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USING CRANK-NICLSON METHOD TO COMPUTE THE NUMERICAL BLOW-UP TIME OF A SEMILINEAR PARABOLIC PROBLEM

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ABSTRACT

In this paper, we study the numerical blow-up solutions of the semilinear heat equation with power type nonlinearities, by using Crank-Nicolson method. We consider a numerical experiment, with quadratic or cubic non linearities, with using the stability condition which was first suggested in [2], with considering different values to the parameter α, which appear in this condition.Moreover, we studied the influence of using large or small values for α on the numerical blow-up times for the problem.

1. INTRODUCTION

It is well known that semilinear parabolic equations arise in many physical situations, where diffusive phenomena and source terms have to be modelled. In [6] Lacy presents a number of physical situations including chemical reactions and electrical heating, where blow-up has physical significance.

This work is concerned with the zero Dirichlet boundary condition of the semilinear heat equation with a special reaction term, which is a power type function:

$$\left.\begin{array}{lr} u_t = u_{xx} + u^p, & 0 < x < 1, \quad p > 1 \\ u(0,t) = u(1,t) = 0, & t > 0 \\ u(x,0) = u_0(x), & 0 < x < 1 \end{array}\right] \tag{1}$$

The problem of semilinear parabolic equation has been introduced in [2, 3, 4, 5]. For instance, in [5] Friedman and McLeod have studied problem (1), under fairly general assumptions on u_0.It has been proved that the solutions of this problem blow up in finite time at only a single point, i.e. there exists T>0, such that:

$$\sup_{x\in[0,1]}|u(x,t)| \to \infty, \quad as\ t \to T^-.$$

For more details about blow-up phenomena, see [10].

In fact, little attention has been devoted to the numerical study for this problem, Abia and Budd [1] considered uniform discretizations of problem (1), and analyzed their blow-up regions and asymptotic behaviour at blow-up points. In order to capture the qualitative behaviour in the blow-up region, Budd, Huang and Russell [3] have considered moving mesh methods for a wide class of problems. Nakagawa [8] and Chen [4] have studied numerical blow-up for two fully discretized finite differences schemes for the problem (1), when the reaction function has the form $f(u) = u^p,\ p > 1$. In [2], it has been considered semi discrete problems based on uniform discretizations, but it was mainly concerned with their blow-up times and their convergence to the blow-up time of (1). It has also considered more general nonlinear terms $f(u)$ and assumes that the function f is at least defined on $[0, \infty)$. Explicit and implicit Euler methods have been used to find the numerical solutions of an experiment, with $f(u) = u^2$, and with a special initial function u_0.

In this work, we use the Crank-Niclson method, to find the numerical blow-up solutions of problem (1), where $p = 2,3$.We will study the influence of using large or small values for α on the numerical blow-up times for the problem.

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2. THE SEMIDISCRETE PROBLEM

For J a positive integer, we set $h = 1/J$ and define the grid $x_j = jh, 0 \leq j \leq J$.

Let S^2 denote the standard second order difference operator. We approximate the solution u of the problem (1) by the solution $U_h(t) = (U_0(t) \ldots\ldots U_J(t))^T$ of the semidiscrete equations

$$\left.\begin{array}{r} \frac{d}{dt}U_j - S^2U_j = f(U_j), \ldots\ldots\ldots\ldots 1 \leq j \leq J-1, t \geq 0, \\ U_0(t) = U_J(t) = 0, \ldots\ldots\ldots\ldots t \geq 0, \\ U_j(0) = U_j^0, \ldots\ldots\ldots\ldots 0 \leq j \leq J \end{array}\right] \qquad (2)$$

3. BLOW-UP IN THE SEMIDISERETE PROBLEM

The solutions of (2) do not exist for all $t \in [0, \infty)$, because they become unbounded. We denote

$$||U_h(t)||_\infty = \max_{j=0,\ldots,J} |U\,j\,(t)|.$$

Definition: Let U_h be a nonnegative solution of (2). We say that U_h achieves blow-up if there exists $T_h^b < \infty$ such that:

1. $||U_h(t)||_\infty < \infty, \quad t \in [0, T_h^b),$
2. $\lim_{t \to T_h^b} ||U_h(t)||_\infty = \infty,$

The time T_h^b called the blow-up time.

