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# CONNECTED TOTAL DOMINATION POLYNOMIALS OF GRAPHS

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## *ABSTRACT*

*In this paper, we introduce the concept of connected total domination polynomial for any graph G. The connected total domination polynomial of a graph G of order n is the polynomial* $D_{ct}(G, x) = \sum_{i=\gamma_{ct}(G)}^{n} d_{ct}(G, i)x^i$, *where $d_{ct}(G, i)$ is the number of connected total dominating sets of G of size i and $\gamma_{ct}(G)$ is the connected total domination number of G. We obtain some properties of $D_{ct}(G, x)$ and its coefficients. Also, we calculate the total domination polynomials for the complete graph $K_n$, the complete bipartite graph $K_{m,n}$, the bi-star $B_{m,n}$, the Barbell graph $B_n$, the Lollipop graph $L_{n,1}$ and the Tadpole graph $T_{n,1}$.*

***Key words:*** *Connected total dominating sets, connected total domination number, connected total domination polynomial.*

## 1. INTRODUCTION

Let G = (V, E) be a simple connected graph of order n. For any vertex $v \in V$, the open neighbourhood of v is the set $N(v)=\{u \in V / uv \in E\}$ and the closed neighbourhood of v is the set $N[v] = N(v) \cup \{v\}$. For a set $S \subseteq V$, the open neighbourhood of S is $N(S) = \bigcup_{v \in S} N(v)$ and the closed neighbourhood of S is $N[S] = N(S) \cup S$. The maximum degree of the graph G is denoted by $\Delta(G)$ and the minimum degree is denoted by $\delta(G)$.

A set S of vertices in a graph G is said to be a dominating set if every vertex $v \in V$ is either an element of S or is adjacent to an element of S.

A set S of vertices in a graph G is said to be a total dominating set if every vertex $v \in V$ is adjacent to an element of S. A total dominating set S of G is called a connected total dominating set if the induced subgraph $\langle S \rangle$ is connected. The minimum cardinality of a connected total dominating set of G is called the connected total domination number and is denoted by $\gamma_{ct}(G)$.

The join of $G_1$ and $G_2$ denoted by $G_1 \vee G_2$ is a graph G with the vertex set $V=V_1 \cup V_2$ and $E=E_1 \cup E_2$ together with all edges joining the elements of $V_1$ to the elements of $V_2$.

The union of $G_1$ and $G_2$ denoted by $G_1 \cup G_2$ is a graph G with the vertex set $V = V_1 \cup V_2$ and $E = E_1 \cup E_2$.

The corona of two graphs $G_1$ and $G_2$ is the graph $G = G_1 o G_2$ formed from one copy of $G_1$ and $|V(G_1)|$ copies of $G_2$, where the $i^{th}$ vertex of $G_1$ is adjacent to every vertex in the $i^{th}$ copy of $G_2$. The corona G o $K_1$ is the graph constructed from a copy of G, where for each vertex $v \in V(G)$ a new vertex $v'$ and a pendant edge $vv'$ are added.

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A bi-star is a tree obtained from the graph $K_2$ with two vertices u and v by attaching m pendant edges in u and n pendant edges in v and denoted by $B_{m,n}$.

The Barbell graph is the simple graph obtained by connecting two copies of complete graph by a bridge and it is denoted by $B_n$.

The Lollipop graph is the graph obtained by joining a complete graph $K_n$ to a path graph $P_1$ with a bridge and it is denoted by $L_{n,1}$.

The Tadpole graph is the graph obtained by joining a cycle graph $C_n$ to a path graph $P_1$ with a bridge and it is denoted by $T_{n,1}$.

## 2. CONNECTED TOTAL DOMINATION POLYNOMIALS

**Definition 2.1:** Let G be a simple graph of order n with no isolated vertices. Let $d_{ct}(G, i)$ be the family of connected total dominating sets of G with cardinality i and let $d_{ct}(G, i) = |D_{ct}(G, i)|$. Then the connected total domination polynomial $D_{ct}(G, x)$ of G is defined as $D_{ct}(G,x) = \sum_{i=\gamma_{ct}(G)}^{n} d_{ct}(G,i)x^i$, where $\gamma_{ct}(G)$ is the connected total domination number of G.

**Example 2.2:** Consider the graph G given in Figure 2.1.

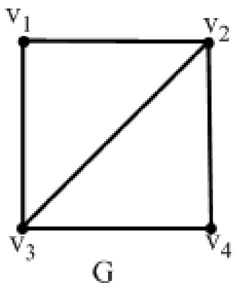

**Figure-2.1**

The connected total dominating sets of G of cardinality 2 are $\{v_1, v_2\}$, $\{v_1, v_3\}$, $\{v_2, v_3\}$, $\{v_2, v_4\}$ and $\{v_3, v_4\}$.

Therefore, $d_{ct}(G, 2) = 5$.

