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MINIMAL WEAKLY OPEN SETS AND MAXIMAL WEAKLY CLOSED SETS IN TOPOLOGICAL SPACES

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ABSTRACT

In this paper, a new class of sets called minimal weakly open sets and maximal weakly closed sets in topological spaces are introduced which are subclasses of weakly open sets and weakly closed sets respectively. We prove that the complement of minimal weakly open set is a maximal weakly closed set and some properties of the new concepts have been studied

Keywords: *Minimal open set, Maximal closed set, Minimal weakly open set, Maximal weakly closed set.*

Mathematics subject classification (2000): *54A05.*

1. INTRODUCTION

In the year 2001 and 2003, F.Nakaoka and N.oda[1] [2] [3] introduced and studied minimal open (resp.minimal closed) sets which are sub classes of open (resp.closed) sets. The complements of minimal open sets and maximal open sets are called maximal closed sets and minimal closed sets respectively. In the year 2000 M. Sheik john [4] introduced and studied weakly closed sets and weakly open sets in topological spaces.

Definition 1.1: [1] A proper non-empty open subset U of a topological space X is said to be minimal open set if any open set which is contained in U is φ or U.

Definition 1.2: [2] A proper non-empty open subset U of a topological space X is said to be maximal open set if any open set which is contained in U is X or U.

Definition 1.3: [3] A proper non-empty closed subset F of a topological space X is said to be minimal closed set if any closed set which is contained in F is φ or F.

Definition 1.4: [3] A proper non-empty closed subset F of a topological space X is said to be maximal closed set if any closed set which is contained in F is X or F.

Definition 1.5: [4] A subset A of (X, τ) is called weakly closed set if Cl(A) $\subseteq$ U whenever A $\subseteq$ U and U is semi-open in X.

Definition 1.6: [4] A subset A in (X, τ) is called weakly open set in X if A^c is weakly closed set in X.

2. MINIMAL WEAKLY OPEN SETS

Definition 2.1: A proper non-empty weakly open subset U of X is said to be minimal weakly open set if any weakly open set which is contained in U is φ or U.

Remark 2.2: Minimal open sets and Minimal weakly open sets are independent of each other as seen from the following example.

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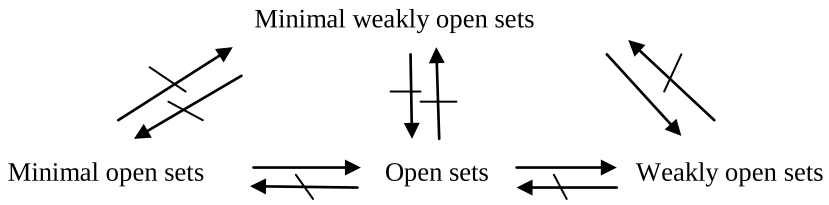
Example 2.3: Let X= {a,b,c} be with the topology τ ={X, φ,{a,b}}
Open sets are = {X, φ ,{a,b}}
Minimal open sets are = {a,b}
Weakly open sets are = {X, φ,{a},{b},{a,b}}
Minimal weakly open sets are = {{a},{b}}

Remark 2.4: From the Known results and by the above example we have the following implication.



Theorem 2.5:
(i) Let U be a minimal weakly open set and W be a weakly open set then U∩W= φ or U⊂W.
(ii) Let U and V be minimal weakly open sets then U∩V= φ or U=V.

Proof:
(i) Let U be a minimal weakly open set and W be a weakly open set. If U∩W= φ, then there is nothing to prove but if U∩W≠ φ then we have to prove that U⊂W. Suppose U∩W≠ φ then U∩W⊂U and U∩W is Weakly open as the finite intersection of weakly open sets is a weakly open set. Since U is a minimal weakly open set, we have U∩W=U therefore U⊂W.

(ii) Let U and V be minimal weakly open sets suppose U∩V≠ φ then we see that U⊂V and V⊂U by (i) therefore U=V.

Theorem 2.6: Let U be a minimal weakly open set if x is an element of U then U⊂W for any open neighbourhood W of x.

Proof: Let U be a minimal weakly open set and x be an element of U. Suppose there exits an open neighbourhood W of x such that U⊄W then U∩W is a weakly open set such that U∩W⊂U and U∩W≠ φ. Since U is a minimal weakly open set, We have U∩W=U that is U⊂W. This contradicts our assumption that U⊄W. Therefore U⊂W for any open neighbourhood W of x.