Theorem [2]: Let U_h be a nonnegative solution of (2), If U_h achieves blow-up,then $T_h^b < \infty$.

The next theorem establishes that, for each fixed time interval [0, T] where u is defined, the solution of the semidiscrete problem (2) approximates u, as $h \to 0$.

Theorem [2].Assume that:
(a) problem (1) has a solution $u \in C^{4,1}([0,1] \text{ X } [0,T])$;
(b) the initial condition U_h^0at (2) satisfies

$$||U_h^0 - u_h(0)||_\infty = O(1), \quad h \to 0.$$

Then, for h sufficiently small, problem (2) has a unique solution

$$U_h \in C^1([0,T], R^{J+1})$$

Such that

$$\max_{t \in [0,T]} ||U_h(t) - u_h(t)||_\infty = 0(||U_h^0 - u_h(0)||_\infty + h^2), \qquad h \to 0.$$

The next theorem establishes the convergence of the blow-up time of the approximate semidiscrete problems to the blow-up time of the theoretical solution.

Theorem [2]: Assume that there exists $T_b < \infty$ such that $u \in C^{4,2}([0,1]x\,[0,T_b), R)$ where$u$ is the solution of problem (1).

If $||U_h(0) - u_h(0)||_\infty = \mathrm{O}(1)$as $h \to 0$, then the solution of (2), U_h, achieves blow-up, for h sufficiently small, at $T_b$and $\lim_{h \to 0} T_b^h = T_b$.

4. NUMERICAL EXPERIMENTS

In this section, we present some numerical approximations to the blow-up time of problem (1), with the initial function,$u_0(x) = 20 sin\,\pi x$, which has been considered in [2],It is clear that u_0 takes its maximum value at the point $x = 1/2$, therefore according to the known blow-up results for the problem of semilinear heat equation (see [5]) the blow-up in problem (1) occurs only at a single point, which is $x = 1/2$.

We study two special cases for the power, firstly $p = 2$ secondly $p = 3$. We obtained such numerical approximations by integrating numerically with respect to time the semidiscrete problem (2) with the initial condition given by the nodal values of u_0. The experiments were solved numerically in [2], by using two finite difference methods.
Firstly explicit Eulermethod:

$$U_j^{n+1} = U_j^n - \Delta t_n S^2 U_j^n + \Delta t_n (U_j^n)^p, \quad 1 \le j \le J-1,$$
$$U_0^{n+1} = U_J^{n+1} = 0.$$

where S is the centre finite difference operator.

Recall that, $2\Delta t_n/h^2 \le 1$, the well-known stability condition of the explicit Euler method for the heat equation. Secondly, using implicit Euler method:

$$U_j^{n+1} - \Delta t_n S^2 U_j^{n+1} = U_j^n + \Delta t_n (U_j^n)^p, \quad 1 \le j \le J-1,$$
$$U_0^{n+1} = U_J^{n+1} = 0.$$

Here, the numerical experiment conducted with the Crank-Nicolson method given by

$$U_j^{n+1} - \frac{\Delta t_n}{2} S^2 U_j^{n+1} - \frac{\Delta t_n}{2}(U_j^{n+1})^p = U_j^n + \frac{\Delta t_n}{2} S^2 U_j^n + \frac{\Delta t_n}{2}(U_j^n)^p, \quad 1 \le j \le J-1,$$
$$U_0^{n+1} = U_J^{n+1} = 0.$$

The time step for all these methods was taken as:

$$\Delta t_n = \min\left(\frac{h^2}{2}, \frac{h^\alpha}{||U^n||_\infty}\right), \quad n \ge 0 \dots\dots (3).$$

where, this time step was first suggest in [2].