The connected total dominating sets of G of cardinality 3 are $\{v_1, v_2, v_3\}$, $\{v_1, v_2, v_4\}$,$\{v_1, v_3, v_4\}$ and $\{v_2, v_3, v_4\}$.

Therefore, $d_{ct}(G, 3) = 4$.

The connected total dominating set of G of cardinality 4 is $\{v_1, v_2, v_3, v_4\}$.

Therefore, $d_{ct}(G, 4) = 1$.

Since, the minimum cardinality is 2, $\gamma_{ct}(G) = 2$.

$$D_{ct}(G, x) = \sum_{i=\gamma_{ct}(G)}^{|V(G)|} d_{ct}(G,i)x^i .$$

$$= \sum_{i=2}^{4} d_{ct}(G,i)x^i .$$

$$= d_{ct}(G, 2)x^2 + d_{ct}(G, 3)x^3 + d_{ct}(G, 4)x^4.$$

$$= 5x^2 + 4x^3 + x^4.$$

$$= x^4 + 4x^3 + 5x^2.$$

$$D_{ct}(G, x) = (1 + x)^4 - (1 + 4x + x^2).$$

**Theorem 2.3:** For any path $P_n$ on $n \geq 4$ vertices, $D_{ct}(P_n, x) = x^{n-2}(1 + x)^2$.

**Proof:** Let $V(P_n)=\{v_1, v_2, v_3, ..., v_n\}$. Clearly, $\{v_2,..., v_{n-1}\}$ is a total dominating set and $\{v_2, .., v_{n-1}\}$ is connected. Also, for any set S of cardinality less than n – 2, $\langle S\rangle$ is not connected.

Therefore, $\{v_2,.., v_{n-1}\}$ is the only connected total dominating set of cardinality n–2. Therefore, the connected total domination number of $P_n$ is n – 2 and $d_{ct}(P_n, n-2) = 1$.

Also, there are only two connected total dominating sets of order n–1 namely $\{v_1, v_2... v_{n-1}\}$ and $\{v_2, v_3,..., v_n\}$. Therefore, $d_{ct}(P_n, n-1) = 2$ and there is only one connected total dominating set of order n.

Therefore $d_{ct}(P_n, n) = 1$.

Hence, $D_{ct}(P_n, x) = x^{n-2} + 2x^{n-1} + x^n$ .
$= x^{n-2}(1 + x)^2$.

**Theorem 2.4:** For any cycle $C_n$ with n vertices, $D_{ct}(C_n, x) = x^n + nx^{n-1}+ nx^{n-2}$.

**Proof:** Let $C_n$ be a cycle with n vertices. Clearly $\{v_1, v_2,..., v_{n-3}, v_{n-2}\}$, $\{v_2, v_3,..., v_{n-2}, v_{n-1}\}$, $\{v_3, v_4,...,v_{n-1}, v_n\}$, $\{v_4, v_5,..., v_n, v_1\}$,...,$\{v_{n-1}, v_n,..., v_{n-5}, v_{n-4}\}$ and $\{v_n, v_1,..., v_{n-4}, v_{n-3}\}$ are the n connected total dominating sets of $C_n$ of cardinality n – 2. Therefore, $\gamma_{ct}(C_n) = n - 2$ and $d_{ct}(C_n, n - 2) = n$.

Also, $\{v_1, v_2... v_{n-2},v_{n-1}\}$, $\{v_2, v_3,.. ., v_{n-1},v_n\}$, $\{v_3, v_4,...,v_n,v_1\}$, ...,$\{v_{n-1}, v_n,..., v_{n-2}, v_{n-3}\}$ and $\{v_n,v_1,..., v_{n-3}, v_{n-2}\}$ are the n connected total dominating sets of cardinality n–1.Therefore, $d_{ct}(C_n, n-1) = n$ and there is only one connected total dominating set of cardinality n. Therefore, $d_{ct}(C_n, n) = 1$.

Hence, $D_{ct}(C_n, x) = nx^{n-2} + nx^{n-1} + x^n$.

**Theorem 2.5:** For any complete graph $K_n$ of n vertices, $D_{ct}(K_n, x) = (1 + x)^n - (1 + nx)$.

**Proof:** Let $K_n$ be a complete graph with n vertices. Any two vertices of $K_n$ are connected and dominate totally all the remaining vertices of $K_n$. Therefore, $\gamma_{ct}(K_n) = 2$.

For any $2 \le i \le n$, it is easy to see that $d_{ct}(K_n, i) = \binom{n}{i}$.

Therefore,

$$D_{ct}(K_n, x) = \sum_{i=2}^{n} \binom{n}{i} x^i .$$
$$= \binom{n}{2}x^2 + \binom{n}{3}x^3 + \binom{n}{4}x^4 + \ldots + \binom{n}{n}x^n .$$
$$= \left[\sum_{i=0}^{n} \binom{n}{i} x^i\right] - 1 - nx .$$

$D_{ct}(K_n, x) = (1 + x)^n - (1 + nx)$.