Theorem 2.7: Let U be a minimal weakly open set, if x is an element of U then U⊂W for any weakly open set W containing x.

Proof: Let U be a minimal weakly open set containing an element x.Suppose there exists an weakly open set W containing x such that U⊄ W then U∩W is an weakly open set such that U∩W⊂U and U∩W≠ φ. Since U is a minimal weakly open set,we have U∩W=U that is U⊂W .This contradicts our assumption that U⊄W.Therefore U⊂W for any weakly open set W containing x.

Theorem 2.8: Let U be a minimal weakly open set then U= ∩{W: W is any weakly open set containing x} for any element x of U

Proof: By theorem 2.7 and from the fact that U is a weakly open set Containing x, We have U⊂∩ {W: W is any weakly open set containing x}⊂W. Therefore we have the following result.

Theorem 2.9: Let U be a non-empty weakly open set then the following three conditions are equivalent.
(i) U is a minimal weakly open set
(ii) U⊂w-cl(S) for any non-empty subset S of U.
(iii) w-cl(U)= w-cl(S) for any non-empty subset S of U.

Proof: (i) $\Longrightarrow$ (ii) Let U be a minimal weakly open set and S be a non-empty subset of U. Let x∈U by theorem 2.7 for any weakly open set W containing x, S⊂U⊂W which implies S⊂W. Now S=S∩U⊂S∩W. Since S is non-empty therefore S∩W≠ φ. Since W is any weakly open set containig x by one of the theorem, we know that, for an x∈X, x∈w-cl(A) iff V∩ A ≠ φ. for any every weakly open set V Containing x that is x∈U implies x∈cl(s) which implies U⊂ w-cl(s) for any non-empty subset S of U.

(ii)$\Longrightarrow$(iii) Let S be a non-empty subset of U that is $S\subset U$ which implies
w-cl(S) $\subset$w-cl(U)$\rightarrow$ (a) Again from (ii) $U\subset$w-cl(s) for any non-empty subset S of U.

Which implies w-cl(U) $\subset$w-cl(w-cl(S))=w-cl(S) i.e., w-cl(U) $\subset$w-cl(S) $\rightarrow$ (b), from (a) and (b) w-cl(U)= w-cl(S) for any non empty subset S of U.

(iii)$\Longrightarrow$ (i) from (3) we have w-cl(U) = w-cl(S) for any non-empty subset S of U. Suppose U is not a minimal weakly open set then there exist a non-empty weakly open set V such that $V\subset U$ and $V \neq U$. Now there exists an element $a\in U$ such that $a \notin V$ which implies $a\in V^c$ that is w-cl$\{(a)\} \subset$ w-cl$\{(V^c)\}= V^c$,as V^c is a weakly closed set in X. It follows that w-cl($\{a\}$)$\neq$w-cl(U). This is contradiction to fact that w-cl($\{a\}$)= w-cl(U) for any non empty subset $\{a\}$ of U therefore U is a minimal weakly open set.

Theorem 2.10: Let V be a non-empty finite weakly open set, then there exists at least one (finite) minimal weakly open set U such that $U\subset V$.

Proof: Let V be a non-empty finite weakly open set. If V is a minimal weakly open set, we may set U=V. If V is not a minimal weakly open set, then there exists a (finite) weakly open set V_1 such that $\emptyset \neq V_1\subset V$. If V_1 is a minimal weakly open set, we may set $U=V_1$. If V_1 is not a minimal weakly open set then there exists a (finite) weakly open set V_2 such that $\emptyset \neq V_2\subset V_1$. Continuing this process we have a sequence of weakly open sets $V_K \ldots\subset V_3\subset V_2\subset V_1\subset V$. Since V is a finite set, this process repeats only finitely then finally we get a minimal weakly open set $U=V_n$ for some positive integer n.

Corollary 2.11: Let X be a locally finite space and V be a non-empty weakly open set then there exists at least one(finite) minimal weakly open set such that $U\subset V$.

