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Proximate (p, q) type of an entire function represented by Dirichlet series

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ABSTRACT

In the present paper, we study proximate type and order of entire function with index pair (p, q) $p \geq (q+1) \geq 1$ *and also with the help of means under certain condition for entire Dirichlet series.*

Keywords: *Entire function, proximate (p ,q)- type, (p ,q)-order, Dirichlet Series.*

MSC: *30B50, 30D20, 54E05.*

1. INTRODUCTION

The notation of order and type of entire functions are classical in complex analysis. For the entire function of complex variable, G.Valiron [7] refined these growth scales by introducing comparison functions, called proximate order

In this paper our approach consists of using the extension of the classical notions of (p, q)-order (p, q) –type results of mean value of f(s) introduced by [1] .The concepts of (p, q)-order and (p, q) -type for entire function of complex variable were introduced by Juneja *et al.* [2,3]. The notion of proximate order was developed by Nandan *et al.* [8]. Later Kamthan [4] also studied the properties of proximate order .We defined the proximate (p, q) type of mean value $N_{\delta,k}(\sigma)$. This mean value was introduced by S.K. Vaish [6]

PRELIMINARIES AND AUXILIARY RESULTS

Consider the Dirichlet series,

$$f(s) = \sum_{n=1}^{\infty} a_n \exp(s\lambda_n)$$

where $0 < \lambda_n < \lambda_{n+1} \to \infty$ $s = \sigma + it$ σ ,t being real variable $\{a_n\}_{n=1}^{\infty}$ is sequence of complex numbers .Set

$$M(\sigma) = \max_{t \in R} | f(\sigma + it) |$$

is called maximum modulus of f(s).

The concept of (p, q)- order, lower (p, q) -order, (p, q) -type, lower (p, q)- type of entire function f(s) is introduced by Juneja [2 ,3].

Thus f(s) is said to be of (p, q) order ρ if

$$\limsup_{\sigma \to \infty} \frac{\log^{[p]} M(\sigma)}{\log^{[q]} \sigma} = \rho(p,q) = \rho \tag{1.1}$$

Further, b < ρ < ∞ then f(s) is said to be of (p ,q)- type τ and lower (p, q)- type υ if

$$\lim_{\sigma \to \infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log^{[p-1]} M(\sigma)}{\left(\log^{(q-1)} \sigma\right)^{\rho}} = \begin{matrix} \tau(p,q) = \tau \\ \nu(p,q) = \upsilon \end{matrix} \tag{1.2}$$

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where b=1 if p=q+1 ,b=0 if p>q+1 and $\log^{[p]} x = \log \log^{[p-1]} x$ An entire function is said to be of perfectly regular (p,q)- growth iff $0 < \upsilon = \tau < \infty$

Mean value of f(s) is introduced by S.K.Vaish [6]

$$(I_\delta(\sigma))^\delta = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} |f(\sigma+it)|^\delta \, dt \quad \text{where } 0<\delta<\infty \tag{1.3}$$

$$N_{\delta,k}(\sigma) = \exp\left\{ e^{-k\sigma}\int_0^\infty \log I_\delta(x) e^{kx} dx \right\},\ 0<k<\infty \tag{1.4}$$

For an entire Dirichlet series (p, q) - order ρ (p, q) - type τ and lower (p, q) - type υ defined by Kamthan [4]. We can easily prove that

$$\lim_{\sigma\to\infty} \begin{matrix}\sup\\ \\ \inf\end{matrix} \frac{\log^{[p-1]} I_\delta(\sigma)}{\left(\log^{[q-1]}\sigma\right)^\rho} = \begin{matrix}\tau\\ \\ \upsilon\end{matrix} \tag{1.5}$$

Definition: A real valued positive function $\tau(\sigma)$ defined on $[\sigma_0,\infty)$ $\sigma_0 > 0$ is said to be proximate type of an entire Dirichlet series with index (p,q) -order ρ, (b < ρ< ∞) and (p, q)-type τ (0<τ< ∞), if for given constant α (0<α< ∞) τ(σ) satisfies the following conditions:

i) $\tau(\sigma)$ is continuous and piecewise differentiable for $\sigma > \sigma_0$

ii) $\lim_{\sigma\to\infty} \tau(\sigma) = \tau$

iii) $\lim_{\sigma\to\infty} \Lambda_{[q-1]}(\sigma)\tau' \quad (\sigma)$ = 0 where τ'(σ) &τ'(σ) these are unequal