**Theorem 2.6:** For a complete bipartite graph $K_{m,n}$, the connected total domination polynomial is $D_{ct}(K_{m,n}, x) = [ (1 + x)^m - 1] [ (1 + x)^n - 1]$.

**Proof:** Let $K_{m,n}$ be a complete bipartite graph with partite sets $V_1$ and $V_2$. Then any connected total dominating set of $K_{m,n}$ contains atleast one vertex from $V_1$ and atleast one vertex from $V_2$.

Therefore, $\gamma_{ct}(K_{m,n}) = 2$.

$$D_{ct}(K_{m,n}, x) = \binom{m}{1}\binom{n}{1}x^2 + \left[\binom{m}{1}\binom{n}{2} + \binom{m}{2}\binom{n}{1}\right]x^3 + \ldots + \left[\binom{m}{1}\binom{n}{m+n-1} + \ldots \binom{m}{n+m-1}\binom{n}{1}\right]x^{m+n}$$
$$= \left[\binom{m}{1}x + \binom{m}{2}x^2 + \ldots + \binom{m}{m}x^m\right]\left[\binom{n}{1}x + \binom{n}{2}x^2 + \ldots + \binom{n}{n}x^n\right].$$
$$= \sum_{i=1}^{m}\binom{m}{i}x^i \sum_{i=1}^{n}\binom{n}{i}x^i .$$

$D_{ct}(K_{m,n}, x) = [ (1 + x)^m - 1] [ (1 + x)^n - 1]$.

**Example 2.7:** Consider $K_{2,3}$ given in Figure 2.2.

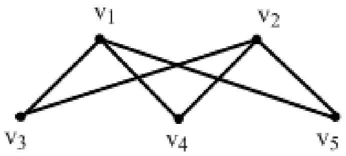


**Figure-2.2**

The connected total dominating sets of $K_{2,3}$ of cardinality 2 are $\{v_1, v_3\}, \{v_1, v_4\}, \{v_1, v_5\}, \{v_2, v_3\}, \{v_2, v_4\}$ and $\{v_2, v_5\}$.

Therefore, $d_{ct}(K_{2,3}, 2) = 6$.

The connected total dominating sets of $K_{2,3}$ of cardinality 3 are $\{v_1, v_2, v_3\}, \{v_1, v_2, v_4\}, \{v_1, v_2, v_5\}, \{v_1, v_3, v_4\}, \{v_1, v_3, v_5\}, \{v_1, v_4, v_5\}, \{v_2, v_3, v_4\}, \{v_2, v_3, v_5\}$ and $\{v_2, v_4, v_5\}$.

Therefore, $d_{ct}(K_{2,3}, 3) = 9$.

The connected total dominating sets of $K_{2,3}$ of cardinality 4 are $\{v_1,v_2,v_3,v_4\}, \{v_1,v_2,v_3,v_5\}, \{v_1,v_2,v_4,v_5\}, \{v_1,v_3,v_4,v_5\}$ and $\{v_2,v_3,v_4,v_5\}$.

Therefore, $d_{ct}(K_{2,3}, 4) = 5$.

The connected total dominating sets of $K_{2,3}$ of cardinality 5 is $\{v_1, v_2, v_3, v_4, v_5\}$.

Therefore, $d_{ct}(K_{2,3}, 5) = 1$.

Since, the minimum cardinality is 2, $\gamma_{ct}(K_{2,3})=2$.

$$D_{ct}(K_{2,3}, x) = \sum_{i=\gamma_{ct}(G)}^{|V(K_{2,3})|} d_{ct}(G, i)x^i .$$
$$= \sum_{i=2}^{5} d_{ct}(K_{2,3}, i)x^i .$$
$$= d_{ct}(K_{2,3}, 2)x^2 + d_{ct}(K_{2,3}, 3)x^3 + d_{ct}(K_{2,3}, 4)x^4 + d_{ct}(K_{2,3}, 5)x^5.$$
$$= 6x^2 + 9x^3 + 5x^4 + x^5.$$
$$= (2x + x^2)(3x + 3x^2 + x^3).$$

$$D_{ct}(K_{2,3}, x) = [(1 + x)^2 - 1][(1 + x)^3 - 1].$$

**Theorem 2.8:** For any star graph $K_{1,n}$ with n+1 vertices, where $n\geq 2$, $D_{ct}(K_{1,n}, x) = x[(1 + x)^n - 1]$.

**Proof:** Let $K_{1,n}$ be a star graph with n+1 vertices and $n\geq 2$. By labeling the vertices of $K_{1,n}$ as $v_1, v_2, .., v_{n+1}$, where $v_1$ is the vertex of degree n, then clearly there are n connected total dominating sets of cardinality two namely $\{v_1, v_2\}, \{v_1, v_3\}, \{v_1, v_4\}, ..., \{v_1, v_{n+1}\}$. Similarly for the connected total dominating sets of cardinality three, we need to select the vertex $v_1$ and two vertices from the set of vertices $\{v_2, v_3, ..., v_{n+1}\}$.

