvectors are composed of the d and q orthogonal components of the bus voltages. In power flow equation two-parameter eigenvalue problems formulation are found in two bus systems with both PV and PQ buses.

Two-parameter eigenvalue problems also yield during dynamic model updating, which is concerned with the correction of finite element models by processing records by dynamic response from the test structures. If we consider a spring-mass model with known mass matrix such that the stiffness parameters of two springs have to be updated based on external measurements of their natural frequencies, then the updated parameters are nothing but the eigenvalues of the associated two-parameter eigenvalue problems [4]. It can also be shown, in Young-Frankel scheme, for the class separable partial differential equations of elliptic type in two independent variables, the optimum value of the over-relaxation parameter ω can be calculated when the eigenvalue of maximum modulus of certain two-parameter eigenvalue problem is known[4].

3. PRELIMINARIES

Definition 1: The Kronecker Product (.⊗.) : $\mathcal{C}^{m\times n} \times \mathcal{C}^{p\times q} \rightarrow \mathcal{C}^{mp\times nq}$ is defined by

$$A \otimes B = \begin{bmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{n1}B & \cdots & a_{nn}B \end{bmatrix} \qquad \text{where } (A)_{ij} = a_{ij}$$

Some properties Kronecker product:

a. (kA) ⊗ B = A⊗(k B) = k(A ⊗B)

b. $A\otimes(B + C) = A\otimes B + A\otimes C$
c. $(A + B) \otimes C = A \otimes C + B \otimes C$
d. If $A \in \mathcal{C}^{m\times n}$, $B \in \mathcal{C}^{p\times q}$, $C \in \mathcal{C}^{n\times k}$ $D \in \mathcal{C}^{q\times r}$ then
$(A \otimes B)(C \otimes D) = (AC \otimes BD)$

In particular if $A, B \in \mathcal{C}^{m\times n}$ and $x, y \in \mathcal{C}^{m}$, then $(Ax \otimes By)$

Definition 2: Let $\lambda_1, \lambda_2, \dots\dots\dots\dots, \lambda_n$ be the eigenvalues of a $n \times n$ square matrix A. λ_1 is called dominant eigenvalue of A if $|\lambda_1| > |\lambda_i|$, $i = 1,2, \dots\dots\dots n$.

Power method: This is an iterative method to find approximating eigenvalues of a matrix A. Let us consider the matrix A has a dominant eigenvalue with corresponding dominant eigenvector of A and x_0 be the initial approximation which is a non-zero vector in R^n. The sequence of iterative scheme is given by

$$x_1 = Ax_0$$
$$x_2 = Ax_1 = A.(Ax_0) = A^2x_0$$
$$x_3 = Ax_2 = A.(A^2x_0) = A^3x_0$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$\dots\dots\dots\dots\dots\dots\dots\dots$$
$$x_k = Ax_{k-1} = \cdots = A^kx_0$$

For large powers of k, and by properly scaling this sequence, we will get a good approximation of the dominant eigenvector of A.

4. KRONECKER PRODUCT METHOD

Equations (2.1) and (2.2) are equivalent to

$$(A_1 x - \lambda B_{11}x - \mu B_{12}x) = 0 \tag{4.1}$$
$$(A_2y - \lambda B_{21}y - \mu B_{22}y) = 0 \tag{4.2}$$

Kronecker multiplying (4.1) on the right by $B_{22}y$ and (4.2) on the left by $B_{12}x$ and equating we get

$$(A_1 x - \lambda B_{11}x - \mu B_{12}x) \otimes B_{22}y = B_{12}x \otimes (A_2y - \lambda B_{21}y - \mu B_{22}y)$$

$$\Rightarrow (A_1 x - \lambda B_{11}x) \otimes B_{22}y = B_{12}x \otimes (A_2y - \lambda B_{21}y)$$

$$\Rightarrow A_1 x \otimes B_{22}y - \lambda (B_{11}x \otimes B_{22}y) = B_{12}x \otimes A_2y - \lambda (B_{12}x \otimes B_{21}y)$$

$$\Rightarrow A_1 x \otimes B_{22}y - B_{12}x \otimes A_2y = \lambda (B_{11}x \otimes B_{22}y - B_{12}x \otimes B_{21}y)$$