iv) $\lim_{\sigma\to\infty} \dfrac{\log^{[p-2]} M(\sigma)}{\exp\left\{\left(\log^{[q-1]}\sigma\right)^\rho \tau(\sigma)\right\}} = \alpha$

where $\Lambda_{[q]} = \prod_{i=0}^{q} \log^{[q]} x$

2. MAIN RESULTS

We wish to prove the following

Theorem 1: If f(s) be an entire function of (p, q)- order ρ and (p, q)- type τ and lower (p, q)- type υ Then

$$\lim_{\sigma\to\infty} \begin{matrix}\sup\\ \\ \inf\end{matrix} \frac{\log^{[p-1]} N_{\delta,k}(\sigma)}{(\log^{[q-1]}\sigma)^\rho} = \begin{matrix}\tau\\ \\ \upsilon\end{matrix} \tag{2.1}$$

Lemma 1: If f(s) be an entire function of (p,q)-order ρ and (p,q)- type τ and lower (p q)-type υ. Then

$$\log I_\delta(\sigma) \sim \log N_{\delta,k}(\sigma). \tag{2.2}$$

The result is proved by A. Nautiyal [5]

Proof of theorem 1: we can easily prove that

$$\log^{[p-1]} I_\delta(\sigma) \sim \log^{[p-1]} N_{\delta,k}(\sigma) \tag{2.3}$$

Dividing (2.3) both sides by $\left(\log^{[p-1]}\sigma\right)^\rho$ and taking limit $\sigma\to\infty$.

Then using (1.5) we get (2.1)

Theorem 2: If f(s) be an entire function with (p, q)- order ρ (b<ρ<∞) , (p, q)- type τ and lower (p, q)- type υ where $0<\upsilon\le\tau<\infty$. Then

$$\liminf_{\sigma\to\infty}\frac{\Lambda_{[q-2]}(\sigma)N'_{\delta,k}(\sigma)}{\left(\log^{[q-1]}\sigma\right)^{[\rho-1]}\Lambda_{[p-2]}N_{\delta,k}(\sigma)}\le\rho\upsilon\le\rho\tau \tag{2.4}$$

$$\le\limsup_{\sigma\to\infty}\frac{\Lambda_{[q-2]}(\sigma)N'_{\delta,k}(\sigma)}{\left(\log^{[q-1]}\sigma\right)^{[\rho-1]}\Lambda_{[p-2]}N_{\delta,k}(\sigma)}$$

Proof: Let

$$\liminf_{\sigma\to\infty}\frac{\Lambda_{[q-2]}(\sigma)N'_{\delta,k}(\sigma)}{\left(\log^{[q-1]}\sigma\right)^{[\rho-1]}\Lambda_{[p-2]}N_{\delta,k}(\sigma)\rho}=\beta \tag{2.5}$$

for β =0 the inequality (2.4) is trivially true. Let β >0 then for given ε >0 and $\sigma>\sigma_0$

$$\frac{N'_{\delta,k}\ (\sigma)}{\Lambda_{[p-2]}\left(N_{\delta,k}(\sigma)\right)}>\frac{\rho(\beta-\varepsilon)\left(\log^{[q-1]}\sigma\right)^{\rho-1}}{\Lambda_{[q-2]}(\sigma)}$$

On integrating both side from σ to σ_0

$$\log^{[p-1]}\left(N_{\delta,k}(\sigma)\right)+0(1)>(\beta-\varepsilon)\left(\log^{[q-1]}\sigma\right)^{\rho}$$

Dividing by $\left(\log^{[q-1]}\sigma\right)^{\rho}$ and taking limit both side, we get

$$\liminf_{\sigma\to\infty}\frac{\log^{[p-1]}\left(N_{\delta,k}(\sigma)\right)}{\left(\log^{[q-1]}\sigma\right)^{\rho}}\ge\beta$$

.
Then

$$\liminf_{\sigma\to\infty}\frac{\log^{[p-1]}\left(N_{\delta,k}(\sigma)\right)}{\left(\log^{[q-1]}\sigma\right)^{\rho}}\ge\ \liminf_{\sigma\to\infty}\frac{\Lambda_{[q-2]}(\sigma)N_{\delta,k}{}'(\sigma)}{\left(\log^{[q-1]}\sigma\right)^{[\rho-1]}\Lambda_{[p-2]}N_{\delta,k}(\sigma)\rho}$$

$$\rho\upsilon\ge\ \liminf_{\sigma\to\infty}\frac{\Lambda_{[q-2]}(\sigma)N_{\delta,k}{}'(\sigma)}{\left(\log^{[q-1]}\sigma\right)^{[\rho-1]}\Lambda_{[p-2]}N_{\delta,k}(\sigma)\rho} \tag{2.6}$$

Again let

$$\limsup_{\sigma\to\infty}\frac{\Lambda_{[q-2]}(\sigma)N_{\delta,k}{}'(\sigma)}{\left(\log^{[q-1]}\sigma\right)^{[\rho-1]}\Lambda_{[p-2]}N_{\delta,k}(\sigma)\rho}=\lambda$$

For λ=0 the inequality is trivially true, if $\lambda>0$ then for given
ε >0 and $\sigma>\sigma_0$.