Therefore, there are $\binom{n}{2}$ connected total dominating sets of cardinality three. Proceeding like this, we obtain the other connected total dominating sets of cardinality 4, 5 ..., n + 1.

Hence,

$$D_{ct}(K_{1,n}, x) = nx^2 + \binom{n}{2}x^3 + \binom{n}{3}x^4 + ... + \binom{n}{n}x^{n+1} .$$
$$= x\left[nx + \binom{n}{2}x^2 + \binom{n}{3}x^3 + ... + \binom{n}{n}x^n\right] .$$
$$= x\left[\sum_{i=0}^{n}\binom{n}{i}x^i - 1\right].$$
$$D_{ct}(K_{1,n}, x) = x[(1 + x)^n - 1] .$$

**Theorem 2.9:** Let $G_1$ and $G_2$ be two graphs with no isolated vertices of orders m and n respectively. Then, $D_{ct}(G_1 \vee G_2, x) = D_{ct}(G_1, x) + D_{ct}(G_2, x) + [(1 + x)^m - 1][(1 + x)^n - 1]$.

**Proof:** From the definition of $G_1 \vee G_2$, if $S_1$ is any connected total dominating set of $G_1$, then $S_1$ is a connected total dominating set of $G_1 \vee G_2$. Similarly if $S_2$ is any connected total dominating set of $G_2$, then $S_2$ is a connected total dominating set of $G_1 \vee G_2$. Also, the sets consist of any one vertex of $G_1$ and any one vertex of $G_2$, form the connected total dominating sets of $G_1 \vee G_2$ of cardinality two. Therefore, the number of connected total dominating sets of cardinality two are $\binom{m}{1}\binom{n}{1}$. Similarly, the number of connected total dominating sets of cardinality three other than the first two cases is $\binom{m}{1}\binom{n}{2} + \binom{n}{1}\binom{m}{2}$. Proceeding like this, we obtain the other connected total dominating sets of cardinality 4, 5,…, m + n.

Therefore,

$$\begin{aligned} D_{ct}(G_1 \vee G_2, x) &= D_{ct}(G_1, x) + D_{ct}(G_2, x) + \binom{m}{1}\binom{n}{1}x^2 + \left[\binom{m}{1}\binom{n}{2} + \binom{n}{1}\binom{m}{2}\right]x^3 \\ &\quad + \left[\binom{m}{1}\binom{n}{3} + \binom{m}{2}\binom{n}{2} + \binom{m}{3}\binom{n}{3}\right]x^4 + \ldots + \left[\binom{m}{1}\binom{n}{m+n-1} + \ldots + \binom{m}{m+n-1}\binom{n}{1}\right]x^{m+n}. \\ &= D_{ct}(G_1, x) + D_{ct}(G_2, x) + \left[\binom{m}{1}x + \binom{m}{2}x^2 + \ldots + \binom{m}{m}x^m\right]\left[\binom{n}{1}x + \binom{n}{2}x^2 + \ldots + \binom{n}{n}x^n\right]. \\ &= D_{ct}(G_1, x) + D_{ct}(G_2, x) + \left[\sum_{i=0}^{m}\binom{m}{i}x^i - 1\right]\left[\sum_{i=0}^{n}\binom{n}{i}x^i - 1\right]. \end{aligned}$$

$D_{ct}(G_1 \vee G_2, x) = D_{ct}(G_1, x) + D_{ct}(G_2, x) + [(1 + x)^m - 1]\,[(1 + x)^n - 1]$

**Example 2.10:** Consider the join $G = G_1 \vee G_2$ of two graphs $G_1 = \{v_1, v_2, v_3\}$ and $G_2 = \{v_4, v_5, v_6, v_7\}$.

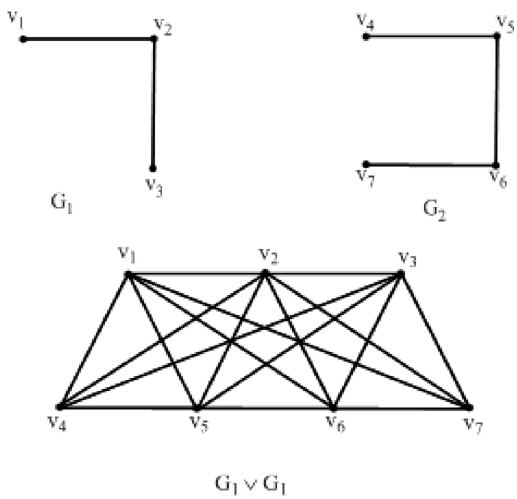


**Figure-2.3**

$d_{ct}(G_1, 2) = 2.$

$d_{ct}(G_1, 3) = 1.$

Therefore, $D_{ct}(G_1, x) = 2x^2 + x^{3.}$
$d_{ct}(G_2, 2) = 1.$

$d_{ct}(G_2, 3) = 2.$

$d_{ct}(G_2, 4) = 1.$

Therefore, $D_{ct}(G_2, x) = x^2 + 2x^3 + x^4$.