We have considered different choices of α in order to examine experimentally, if there exists any rate of convergence for the numerical blow-up times with respect to the mesh size h. The numerical integration was terminated at the first time that $||U^n||_\infty \ge 10^{15}$, and the value $T_J^n = \sum_{m=0}^{n-1} \Delta t_m$, was taken as a numerical approximation to the blow-up time T_b^h of the semidiscrete problem. We refer to the last iteration before numerical blow-up occurs by k.
The problems were solved by using Matlab programming.

Every five rows of the table correspond to the use of indicated value for α in the time stepping procedure. In the columns, we show numerical blow-up times, which arise from using Crank-Nicolson method, corresponding to meshes 10, 20 and 40 subintervals. The errors in the numerical bow-up times, are computed by using

$$E_J = |T_{2J}^k - T_J^k|, \tag{4}$$

for$J$ takes the values 10 and 20.

Table-1: Computed blow-up times, P=2

α	J=10	K	J=20	K	J=40	K
1	0.0844	399	0.0832	896	0.0827	2124
1/2	0.0900	139	0.0853	238	0.0834	510
1/10	0.0948	73	0.0869	126	0.0839	327
1/50	0.0957	65	0.0872	116	0.0840	315
1/100	0.0958	65	0.0873	115	0.0840	313

Table-2: Computed blow-up times, P=3

α	J=10	K	J=20	K	J=40	K
1	0.0042	7	0.0025	10	0.0018	15
1/2	0.0082	6	0.0041	8	0.0023	11
1/10	0.0105	6	0.0048	7	0.0025	11
1/50	0.0106	6	0.0050	7	0.0025	11
1/100	0.0106	6	0.0050	7	0.0025	11

Table-3: errors in the numerical bow-up times $E_J = \left|T_{2J}^k - T_J^k\right|,\ p = 2,\ p = 3$

α	J=10	J=20	α	J=10	J=20
1	0.0012	0.0005	1	0.0017	0.0007
1/2	0.0047	0.0019	1/2	0.0041	0.0018
1/10	0.0079	0.0030	1/10	0.0057	0.0023
1/50	0.0085	0.0032	1/50	0.0056	0.0025
1/100	0.0085	0.0032	1/100	0.0056	0.0025

It is well known that, for each fixed time interval $[0, T]$, where the solution u of (1) is defined and sufficiently smooth, the numerical schemes (explicit &implicit Euler methods) considered approximate u with a rate of convergence of $\mathrm{O}(\mathrm{T} + h^2)$, where $T = \max \ \Delta_{tn}$, while for Crank-Nicolson schemes, it is well known that the numerical solutions convergent to the exact solution with a rate of $\mathrm{O}(\mathrm{T}^2 + h^2)$.Because of the choice of Δ_{tn}, we have a rate of convergence O(h^{α}), as $h \to 0$. The same order of convergence might be expected for the numerical blow-up times.

The next figures show the evolutions of the numerical bow-up solutions of problems (2), which arise from using Crank-Nicolson method, for different values to J and α.