Proof: Let X be a locally finite space and V be a non empty weakly open set. Let $x\in V$ since X is a locally finite space we have a finite open set V_x such that $x\in V_x$ then $V\cap V_x$ is a finite weakly open set. By theorem 2.10 there exist at least one (finite) minimal weakly open set U such that $U \subset V\cap V_x$ that is $U\subset V\cap V_x\subset V$. Hence there exists at least one (finite) minimal weakly open set U such that $U \subset V$.

Corollary 2.12: Let V be a finite minimal open set then there exist at least one(finite) minimal weakly open set U such that $U\subset V$.

Proof: Let V be a finite minimal open set then V is a non-empty finite weakly open set, by theorem 2.10 there exist at least one (finite)minimal weakly open set U such that $U\subset V$.

Theorem 2.13: Let U and U_λ be minimal weakly open sets for any element λ of Λ.If $U\subset \bigcup_{\lambda\in\Lambda} U_\lambda$ then there exists an element $\lambda \in \Lambda$ such that $U= U_\lambda$.

Proof: Let $U \subset \bigcup_{\lambda\in\Lambda} U_\lambda$ then $U\cap (\bigcup_{\lambda\in\Lambda} U_\lambda) = U$ that is $\bigcup_{\lambda\in\Lambda} (U\cap U_\lambda) = U$ also by Theorem 2.5 (ii) $U\cap U_\lambda= \varphi$ for any $\lambda \in \Lambda$ it follows that there exist an element $\lambda \in \Lambda$ such that $U= U_\lambda$.

Theorem 2.14: Let U and U_λ be minimal weakly open sets for any element $\lambda\in\Lambda$. if $U= U_\lambda$ for any element λ of Λ
If $U= U_\lambda$ for any element λ of Λ then $(\bigcup_{\lambda\in\Lambda} U_\lambda) \cap U= \emptyset$.

Proof: Suppose that $(\bigcup_{\lambda\in\Lambda} U_\lambda)\cap U \neq \emptyset$ that is $\bigcup_{\lambda\in\Lambda} (U_\lambda\cap U) = \emptyset$. then there exists an element $\lambda \in \Lambda$ such that $U \cap U_\lambda \neq \emptyset$ by theorem 2.5 (ii) we have $U= U_\lambda$, which contradicts the fact that $U \neq U_\lambda$ for any $\lambda \in \Lambda$ then $(\bigcup_{\lambda\in\Lambda} U_\lambda)\cap U = \emptyset$.

Theorem 2.15: Let U_λ be a minimal weakly open set for any element $\lambda \in \Lambda$ and $U_\lambda \neq U\mu$ for any element λ and μ of Λ with $\lambda\neq \mu$ assume that $|\Lambda| > 2$. Let μ be any element of Λ then $(\bigcup_{\lambda\in\Lambda-\{M\}} U_\lambda) \cap U\mu = \emptyset$.

Proof: Put $U= U\mu$ in theorem 2.14, then we have the result.

Corollary 2.16: Let U_λ be a minimal weakly open set for any element $\lambda \in \Lambda$ and $U_\lambda \neq U_\mu$ for any element λ and μ of Λ with $\lambda \neq \mu$. If η a proper non-empty subset of Λ then $(\bigcup_{\lambda\in\Lambda-\{\eta\}} U_\lambda) \cap (\bigcup_{\gamma\in\eta} U_\lambda) = \emptyset$.

Theorem 2.17: Let U_λ and $U\gamma$ be minimal weakly open sets for any element $\lambda \in \Lambda$ and $\gamma \in \eta$ such that $U_\lambda \neq U_\gamma$ for any element λ of Λ then $\bigcup_{\gamma\in\eta}$ then $\bigcup_{\gamma\in\eta} U_\lambda \not\subset (\bigcup_{\lambda\in\Lambda} U_\lambda)$.

Proof: Suppose that an element γ^1 of η satisfies $U_\lambda = U_{\gamma^1}$ for any element λ of Λ. if $\bigcup_{\gamma\in\eta} U_\lambda \subset (\bigcup_{\lambda\in\Lambda} U_\lambda)$, then we see $U_{\gamma^1} \subset (\bigcup_{\lambda\in\Lambda} U_\lambda)$ by theorem 2.13 there exists an element λ of Λ such that $U_{\gamma^1} = U_\lambda$ which is a contradiction it follows that $\bigcup_{\gamma\in\eta} U \not\subset (\bigcup_{\lambda\in\Lambda} U_\lambda)$.