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$d_{ct}(G_1 \vee G_2, 2) = 15$.

$d_{ct}(G_1 \vee G_2, 3) = 33$.

$d_{ct}(G_1 \vee G_2, 4) = 34$.

$d_{ct}(G_1 \vee G_2, 5) = 21$.

$d_{ct}(G_1 \vee G_2, 6) = 7$.

$d_{ct}(G_1 \vee G_2, 7) = 1$.

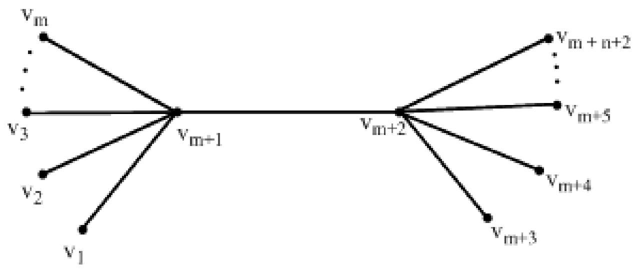
Therefore,

$$D_{ct}(G_1 \vee G_2, x) = 15x^2 + 33x^3 + 34x^4 + 21x^5 + 7x^6 + x^7.$$
$$= (2x^2 + x^3) + (x^2 + 2x^3 + x^4) + (3x + 3x^2 + x^3)\ (4x + 6x^2 + 4x^3 + x^4).$$

$$D_{ct}(G_1 \vee G_2, x) = D_{ct}(G_1, x) + D_{ct}(G_2, x) + [\ (1 + x)^3 - 1\ ]\ [\ (1 + x)^4 - 1].$$

**Theorem 2.11:** Let $B_{m,n}$ be a bi-star with m+n +2 vertices. Then the connected total domination polynomial of $B_{m,n}$ is $D_{ct}(B_{m,n}, x) = x^2(1 + x)^{m+n}$.

**Proof:** Let $B_{m,n}$ be a bi-star with m+n+2 vertices. Label the vertices of $B_{m,n}$ as $v_1, v_2, v_3, \ldots, v_m, v_{m+1}, v_{m+2}, \ldots, v_{m+n+2}$ as given in Figure 2.4.




$B_{m,n}$

**Figure-2.4**

Then the set $\{v_{m+1}, v_{m+2}\}$ is the only unique minimum connected total dominating set of cardinality 2. Therefore, $\gamma_{ct}(B_{m,n}) = 2$ and $d_{ct}(B_{m,n}, 2) = 1$.

It is obvious that the other connected total dominating sets must contain the two vertices $v_{m+1}$ and $v_{m+2}$. Hence, there are $\binom{m+n}{1}$ connected total dominating sets of cardinality three and $\binom{m+n}{2}$ connected total dominating sets of cardinality four. Proceeding like this, we obtain $\binom{m+n}{m+n}$ connected total dominating sets of cardinality m + n + 2.

Hence,

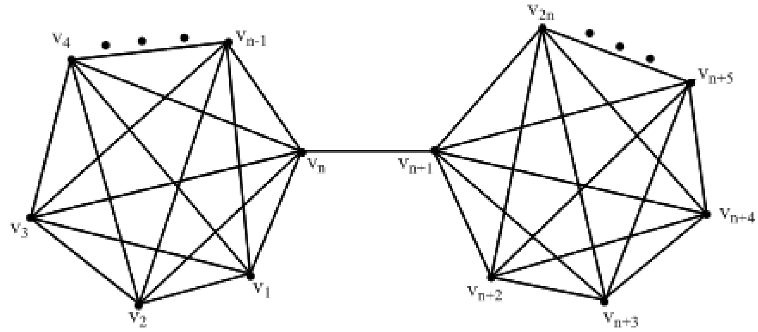
$$D_{ct}(B_{m,n}, x) = x^2 + \binom{m+n}{1}x^3 + \binom{m+n}{2}x^4 + \ldots + \binom{m+n}{m+n}x^{m+n+2}$$
$$= x^2\left[1 + \binom{m+n}{1}x + \binom{m+n}{2}x^2 + \ldots + x^{m+n}\right].$$

$$D_{ct}(B_{m,n}, x) = x^2\,(1 + x)^{m+n}.$$

**Corollary 2.12:** Let $B_{n,n}$ be a bi-star with 2n+2 vertices. Then the connected total domination polynomial of $B_{n,n}$ is $D_{ct}(B_{n,n}, x) = x^2\,(1 + x)^{2n}$.

**Theorem 2.13:** For a Barbell graph $B_n$ with 2n vertices, the connected total domination polynomial is
$$D_{ct}(B_n,x) = x^2(1+x)^{2n-2}.$$

**Proof:** Let $B_n$ be a Barbell graph with 2n vertices. Label the vertices of $B_n$ as $v_1, v_2, v_3, \ldots, v_n, v_{n+1}, \ldots, v_{2n}$ as given in Figure 2.5.