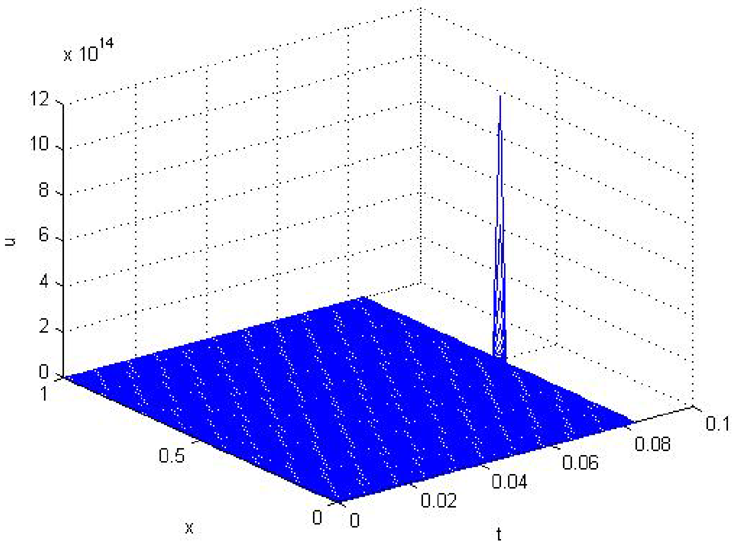

Figure-1: $p = 2,\ J = 40,\ \ \alpha = 1/100,\ \ t \in [0, t_k]$

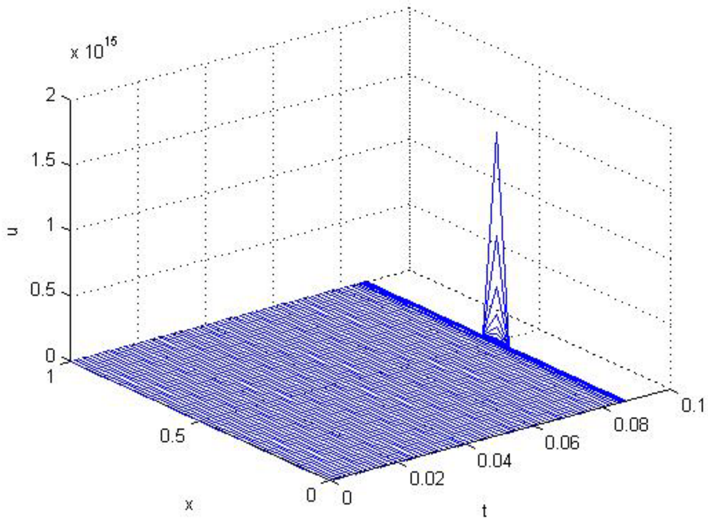

Figure-2: $p = 2,\ J = 20, = 1/100,\ t \in [0, t_k]$

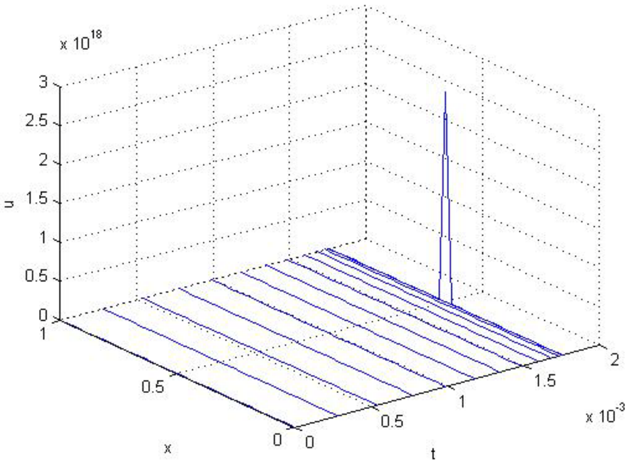

Figure-3: $p = 3, J = 40, \quad \alpha = 1, \ t \in [0, t_k]$

4. CONCLUSIONS

From our Numerical results (Table 1,2 &3), we can point out the following conclusions:

1. Decreasing the values of α, leads almost to decreasing the number of iterations, k, until the numerical blow up occurs, and increasing the numerical blow-up times.
2. For a fixed value to J (for instance J=10), we have found that the corresponding numerical blow-up time is larger than the numerical blow-up time, with respect to $2J$.
3. Taking large values for J (meaning small values for h), with small values for α, gives similar results when α is large and J is certainly small.
4. The table of errors in the computed blow-up times, that was computed using (4), shows that, for a fixed value of J, increasing the value of α, leads to decreasing the errors. On the other hand, for a fixed values for α and J, we have$E_{2J} < E_J$.

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