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COMMON FIXED POINT FOR COMPATIBLE MAPPINGS ON 2-MENGER SPACES

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ABSTRACT

The aim of the present paper is to obtain i) a common fixed point theorem for compatible mappings by using the concept of asymptotic regularity and ii) a common fixed point theorem using the concept of joint reciprocal continuity in a 2-Menger space.

Keywords: *Fixed point, 2-Menger space, compatible mappings, asymptotic regularity and joint reciprocal continuity in a 2- Menger space.*

AMS subject classification 2000: *47H10, 54H25.*

1. INTRODUCTION

The study of 2-metric spaces was initiated by Gähler [3] and some fixed point theorems in 2-Metric spaces were proved in Hadžić [7], Rhoades [9] and Iseki [8]. The probabilistic 2-metric spaces were first introduced in Golet ([4], [5]), proved a fixed point theorem in probabilistic metric spaces. Some fixed point theorems in a 2-Menger space are proved in Golet [6] and Hadžić [7]. Badshah and Gopal Meena [1] proved a fixed point theorem for a pair of self-maps on a 2-metric space.

In this paper we introduce the notion of a 2-Menger space and obtain i) a common fixed point theorem (Th 2.1) for compatible mappings by using the concept of asymptotic regularity and ii) a common fixed point theorem using the concept of joint reciprocal continuity in a 2-Menger space. Supporting example is also provided (example 2.3).

1.1 Notations: The set of all real numbers is denoted by R and set of all non-negative real numbers is denoted by R^+.

1.2 Definition (Sehgal and Bharucha-Reid [10]): A mapping $F: R \to [0,1]$ is said to be a distribution function if it is non-decreasing, left-continuous with $\inf_{t\in R} F(t) = 0$ and $\sup_{t\in R} F(t) = 1$.

The set of all distribution functions is denoted by $\mathfrak{D}$ and $\mathfrak{D}^+ = \{F \in \mathfrak{D} | F(0) = 0\}$.

1.3 Definition (Gähler [3]): A 2-metric space is an ordered pair (X, d) where X is an abstract set and $d: X^3 \to R^+$ such that

i) For distinct points $x, y \in X$ there exists a point $z \in X$ such that $d(x, y, z) \neq 0$
ii) $d(x, y, z) = 0$ if at least two of x, y and z are equal
iii) $d(x, y, z) = d(x, z, y) = d(y, z, x)\ \forall x, y, z \in X$
iv) $d(x, y, z) \leq d(x, y, u) + d(x, u, z) + d(u, y, z)\ \forall x, y, z, u \in X$.

The function d is called a 2-metric for the space X and the pair (X, d) denotes a 2-metric space.

The following definitions on the concept of 2-Menger spaces are given by Golet [6].

1.4 Definition (Golet [6]): A probabilistic 2-metric space (P-2-M space) is an ordered pair (X, F) where $F: X^3 \to \mathfrak{D}^+$ is such that

i) $F_{x\,y\,z}(t) = 1\ \forall\ t > 0$ if and only if at least two of the three points x, y and z are equal,
$F_{x\,y\,z}(t) = 0\ \forall t \leq 0\ \ \forall x, y, z \in X$
ii) For distinct points $x, y \in X$ there exists a point $z \in X$ such that $F_{x\,y\,z}(t) \neq 1$ if $t > 0$

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iii) $F_{x\,y\,z}(t) = F_{x\,z\,y}(t) = F_{y\,z\,x}(t)$
iv) $If \quad F_{x\,y\,w}(t_1) = 1, F_{x\,w\,z}(t_2) = 1 \; and \; F_{w\,y\,z}(t_3) = 1 \; then$

$$F_{x\,y\,z}\,(t_1 + t_2 + t_3) = 1$$

1.5 Definition (Golet [6]): A mapping $*: [0,1]^3 \to [0,1]$ is said to be 2- t- norm if
i) $*\,(a, 1,1) = a$
ii) $*\,(a, b, c) = *\,(a, c, b) = *\,(c, b, a)$
iii) $*\,(a, b, c) \le *\,(d, e, f)$ if $a \le d, b \le e \; and \; c \le f$
iv) $*\,(*\,(a, b, c), d, e) = *\,(a, *\,(b, c, d), e) = \; *\big(a, b, *\,(c, d, e)\big) \; \forall a, b, c, d, e \in [0,1]$

1.6 Example: If $*$ is defined as $* = \min(a, b, c)\,, a, b, c \in [0,1]$ then $*$ is a 2- t-norm.

1.7 Definition (Golet [6]): A 2-Menger space is a triplet $(X, F,*)$ where (X, F) is a P -2 -M space, $*$ is a 2- t-norm satisfying the following inequality:

$$F_{x\,y\,z}\,(t_1{+}t_2{+}t_3) \ge *\left(F_{x\,y\,w}(t_1), F_{x\,w\,z}(t_2), F_{w\,y\,z}(t_3)\right) \; \forall x, y, z, w \in X.$$