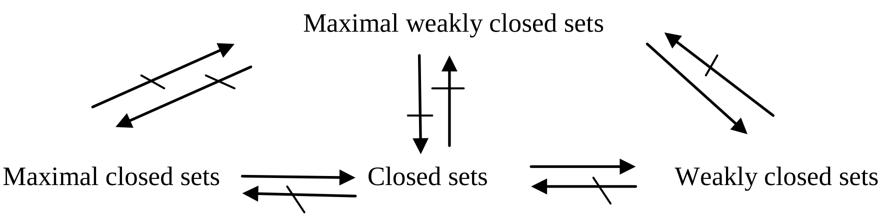
3. MAXIMAL WEAKLY CLOSED SETS

Definition 3.1: A proper non-empty weakly closed subset F of X is said to be Maximal weakly closed set if any weakly closed set which is contained in F is *X* or F.

Remark 3.2: Maximal closed sets and Maximal weakly closed sets are independent each other as seen from the following implication.

Example 3.3: Let X={a,b,c} and τ ={X, ∅,{a,b}} be a topological space.
Closed sets are ={X, ∅,{c}}
Maximal closed sets are = {c}
Weakly closed sets are ={X, ∅ ,{c},{b,c},{a,c}}
Maximal weakly closed sets are ={{a,c},{b,c}}

Remark 3.4: From the known results and by the above example 3.3 we have the following implication.



Theorem 3.5: A proper non-empty subset F of X is Maximal weakly closed set iff X-F is a minimal weakly open set.

Proof: Let F be a Maximal weakly closed set, suppose X-F is not a minimal weakly open set then there exists a weakly open set U≠ X − F such that ∅ ≠U ⊂ X-F that is F⊂X-U and X-U is a weakly closed set. This contradicts our assumption that F is a minimal weakly open set. Conversely, let X-F be a minimal weakly open set. Suppose F is not a Maximal weakly closed set then there exist a weakly closed set E ≠ F such that F⊂E ≠ X that is ∅ ≠ X-E ⊂ X-F and X-E is a weakly open set. This Contradicts our assumption that X-F is a minimal weakly open set. Therefore F is a Maximal weakly closed set.

Theorem 3.6:
(i) Let F be a Maximal weakly closed set and W be a weakly closed set Then F∪W=X or W⊂F.
(ii) Let F and S be Maximal weakly closed sets then F∪S=X or F=S

Proof:
(i): Let F be a Maximal weakly closed set and W be a Weakly closed set if F∪W=X then there is nothing to prove but if F∪W≠ X, then we have to prove that W⊂F. Suppose F∪W≠ X then F ⊂F∪W and F∪W is weakly closed as the finite union of weakly closed set is a weakly closed set we have F∪W=X Therefore F∪W=F which implies W⊂F.
(ii): Let F and S be Maximal weakly closed sets. Suppose F∪S≠ X then we see that F⊂S and S⊂F by (i) therefore F=S.

Theorem 3.7: Let F be a Maximal weakly closed set. If x is an element of F then for any weakly closed set S containing x, F∪S=X.

Proof: Proof is similar to 2.7 theorem.

Theorem 3.8: Let F_α, F_η ,$F\gamma$ be Maximal weakly closed sets such that $F_\alpha \neq F\eta$ if $F_\alpha \cap F\eta \subset F\gamma$. then either $F_\alpha = F\gamma$ or $F\eta = F\gamma$.

Proof: Given that $F_\alpha \cap F\eta \subset F\gamma$, if $F_\alpha = F\gamma$ then there is nothing to prove but if $F_\alpha \neq F\gamma$ then We have to prove $F\eta = F\gamma$.