**$B_n$**
**Figure-2.5**

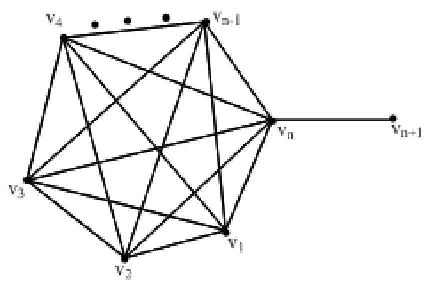
Then the set $\{v_n, v_{n+1}\}$ is the unique minimum connected total dominating set of cardinality 2. Therefore, $\gamma_{ct}(B_n) = 2$ and $D_{ct}(B_n,2) = 1$.

It is obvious that the other connected total dominating sets must contain the two vertices $v_n$ and $v_{n+1}$.
Hence there are $\binom{2n-2}{1}$ connected total dominating sets of cardinality three and $\binom{2n-2}{2}$ connected total dominating sets of cardinality four. Proceeding like this we obtain $\binom{2n-2}{2n-2}$ connected total dominating sets of cardinality 2n.

Hence,
$$D_{ct}(B_n,x) = x^2 + \binom{2n-2}{1}x^3 + \binom{2n-2}{2}x^4 + \binom{2n-2}{3}x^5 + \ldots + \binom{2n-2}{2n-2}x^{2n}.$$
$$= x^2\left[1+\binom{2n-2}{1}x + \binom{2n-2}{2}x^2 + \binom{2n-2}{3}x^3 + \ldots + \binom{2n-2}{2n-2}x^{2n-2}\right].$$
$$D_{ct}(B_n,x) = x^2\left[(1+x)^{2n-2}\right].$$

**Theorem 2.14:** For a Lollipop graph $L_{n,1}$ with n+1 vertices, the connected total domination polynomial is $D_{ct}(L_{n,1}, x) = x\left[(1+x)^n - 1\right]$.

**Proof:** Let $L_{n,1}$ be a Lollipop graph with n + 1 vertices. Label the vertices of $L_{n,1}$ as $v_1, v_2, \ldots, v_n, v_{n+1}$, where $v_n$ is the vertex of degree n + 1 and $v_{n+1}$ is the vertex of degree 1 as given in Figure 2.6.




**$L_{n,1}$**
**Figure-2.6**

Then, clearly there are n connected total dominating sets of cardinality two namely $\{v_1, v_n\}, \{v_2, v_n\}, \ldots, \{v_{n-1}, v_n\}, \{v_n, v_{n+1}\}$. Similarly, for the connected total dominating sets of cardinality 3 we need to select the vertex $v_n$ and two vertices from the set of vertices $\{v_1, v_2, \ldots, v_{n-1}, v_{n+1}\}$. Therefore, there are $\binom{n}{2}$ connected

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total dominating sets of cardinality three. Proceeding like this, we obtain the other connected total dominating sets of cardinality 4, 5,…, n +1.

Hence,

$$D_{ct}(L_{n,1}, x) = n\,x^2 + \binom{n}{2} x^3 + \binom{n}{3} x^4 + \binom{n}{4} x^5 + \dots + \binom{n}{n} x^{n+1}.$$

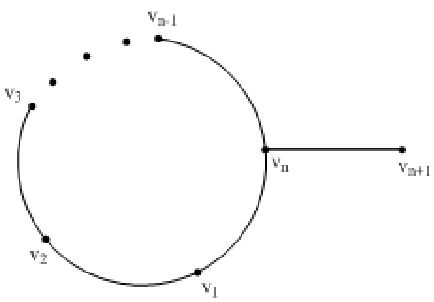
$$= x\left[nx + \binom{n}{2} x^2 + \binom{n}{3} x^3 + \binom{n}{4} x^4 + \dots + \binom{n}{n} x^n\right].$$

$$= x\left[\sum_{i=0}^{n}\binom{n}{i} x^i - 1\right].$$

$$D_{ct}(L_{n,1}, x) = x\,[(1 + x)^n - 1].$$

**Theorem 2.15:** For a Tadpole graph $T_{n,1}$ with n+1 vertices, the connected total domination polynomial is $D_{ct}(T_{n,1}, x) = (n-2)\,x^{n-2} + (2n-3)\,x^{n-1} + nx^n + x^{n+1}$.

**Proof:** Let $T_{n,1}$ be a Tadpole graph with n+1 vertices. Label the vertices of $T_{n,1}$ as $v_1, v_2, \ldots, v_n, v_{n+1}$ where $v_n$ is the vertex of degree 3 and $v_{n+1}$ is the vertex of degree 1 as given in Figure 2.7.




