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THE DEGREE OF APPROXIMATION OF CONJUGATES OF ALMOST LIPSCHITZ FUNCTIONS BY (N, p, q) (E, q) MEANS

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ABSTRACT

The degree of approximation of f (x), conjugate of a function f ∈ almost lip α, by (N, p, q) (E, q) means of the conjugate series is determined.

Keywords: *Almost Lipschitz function; Degree of Approximation; (N, p, q) means; (E, q) means; conjugate series.*

1. INTRODUCTION AND DEFINITIONS

The degree of approximation by Cesaro means and by Nörlund means of the Fourier series of a function f ∈ Lip α have been studied by Alexits [1], Sahney and Goel [8], Chandra [3], Qureshi ([4], [5]), and Qureshi and Neha [7], But till now no work seems to have been done to obtain the degree of approximation of f(x), conjugate of a function f ∈ L^a ip α, by product of generalized Nörlund mean (N, p, q) and Euler's means of order q, (E, q) in an attempt to make an advance study in this direction, the object of this paper is to determine the degree of approximation of conjugates of almost Lipschitz functions by (N, p, q) (E, q) means of the conjugate series of the Fourier series.

Let f(x) be a function with period 2π and integrable in the sense of Lebesgue over (-π, π). Let its Fourier series be given by

$$f(x) \sim \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} A_n(x)$$

Then

$$\sum_{n=1}^{\infty}(a_n \sin nx - b_n \cos nx) = -\sum_{n=1}^{\infty} B_n(x) \tag{1}$$

is called the conjugate series of Fourier series.

The degree of approximation of a function f: R → R by a trigonometric polynomial t_n of order n is defined by Zygmund [9]

$$\| t_n - f \|_{\infty} = \sup\{|t_n(x) - f(x)| : x \in R\}$$

Let 0 < α ≤ 1 and let f: R → R be almost Lipschitz of order α, f∈ L^a ip α, in the sense that there is a constant $M=M_f \geq 0$ and for each x∈R; there is a subset $A_x \subset \left[0, \frac{\pi}{2}\right]$ of measure zero, such that $t \in \left[0, \frac{\pi}{2}\right] / A_x$ implies $|f(x + 2t) - f(x - 2t)| \leq Mt^{\alpha}$

Every Lip α function is trivially L^a ip α, but the class L^a ip α greatly extends the class Lip α. For example, let g denote the characteristic function of the irrationals, Take $A_x = \{t \in \left[0, \frac{\pi}{2}\right]$: at least one of (x + 2t) and (x - 2t) is rational}.

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So that A_x being countable has measure zero. For each x and $t \in \left[0, \frac{\pi}{2}\right] / Ax,$ both (x+2t) and (x - 2t) are irrational and so |g (x + 2t) - g (x - 2t) | = 0. Hence g is L^a ip α for every α. But obviously g is not Lipschitz of any non-zero order.

For $0 < t \le \frac{\pi}{2}$, *Since* $\sin t \ge \frac{2t}{\pi}$

so for each x $\in$ R, we have

$$\left|\psi_x\ (t)\ \cos t\right| \le Mt^{\alpha}\ \frac{\pi}{2t} = M\frac{\pi}{2}\ t^{\alpha-1},\ t \in \left[0, \frac{\pi}{2}\right] \setminus A_x$$

where $\psi_x(t) = f\ (x+2t) - f\ (x-2t).$

Since A_x has measure zero, it follows at once that f has its conjugate function f, zygmund [9] defined and finite for each x$\in$R by the improper Lebesgue integral

$$\tilde{f}\ (x) = \frac{-1}{\pi}\int_0^{\pi/2} \psi_x\ (t)\ \cot\ t\ dt = -\frac{1}{\pi}\lim_{\in\to o_+}\int_{\in}^{\pi/2} \psi_x\ (t)\ \cot\ t\ dt.$$

If $(E,\ q) = E_n^q = \frac{1}{(1+q)^n}\sum_{k=0}^{n}\binom{n}{k} q^{n-k}\ s_k \to s$ as n $\to \infty$ then the infinite series $\sum_{n=0}^{\infty} u_n$ is said to be summable (E, q) to a definite number s. Hardy [4].