Now we have

$$
\begin{aligned}
F_\eta \cap F\gamma &= F_\eta \cap (F\gamma \cap X) \\
&= F_\eta \cap (F\gamma \cap (F_\alpha \cap F\eta) \text{ (by theorem 3.6 (ii))} \\
&= F_\eta \cap (F_\gamma \cap F_\alpha) \cup (F\gamma \cap F\eta) \\
&= (F_\eta \cap F\gamma \cap F_\alpha) \cup (F_\eta \cap F\gamma \cap F_\eta) \\
&= (F_\alpha \cap F\eta) \cup (F\gamma \cap F\eta) \text{ (by } F_\eta \cap F\gamma \cap F_\alpha \text{)} \\
&= (F_\alpha \cup F\gamma) \cap F_\eta \\
&= X \cap F_\eta \text{ (since } F_\alpha \text{, and } F\gamma \text{ are Maximal weakly closed sets by thm 3.6(ii) } F_\alpha \cup F\gamma = X \\
&= F_\eta \text{ that is } F_\eta \cap F\gamma = F_\eta \text{ implies } F_\eta \subset F\gamma,
\end{aligned}
$$

since F_η , $F\gamma$ are maximal weakly closed sets we have $F\eta = F\gamma$.

Theorem 3.9: Let F_α, F_η ,$F\gamma$ be Maximal weakly closed sets which are different from each other then
$(F_\alpha \cap F\eta) \not\subset (F_\alpha \cap F_\gamma)$.

Proof: Let $((F_\alpha \cap F\eta) \subset (F_\alpha \cap F\gamma)$ which implies $(F_\alpha \cap F\eta) \cup (F\gamma \cap F\eta) \subset (F_\alpha \cap F\gamma) \cup (F\gamma \cap F_\eta)$ which implies $(F_\alpha \cup F\gamma) \cap (F\gamma \cup F\eta) \cap (F_\alpha \cup F_\eta)$ since by theorem 3.6 (ii) $F_\alpha \cap F\gamma = X$ and $F_\alpha \cap F\eta = X$ which implies $X \cap F_\eta \subset F\gamma \cap X$ which implies $F_\eta \subset F\gamma$. From the definition of Maximal weakly closed set it follows that $F_\eta = F\gamma$. This is contradiction to the fact that Let F_α, F_η ,$F\gamma$ are different from each other. Therefore $(F_\alpha \cap F\eta) \not\subset (F_\alpha \cap F_\gamma)$.

Theorem 3.10: Let F be a Maximal weakly closed set and x be an element of F then $F=\cup\{S$: S is a weakly closed set containing x such that $F \cup S \neq X\}$.

Proof: By theorem 3.7 and from the fact that F is a weakly closed set containing x we have $F \subset \cup\{S$:S is a weakly closed set containing x such that $F \cup S \neq X\} \subset F$ therefore we have the result.

Theorem 3.11: Let F be a Proper non-empty co-finite weakly closed subset then there exists (co-finite) Maximal weakly closed set E such that $F \subset E$.

Proof**:** Let F be a non-empty co-finite weakly closed set. If F is a Maximal weakly closed set, we may set E=F. If F is not a Maximal weakly closed set, then there exists a (co-finite) weakly closed set F_1 such that $F \subset F_1 \neq X$. If F_1 is a Maximal weakly closed set, we may set $E = F_1$ .If F_1 is not a Maximal weakly closed set ,then there exists a (co-finite) weakly closed set sets F_2 such that $F \subset F_1 \subset F_2 \neq X$ continuing this process we have a sequence of weakly closed sets $F \subset F_1 \subset F_2 \subset \ldots \subset F_K \subset \ldots$ since F is a co-finite set, this process repeats only finitely then finally we get a minimal weakly open set $E=E_n$ for some positive integer n.

Theorem 3.12: Let F be a Maximal weakly closed set. If x is an element of X-F then $X-F \subset E$ for any weakly closed set containing set E containing x

Proof: Let F be a Maximal weakly closed set and $x \in X–F$. $E \not\subset F$ for any weakly closed set E containing x then $E \cup F=X$ by theorem 3.6(ii) .Therefore $X–F \subset E$.

REFERENCES

1. F.Nakaoka and F.Oda, Some Application of minimal open sets, Int.J.Math.Math.sci.Vol 27, No.8, 471-476(2001).
2. F.Nakaoka and F.Oda, Some Properties of Minimal open sets, Int.J.Math.Math.sci.Vol 21, No.21, 1331-1340(2001).
3. F.Nakaoka and F.Oda,on maximal closed sets, Proceeding of topological spaces and It's application, 5, 19-21(2003).

4. M. Sheik John, A study on generalization of closed sets on continuous maps in topological and bitopological spaces, Ph.d thesis Bharathiar University, Coimbatore, (2002).

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