**$T_{n,1}$**
**Figure-2.7**

It is obvious that every connected total dominating set must contain the vertex $v_n$. Hence there are n–2 connected total dominating sets of cardinality n–2 and n–2 is the minimum cardinality.

Therefore, $\gamma_{ct}(T_{n,1}) = n - 2$ and $d_{ct}(T_{n,1}, n-2) = n - 2$.

Also, there are 2n–3 connected total dominating sets of cardinality n–1, n connected total dominating sets of cardinality n and there is only one connected total dominating set of cardinality n + 1.

Hence, $D_{ct}(T_{n,1}, x) = (n-2)\,x^{n-2} + (2n-3)\,x^{n-1} + nx^n + x^{n+1}$.

## 3. COEFFICIENTS OF CONNECTED TOTAL DOMINATION POLYNOMIAL

**Theorem 3.1:** Let G be a graph with $|V(G)| = n$. Then

- (i) $d_{ct}(G, n) = 1$ and $d_{ct}(G, n-1) = n - 1$ if $2 \le \delta(G)$.
- (ii) $d_{ct}(G, i) = 0$ if and only if $i < \gamma_{ct}(G)$ or $i > n$.
- (iii) $D_{ct}(G, x)$ has no constant and first degree terms.
- (iv) Let G be a graph and H be any induced sub graph of G. Then $\deg(D_{ct}(G, x)) \ge \deg(D_{ct}(H, x))$.
- (v) $D_{ct}(G, x)$ is a strictly increasing function in $[0, \infty)$.
- (vi) Zero is a root of $D_{ct}(G, x)$ with multiplicity $\gamma_{ct}(G)$.

**Proof:**

(i) Since G has n vertices, there is only one way to choose all the vertices. Therefore, $d_{ct}(G, n) = 1$.
If we delete one vertex from V(G), then the subgraphs induced by the remaining n–1 vertices are connected and dominate totally only if $\delta(G) \ge 2$. Therefore, $d_{ct}(G, n-1) = n - 1$ if $\delta(G) \ge 2$.

(ii) Since $D_{ct}(G, i) = \phi$ if $i < \gamma_{ct}(G)$ and $D_{ct}(G, n+k) = \phi$, $k = 1, 2, 3, \ldots$, we have
$d_{ct}(G, i) = 0$ if $i < \gamma_{ct}(G)$ and $i > n$.

(iii) A single vertex cannot dominate totally itself. So, the set of all vertices of G is dominated totally by atleast two of the vertices of G. Hence, the total domination polynomial has no constant term as well as first degree terms. Therefore, the connected total domination polynomial also has no constant term as well as first degree terms.

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(iv) Since the number of vertices in H ≤ number of vertices in G, deg $(D_{ct}(H, x)) \leq \deg(D_{ct}(G, x))$.
The proof of (v) and (vi) follows from the definition of connected total domination polynomial.

**Theorem 3.2:** For every natural number n, the connected total domination polynomial $D_{ct}(K_{1,n}, x)$ has no non-zero real root for odd n and only one non-zero root for even number n.

**Proof:** The connected total domination polynomial of $K_{1,n}$ is $D_{ct}(K_{1,n}, x) = x[(1+x)^n - 1]$.
Let $f_n(x) = (1+x)^n - 1$. It is enough to prove that $f_n(x) = 0$ has no non-zero real root for odd n and only one non-zero real root for even n. The roots of $f_n(x) = 0$ are $x = \left(\cos\left(\frac{2k\pi}{n}\right) - 1\right) + i\sin\left(\frac{2k\pi}{n}\right)$, k = 0, 1, 2,…, n–1. Clearly $|x| \leq 2$. Therefore, all the roots of this polynomial lie within a circular disc of radius 2. When n is odd, $\sin\left(\frac{2k\pi}{n}\right) \neq 0$, k = 0, 1, 2,…, n–1. Therefore all the roots are complex. When n is even, $\sin\left(\frac{2k\pi}{n}\right) = 0$, only for $k = \frac{n}{2}$ The corresponding root is a non-zero real. Therefore, $f_n(x)$ has only one non-zero real root for even n.

## 4. CONNECTED TOTAL DOMINATION POLYNOMIAL OF $G \circ K_1$

**Theorem 4.1:** Let G be any connected graph with n vertices. Then,

$$D_{ct}(G \circ K_1, x) = x^n(1+x)^n.$$

**Proof:** Since G has n vertices, $G \circ K_1$ has 2n vertices. Clearly $\{v_1, v_2, \ldots, v_n\}$ is the minimal connected total dominating set of $G \circ K_1$. Therefore, $\gamma_{ct}(G \circ K_1) = n$. It is easy to see that there are $\binom{n}{i}$ possibilities of connected total dominating sets of cardinality n + i. Hence,

$$\begin{aligned} D_{ct}(G \circ K_1, x) &= x^n + \binom{n}{1} x^{n+1} + \binom{n}{2} x^{n+2} + \ldots + x^{2n}. \\ &= x^n \left(1 + \binom{n}{1} x + \binom{n}{2} x^2 + \ldots + x^n\right). \\ &= x^n \left[\sum_{i=0}^{n} \binom{n}{i} x^i\right]. \end{aligned}$$

$D_{ct}(G \circ K_1, x) = x^n(1+x)^n$.