For two sequences $\{p_n\}$ and $\{q_n\}$, we write $t_n^{p,q} = \frac{1}{R_n}\sum_{k=0}^{n} p_{n-k}\ q_k\ s_k,$

$R_n = \sum_{k=0}^{n} p_k\ q_{n-k} \ne 0$ for all n, then the generalized, Nörlund transformation of the sequence $\{s_n\}$ is the sequence $\{t_n^{p,q}\}$. If $t_n^{p,q} \to s$ as $n \to \infty$, then the series $\sum_{n=0}^{\infty} u_n$ or the sequence $\{s_n\}$ is said to be summable to s by generalized Nörlund method (N, p, q) and is denoted by $s_n \to s$ (N, p, q) Borwein [2].

The necessary and sufficient conditions for a (N, p, q) method to be regular are

$\sum_{k=0}^{n} \left|p_{n-k}\ q_k\right| = O\ (|R_n|)$ *and* $p_{n-k} = o\ (|R_n|)$ as n $\to \infty$, for every fixed k $\ge$ 0 for which $q_k \ne 0$.

The product of the (N, p, q) summability with a (E, q) summability defines the (N, p, q) (E, q). Thus the (N, p, q) (E, q) transform of $\{s_n\}$ is given by

$$= \frac{1}{R_n}\sum_{k=0}^{n} p_{n-k}\ q_k\ E_k^q$$

$$= \frac{1}{R_n}\sum_{k=0}^{n} p_k\ q_{n-k}\ E_{n-k}^q$$

$$= \frac{1}{R_n}\sum_{k=0}^{n} p_{n-k}\ q_k\ \frac{1}{(1+q)^k}\sum_{r=0}^{k}\binom{k}{r} q^{k-r}\ s_r$$

If the (N, p, q) (E, q) transform of $\{s_n\} \to$ s, as n $\to \infty$ then the series $\sum_{n=0}^{\infty} u_n$ or the sequence $\{s_n\}$ is said to be summable to the sum s by (N, p, q) (E, q) method and we write the (N, p, q) (E, q) transform of $\{s_n\} \to$ s (N, p, q) (E, q), as n $\to \infty$ we shall use following notations.

$$\tilde{N}_n\ (t) = \frac{1}{\pi R_n} \sum_{k=0}^{n} p_k\, q_{n-k} \left(\frac{\cos^{n-k} t \cos (n-k+1)t}{\sin t} \right)_{dt} \tag{2}$$

$$\tau = \left[\frac{1}{t}\right] = \text{integral part of } \frac{1}{t}.$$

2. THEOREM

Let (N, p, q) be a regular generalized Nörlund method generated by non-negative, monotonic non-increasing sequence $\{p_n\}$ and $\{q_n\}$ of real constants such that

$$R_n = \sum_{k=0}^{n} p_k\ q_{n-k} \neq 0. \text{ If } f : R \to R \text{ is } 2\pi$$

periodic Lebesgel integrable on $[-\pi, \pi]$ and is almost Lipschitz function of order α, $0 < \alpha \le 1$, $f \in L^a$ ip α, then the degree of approximation of the conjugate $\tilde{f}$ by (N, p, q) (E, q) product means $= \frac{1}{R_n} \sum_{k=0}^{n} p_{n-k}\ q_k \frac{1}{(1+q)^k} \sum_{r=0}^{k} q^{k-r}\ \tilde{s}_r$ of the conjugate series of Fourier series (1) is given by

$$\left\| \frac{1}{R_n} \sum_{k=0}^{n} p_{n-k}\ q_k \frac{1}{(1+q)^k} \sum_{r=0}^{k} \binom{k}{r} q^{k-r}\ \tilde{s}_r - \tilde{f} \right\|_\infty = \begin{cases} 0\left(\frac{1}{n^\alpha}\right), & 0 < \alpha < 1 \\ 0\left(\frac{\log n\pi e}{n}\right), & \alpha = 1 \end{cases}$$

3. LEMMA

For the proof of the theorem we require the following lemma:

Lemma: Let $\tilde{N}_n\ (t)$ be given by (2), then $\tilde{N}_n\ (t) = 0\left(\frac{R_\tau}{tR_n}\right), \frac{1}{n} < t \le \frac{\pi}{2}$