Proceed in similar way we get

$$\rho\tau\ \le\limsup_{\sigma\to\infty}\frac{\Lambda_{[q-2]}(\sigma)N_{\delta,k}{}'(\sigma)}{\left(\log^{[q-1]}\sigma\right)^{[\rho-1]}\Lambda_{[p-2]}N_{\delta,k}(\sigma)} \tag{2.7}$$

Thus from (2.6) & (2.7) we get

$$\liminf_{\sigma\to\infty}\frac{\Lambda_{[q-2]}(\sigma)N_{\delta,k}{}'(\sigma)}{\left(\log^{[q-1]}\sigma\right)^{[\rho-1]}\Lambda_{[p-2]}N_{\delta,k}(\sigma)}\le\rho\upsilon\le\rho\tau$$

$$\le\limsup_{\sigma\to\infty}\frac{\Lambda_{[q-2]}(\sigma)N_{\delta,k}{}'(\sigma)}{\left(\log^{[q-1]}\sigma\right)^{[\rho-1]}\Lambda_{[p-2]}N_{\delta,k}(\sigma)}$$

Theorem 3: If f(s) be an entire function of (p, q) - order ρ and (p, q)- type $\tau\ (0<\tau<\infty)$. If we define the function

$$L(\sigma)=\frac{\log\left[\alpha^{-1}\log^{[p-2]}N_{\delta,k}(\sigma)\right]}{(\log^{[q-1]}\sigma)^{\rho}} \tag{3.1}$$

Then $L(\sigma)$ is proximate type of f(s), where α is constant

Proof: For a given constant α. We have defined (2.8). The function $L(\sigma)$ is continuous for $\sigma>\sigma_0$ since $I_\delta(\sigma)$ is almost differentiable function σ with an increasing derivative .Then $N_{\delta, k}(\sigma)$ is also almost differentiable everywhere.

$\log^{[p-1]} N_{\delta,k}(\sigma)$ is differentiable almost everywhere. Consequently $L(\sigma)$ is piecewise differentiable we have
$\lim_{\sigma\to\infty} L(\sigma)=\tau$

Differentiating equation (3.1) w .r .t σ we get

$$L'(\sigma)=\frac{-\rho\log\left[\alpha^{-1}\log^{[p-2]}N_{\delta,k}(\sigma)\right]}{\Lambda_{[q-2]}(\sigma)\left(\log^{[q-1]}\sigma\right)^{\rho+1}}+\frac{N_{\delta,k}'(\sigma)}{\Lambda_{[p-2]}\left(N_{\delta,k}(\sigma)\right)\left(\log^{[q-1]}\sigma\right)^{\rho}}$$

$$\Lambda_{[q-1]}(\sigma)L'(\sigma)=\frac{N_{\delta,k}'(\sigma)\Lambda_{[q-2]}(\sigma)}{\Lambda_{[q-2]}\left[N_{\delta,k}(\sigma)\right]\left(\log^{[q-1]}\sigma\right)^{\rho-1}}-\frac{\rho\log^{[p-1]}N_{\delta,k}(\sigma)}{\left(\log^{[q-1]}\sigma\right)^{\rho}}+0(1)$$

by theorem (2) and (2.1) we get

$$\lim_{\sigma\to\infty}\Lambda_{[q-1]}(\sigma)L'(\sigma)=0$$

From definition of $L(\sigma)$

$$\left\{\alpha^{-1}\log^{[p-2]}N_{\delta,k}(\sigma)\right\}=\exp\left\{L(\sigma)\left(\log^{[q-1]}\sigma\right)^{\rho}\right\}$$

$$\Rightarrow \lim_{\sigma\to\infty}\sup\frac{\log^{[p-2]}N_{\delta,k}(\sigma)}{\exp\{\mathrm{L}(\sigma)(\log^{[q-1]}\sigma)^{\rho}\}}=\alpha$$

It follows that $L(\sigma)$ satisfies all properties (i) to (iv).Hence $L(\sigma)$ is a proximate type of f(s).

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