$$\Rightarrow (A_1 \otimes B_{22} - B_{12} \otimes A_2)(x \otimes y) = \lambda (B_{11} \otimes B_{22} - B_{12} \otimes B_{21})(x \otimes y)$$

$$\Rightarrow \Delta_1 z = \lambda\Delta_0 z \tag{4.3}$$

Similarly, by Kronecker multiplying (4.1) on the right by $B_{21}y$ and (4.2) on the left by $B_{11}x$ and equating we get

$$\Delta_2 z = \mu\Delta_0 z \tag{4.4}$$

where

$$\Delta_0 = B_{11} \otimes B_{22} - B_{12} \otimes B_{21} \tag{4.5}$$
$$\Delta_1 = A_1 \otimes B_{22} - B_{12} \otimes A_2 \tag{4.6}$$
$$\Delta_2 = B_{11} \otimes A_2 - A_1 \otimes B_{21} \tag{4.7}$$
$$\text{and } z = x \otimes y \tag{4.8}$$

4. NUMERICAL EXAMPLE

Consider the following simple example

$$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = \lambda \begin{pmatrix} 15 & 2 \\ 2 & 20 \end{pmatrix} + \mu \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix} \tag{5.1}$$
$$\begin{pmatrix} 8 & 0 \\ 0 & 9 \end{pmatrix} = \lambda \begin{pmatrix} -4 & 0 \\ 0 & -5 \end{pmatrix} + \mu \begin{pmatrix} 15 & 0 \\ 0 & 25 \end{pmatrix} \tag{5.2}$$

The first one-parameter problem of equation (4.3) will be

$$\begin{pmatrix} 46 & 0 & 0 & 0 \\ 0 & 68 & 0 & 0 \\ 0 & 0 & 46 & 0 \\ 0 & 0 & 0 & 68 \end{pmatrix} z = \lambda \begin{pmatrix} 217 & 0 & 30 & 0 \\ 0 & 365 & 0 & 50 \\ 30 & 0 & 292 & 0 \\ 0 & 50 & 0 & 490 \end{pmatrix} z \tag{5.3}$$

Similarly, the second one-parameter problem of the equation (4.4) is given by

$$\begin{pmatrix} 128 & 0 & 16 & 0 \\ 0 & 145 & 0 & 18 \\ 16 & 0 & 168 & 0 \\ 0 & 18 & 0 & 190 \end{pmatrix} z = \mu \begin{pmatrix} 217 & 0 & 30 & 0 \\ 0 & 365 & 0 & 50 \\ 30 & 0 & 292 & 0 \\ 0 & 50 & 0 & 490 \end{pmatrix} z \tag{5.4}$$

$$\begin{pmatrix} 0.2150 & 0 & -0.0221 & 0 \\ 0 & 0.1889 & 0 & -0.0193 \\ -0.0221 & 0 & 0.1598 & 0 \\ 0 & -0.0193 & 0 & 0.1407 \end{pmatrix} z = \lambda I_3 \mathrm{z} \tag{5.5}$$

$$\begin{pmatrix} 0.5907 & 0 & -0.0059 & 0 \\ 0 & 0.3978 & 0 & -0.0039 \\ -0.0059 & 0 & 0.5759 & 0 \\ 0 & -0.0039 & 0 & 0.3881 \end{pmatrix} z = \mu I_3 \mathrm{z} \tag{5.6}$$

To find the approximating eigenvalue and eigenvector of (5.5) and (5.6) by Power Method discussed in section 3, MATLAB programme has been used. Calculated approximate dominant eigen-pair is **(0.2221, 0.5901)** and the corresponding common eigenvector is

$$[\mathbf{1} \quad \mathbf{0} \quad -\mathbf{0.32} \quad \mathbf{0}]^T.$$

5. CONCLUSION

The Kronecker Product Method discussed above for two-parameter eigenvalue problem in matrix equations, we have seen that the problem has been transformed into two generalized eigenvalue problems of one parameter on tensor product space $H_1 \otimes H_2$ of dimension N = $n_1 . n_2$. Due to significant increase in the dimension of N of the problem, the computational cost as well as time will be high for matrix equations of high dimension and hence this method is suitable for low dimensional matrix equation. Though we have considered the two-parameter case, this method can be extended to more than two-parameter eigenvalue problems.

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