**Example 4.2:** Consider $G \circ K_1$ given in Figure 4.1

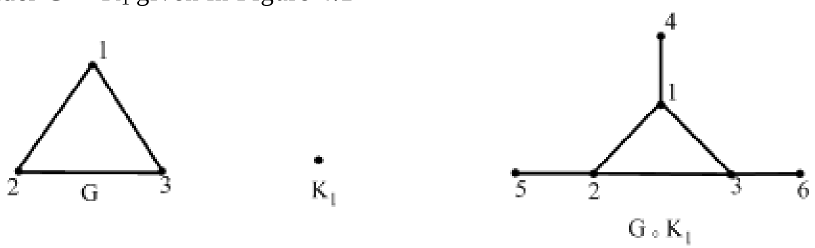


**Figure-4.1**

$d_{ct}(G \circ K_1, 3) = 1$.

$d_{ct}(G \circ K_1, 4) = 3$.

$d_{ct}(G \circ K_1, 5) = 3$.

$d_{ct}(G \circ K_1, 6) = 1$.

$$\begin{aligned} D_{ct}(G \circ K_1, x) &= x^3 + 3x^4 + 3x^5 + x^6. \\ &= x^3(1 + 3x + 3x^2 + x^3). \end{aligned}$$

$D_{ct}(G \circ K_1, x) = x^3(1+x)^3$.

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**Theorem 4.3:** Let G be any connected graph of order n. Then $D_{ct}(G \circ \overline{K_m}, x) = x^n(1 + x)^{mn}$.

**Proof:** Since G has n vertices, $G \circ \overline{K_m}$ has n(m+1) vertices. Clearly $\{v_1, v_2,\ldots,v_n\}$ is the minimal connected total dominating set of $G \circ \overline{K_m}$. Therefore, $\gamma_{ct}(G \circ \overline{K_m}) = n$. It is obvious that there are $\binom{mn}{i}$ connected total dominating sets of cardinality n + i, where $1 \le i \le mn$. Hence

$$D_{ct}(G \circ \overline{K_m}, x) = x^n + \binom{mn}{1} x^{n+1} + \binom{mn}{2} x^{n+2} + \ldots + x^{nm+n}.$$

$$= x^n\left(1 + \binom{mn}{1}x + \binom{mn}{2}x^2 + \ldots + x^{mn}\right).$$

$$= x^n \sum_{i=0}^{n} \binom{mn}{i} x^{mn}.$$

$$D_{ct}(G \circ \overline{K_m}, x) = x^n(1 + x)^{mn}.$$

**Example 4.4:** Consider $G \circ \overline{K_3}$ given in Figure 4.2.

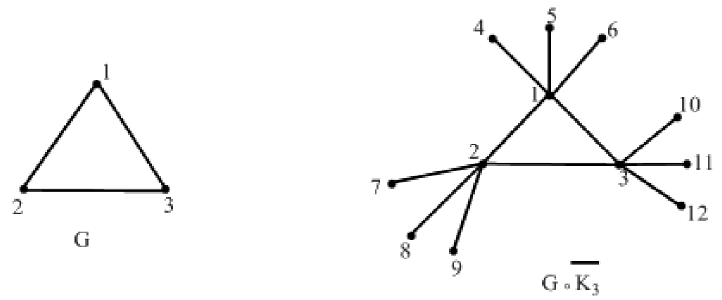


**Figure-4.2**

$D_{ct}(G \circ \overline{K_3}, x) = x^3(1 + x)^9$.

## REFERENCES

1. Alikhani.S and Hamzeh Torabi, 2010, On Domination Polynomials of complete partite Graphs, World Applied Sciences journal, 9(1): 23-24.
2. Alikhani.S and Peng.Y.H, 2008, Domination sets and Domination Polynomial of cyles, Global Journal of Pure and Applied Mathematics, Vol.4.No.2.
3. Alikhani.S and Peng.Y.H, 2009, Introduction to Domination Polynomial of a graph, arXiv: 0905. 2251 v1 [math.CO] 14 may.
4. Bondy J.A and Murthy U.S.R, 1984, Graph theory with application, Elsevier Science Publishing Co, Sixth printing.
5. Frucht.R and Harary.F, 1970, On Corona of two graph, Aequationes Math.4, 322-324.
6. Gray Chartand, Ping Zhang, 2005, Introduction to Graph Theory, Mc Graw Hill, Higher Education.
7. Maryam Atapour and Nasrin Soltankhah, 2009, On Total Domination Sets in Graphs, Int. J. Contemp. Math. Sciences, Vol.4, No.6, 253-257.

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