Proof: for $\frac{1}{n} < t \le \frac{\pi}{2}$, $\sin t \ge \frac{2t}{\pi}$, then

$$\left|\tilde{N}_n\ (t)\right| \le \frac{1}{2tR_n} \left| \sum_{k=0}^{\tau-1} p_k\, q_{n-k} \cos^{n-k}(t) \cos (n-k+1)\ t + \sum_{k=t}^{n} p_k\, q_{n-k} \cos^{n-k}(t) \cos (n-k+1t \right|$$

$$\le \frac{1}{2tR_n} \left[\sum_{k=0}^{\tau-1} p_k\, q_{n-k} \left|\cos^{n-k}(t) \cos (n-k+1)\ t\right| + \sum_{k=t}^{n} p_k\, q_{n-k} \left|\cos^{n-k}(t) \cos (n-k+1t\right| \right]$$

$$\le \frac{1}{2tR_n} \left[\sum_{k=0}^{\tau-1} p_k\, q_{n-k} + 2p_s\ q_o\ \max \left| \frac{\cos\left(\frac{3n-2\tau-1}{2}\right) \sin \frac{nt}{2}}{\sin \frac{t}{2}} \right| \right]$$

$$\le \frac{1}{2tR_n} \left[\sum_{k=0}^{\tau-1} p_k\, q_{n-k} + O\left(\frac{p_\tau\ q_o}{t}\right) \right]$$

Since $\sum_{k=0}^{\tau-1} p_k p_{n-k} \le R_\tau$ and $\frac{p_\tau\, q_o}{t} \le P_\tau\ Q_\tau \le R_\tau$

where $R_\tau = \sum_{k=0}^{\tau} p_k\, q_{\tau-k}$, therefore

$$\tilde{N}_n\,(t) = O\left(\frac{R_\tau}{tR_n}\right)$$

4. PROOF OF THE THEOREM

r^{th} partial sum, $\tilde{S}_r(x)$, of the conjugate series (1) can be written as

$$\tilde{S}_r(x) = \tilde{f}(x) + \frac{1}{\pi}\int_0^{\pi/2} \psi_x(t)\, \frac{\cos(2r+1)^t}{\sin t}\, dt$$

So the (E, q) means of the $\tilde{S}_r(x)$ are

$$E_k^q = \frac{1}{(1+q)^k}\sum_{r=0}^{k}\binom{k}{r} q^{k-r}\, \tilde{S}_r(x),\ r = 0,1,2,..$$

$$= \frac{q^{k-r}}{(1+q)^k}\sum_{r=0}^{k}\binom{k}{r}\left[\tilde{f}(x) + \frac{1}{\pi}\int_0^{\pi/2}\psi_x(t)\,\frac{\cos(2r+1)t}{\sin t}\,dt\right]$$

$$= \tilde{f}(x) + \frac{q^{k-r}}{\pi(1+q)^k}\int_0^{\pi/2}\frac{\psi_x(t)}{\sin t}\sum_{r-0}^{k}\binom{k}{r}\cos(2r+1)_t\ dt$$

$$E_k^q(x) - \tilde{f}(x) = \frac{q^{k-r}}{\pi(1+q)^k}\int_0^{\pi/2}\frac{\psi_x(t)}{\sin\ t}\sum_{r=0}^{k}\binom{k}{r}\cos(2r+1)t\ dt$$

$$= \frac{q^{k-r}}{\pi(1+q)^k}\int_0^{\pi/2}\frac{\psi_x(t)}{\sin\ t}\left[\mathrm{Re}\left[\sum_{r=0}^{k}\binom{k}{r}e^{i}(2r+1)t\right]\right] dt$$

$$= \frac{q^{k-r}}{\pi(1+q)^k}\int_0^{\pi/2}\frac{\psi_x(t)}{\sin\ t}\left(\mathrm{Re}\left[\sum_{r=0}^{k}\binom{k}{r}e^{i2rt}\ e^{it}\right]\right) dt$$

$$= \frac{q^{k-r}}{\pi(1+q)^k}\int_0^{\pi/2}\frac{\psi_x(t)}{\sin\ t}\left(\mathrm{Re}\left[\left(1+e^{2it}\right)^k\ e^{it}\right]\right) dt$$

$$= \frac{q^{k-r}}{\pi(1+q)^k}\int_0^{\pi/2}\frac{\psi_x(t)}{\sin\ t}\left(2^k\ \cos^k t\ \cos kt\ \cos t\right) dt$$

$$= \frac{q^{k-r}\ 2^k}{\pi(1+q)^k}\int_0^{\pi/2}\frac{\psi_x(t)}{\sin\ t}\left(\cos^k t\ \cos(k+1)t\right) dt$$

Now,

$$\frac{1}{R_n}\sum_{k=0}^{n} p_k\, q_{n-k}\left[E_{n-k}^q(x) - \tilde{f}(x)\right]$$

$$= \frac{1}{R_n}\sum_{k=0}^{n} p_k\, q_{n-k}\left[\frac{q^{n-k-r}\,2^{n-k}}{\pi(1+q)^{n-k}}\int_0^{\pi/2}\frac{\psi_x(t)}{\sin t}\left(\cos^{n-k} t\cos(n-k+1)t\right)dt\right]$$

$$= \int_0^{\pi/2}\psi_x(t)\,\frac{q^{n-k-r}\ 2^{n-k}}{\pi R_n\ (1+q)^{n-k}}\sum_{k=0}^{n} p_k\, q_{n-k}\left(\frac{\cos^{n-k} t\cos(n-k+1)\ t}{\sin\ t}\right) dt$$

$$= \int_0^{\pi/2}\psi_x(t)\,\frac{q^{n-k-r}\ 2^{n-k}}{(1+q)^{n-k}}\ \tilde{N}_n(t)\ dt$$

$$= \frac{q^{n-k-r}\ 2^{n-k}}{(1+q)^{n-k}}\left[\int_0^{1/n}\psi_x(t)\ \tilde{N}_n(t)\ dt + \int_{1/n}^{\pi/2}\psi_n(t)\ \tilde{N}_n(t)\ dt\right]$$

$$= \frac{q^{n-k-r}\ 2^{n-k}}{(1+q)^{n-k}}\left[I_1 + I_2\right],\ say \tag{3}$$

For $0 < t \le \dfrac{1}{n}$

$$|I_1| \le \int_0^{1/n} \left(|\psi_x(t)|\,|\tilde{N}_n|\right) dt$$

$$\le \int_0^{1/n} \left(Mt^{\alpha} \left| \frac{1}{\pi R_n} \sum_{k=0}^{n} p_k\, q_{n-k} \frac{\cos^{n-k} t \, \cos(n-k+1)t}{\sin\, t} \right| \right) dt$$

$$\le \int_0^{1/n} \left(Mt^{\alpha} \frac{1}{\pi R_n} \sum_{k=0}^{n} p_k\, q_{n-k} \left(\frac{|\cos^{n-k} t|.\,|\cos\,(n-k+1)t\,|}{\sin\, t} \right) \right) dt$$

$$= \left[\int_0^{1/n} t^{\alpha-1}\ dt \right]$$

$$= 0\left(\frac{1}{n^{\alpha}}\right) \tag{4}$$

for $\dfrac{1}{n} < t \le \dfrac{\pi}{2}$,

$$|I_2| \le \int_{1/n}^{\pi/2} \left(|\psi_x(t)|\,|\tilde{N}_n\ (t)|\right) dt$$

$$= 0\left[\int_{1/n}^{\pi/2} \left(t^{\alpha}.\, \frac{R_{\tau}}{tR_n} \right) \right] dt$$

$$= 0\left[\int_{1/n}^{\pi/2} \left(t^{\alpha-1}\, \frac{R_{\tau}}{R_n} \right) \right] dt$$

$$= 0\left[\int_{2/\pi}^{n} \frac{1}{4^{\alpha-1}}\, \frac{R(u)}{R_n}\, \frac{du}{u^2} \right],\ taking\ t = \frac{1}{u}$$

$$= 0\left[\int_{2/\pi}^{n} \left(\frac{R_{(u)}}{R_n}\, \frac{du}{u^{\alpha+1}} \right) \right]$$

$$= 0\left[\frac{1}{n} \int_{2/\pi}^{n} \frac{du}{u^{\alpha}} \right],\ \left\{ \frac{R_{(u)}}{u} \right\} \quad \text{is monotonic decreasing}$$

$$= \begin{cases} 0\left(\dfrac{1}{n^{\alpha}}\right);\ 0 < \alpha < 1 \\ 0\left(\dfrac{\log\, n\pi}{n}\right);\ \alpha = 1 \end{cases} \tag{5}$$

Combining from (3) to (5), we have

$$\left| \frac{1}{R_n} \sum_{k=0}^{n} p_{n-k}\, q_k\, \frac{1}{(1+q)^k} \sum_{r=0}^{k} \binom{k}{r} q^{k-r}\, \tilde{s}_r - \tilde{f}\ \right| = \begin{cases} 0\left(\dfrac{1}{n^{\alpha}}\right)\ ;\ 0 < \alpha < 1 \\ 0\left(\dfrac{\log\, n\pi e}{n}\right);\ \alpha = 1 \end{cases}$$

$$\left\| \frac{1}{R_n} \sum_{k=0}^{n} p_{n-k}\, q_k\, \frac{1}{(1+q)^k} \sum_{r-0}^{k} \binom{k}{r} q^{k-r}\, \tilde{s}_r - \tilde{f}\ \right\|_{\infty}$$

$$= \sup\ \left\{ \left| \frac{1}{R_n} \sum_{k=0}^{n} p_{n-k}\, q_k\, \frac{1}{(1+q)^k} \sum_{r=0}^{k} \binom{k}{r} q^{k-r}\, \tilde{s}_r\ (x) - \tilde{f}\,(x) \right| : x \in R \right\}$$

$$= \begin{cases} 0\left(\dfrac{1}{n^{\alpha}}\right); \ 0 < \alpha < 1 \\ 0\left(\dfrac{\log n\pi e}{n}\right); \ \alpha = 1 \end{cases}$$

5. APPLICATIONS

The following corollaries can be obtained from our theorem

Cor.1: Taking $q_n = 1 \ \forall \ n \geq 0$, the degree of approximation of $\tilde{f}(x)$, conjugate of a function $f \in L^a$ ip α, $0 < \alpha < 1$, by (N, p_n) (E, 1) means of the conjugate series of Fourier series (1) is given by

$$\left\| \frac{1}{R_n} \sum_{k=0}^{n} p_{n-k} \ q_k \ \frac{1}{(1+q)^k} \sum_{r=0}^{k} \binom{k}{r} q^{k-r} \ \tilde{s}_r \ - \tilde{f} \right\|_{\infty} = O\left(\frac{1}{n^{\alpha}}\right)$$

Cor. 2: Taking $p_n = q_n = 1 \ \forall \ n \geq 1$, the degree of approximation of $\tilde{f}(x)$, conjugate of a function $f \in L^a$ ip α, $0 < \alpha < 1$, by (C, 1) (E, 1) means of the conjugate series of Fourier series (1) is given by

$$\left\| \tilde{t}_n^{CE} - \tilde{f} \right\|_{\infty} = O\left(\frac{\log (n\pi e)}{n^{\alpha}}\right)$$

REFERENCES

(1) Alexits, G., Convergence problems of orthogonal series, pergamon press, London (1961).
(2) Borwein, D., on products of sequences, J. London Math. Soc., 33 (1958), 352-57.
(3) Chandra, Prem, on degree of approximation of functions belonging to the Lipschitz class, Nanta Math, 8 (1975) no.1, 88.
(4) Hardy, G.H., Divergent series, first edition, Oxford press (1949), 70.
(5) Lal, S. and Prasad, H.P., The degree of approximations of conjugates of Almost Lipschitz functions by (N, p, q) (E, 1) means. International Mathematical Forum, 5, 2010, no. 34, 1663-1671.
(6) Qureshi, K., On degree of approximation of a periodic function f by almost Nörlund means, Tamkang J. Math., 12 (1981) no.1, 35.
(7) Qureshi, K., on degree of approximation of functions belonging to the class Lip α, Indian J. pure app. Math; 13 (1982) no. 8.
(8) Qureshi, K. and Neha H.K., A class of functions and their degree of approximation, Ganita, 41 (1990) no. 1, 37.
(9) Sahney, B.N. and Goel, D.S., on the degree of approximation of continuous functions, Rachi Univ. Math; J, 4 (1973), 50-53.
(10) Zygmund, A., Trigonometric series, 2nd rev. ed., I, Cambridge Univ. Press Cambridge (1968), 51.

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