1.8 Definition (Golet [6])**:** Let $(X, F,*)$ be a 2-Menger space and $*$ be a continuous 2-t-norm, then $(X, F,*)$ is Hausdroff in the topology induced by the family of neighborhoods,$\mathrm{U}_x\big(\varepsilon, \lambda, a_1, a_{2,} \dots, a_n\big), x, a_i \in X, \varepsilon > 0,$
$i = 1,2, \dots, n$ and $\lambda \in (0,1)$ where $\mathrm{U}_x\big(\varepsilon, \lambda, a_1, a_{2,} \dots, a_n\big) = \big\{y \in X \big| F_{x\,y\,a_i}(\varepsilon) > 1 - \lambda, 1 \le i \le n\big\}$

$$= \cap_{i=1}^{n}\big\{y \in X \big| F_{x\,y\,a_i}(\varepsilon) > 1 - \lambda, 1 \le i \le n\big\}\,.$$

1.9 Definition (Golet [6])**:** Let $(X, F,*)$ be a 2-Menger space and $*$ be a continuous 2- t-norm. A sequence $\{x_n\}$ in X is said to converge to a point $x \in X$ if for every $\varepsilon > 0 \; and \; \lambda \in (0,1)$, there exists an integer $\mathrm{M}(\varepsilon, \lambda)$ such that

$$F_{x_n\,x\,a}(t) > 1 - \lambda \;\; whenever \; \mathrm{m}, n \ge \mathrm{M}(\varepsilon, \lambda) \;\; and \; a \in X.$$

1.10 Definition (Golet [6])**:** A sequence $\{x_n\}$ in a 2-Menger space $(X, F,*)$ is said to be a Cauchy sequence if for every$\varepsilon > 0 \; and \; \lambda \in (0,1)$, there exists an integer $\mathrm{M}(\varepsilon, \lambda)$ such that

$$F_{x_n\,x_m\,a}(t) > 1 - \lambda \;\; whenever \;\; m, n \ge \mathrm{M}(\varepsilon, \lambda) \; and \; a \in X.$$

1.11 Definition (Golet [6])**:** A 2-Menger space $(X, F,*)$ is said to be complete if each Cauchy sequence in X converges to a point of X.

1.12 Definition: A sequence $\{x_n\}$ in a 2-Menger space $(X, F,*)$ is said to be asymptotically regular with respect to the pair (S, T) of self-mappings on X if

$$\lim_{n\to\infty} F_{Sx_n\,Tx_n\,a}\,(t) = 1 \; \forall a \in X.$$

1.13 Definition (Chang [2]): Two self-mapping $S \; and \; T$ on 2-Menger space $(X, F,*)$ is said to be compatible if $\lim_{n\to\infty} F_{STx_n\,TSx_n\,a}\,(t) = 1 \;\; \forall t > 0$, whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} Sx_n = z = \lim_{n\to\infty} Tx_n \;\; for \; some \; z \in X.$$

1.14 Example: Let $X = R$ and define $d: X^3 \to R$ by

$$d(x, y, z) = \begin{cases} 0\,, & if \; at \; least \; two \; of \; the \; three \; points \; x, y, z \; are \; equal \\ 2\,, & otherwise \end{cases} \quad .$$

Then (X, d) is a 2-metric space. Define $F : X^3 \to \mathfrak{D}^+$ by

$$F_{x\,y\,z}\,(t) = \frac{t}{t + d(x,y,z)}$$, then (X, F) is a probabilistic 2-metric space.

If $* : [0,1]^3 \to [0,1]$ is defined as $* = \min\{r, s, t\}\,, \; r, s, t \in [0,1]$, then $(X, F,*)$ is a 2-Menger space.

1.15 Notation: Write
$\Psi = \{\psi |\, \psi: [0,1] \to [0,1], \psi \; is \; continous\,, \psi(1) = 1 \; and \; \psi(t) > t \; \forall t \in (0,1)\}$.

1.16 Example: Define $\psi: [0,1] \to [0,1] \;\; as \;\; \psi(t) = \frac{t+1}{2}.\; Then \; \psi \in \Psi$.

2. MAIN RESULTS

In this section first we prove our first main result using the concept of asymptotic regularity.

2.1Theorem: Let $P, S \; and \; T$ be self-mappings of a complete 2-Menger space$(X, F, *)$, where $*$ is a continuous 2- t-norm, satisfying the following conditions:

i) $F_{Px \; Py \; a}(t) \geq \psi\left(min\left\{F_{Px \; Sx \; a}(t), F_{Py \; Sy \; a}(t)\right\}\right)$, $\forall x, y, a \in X \; and \; for \; some \; \psi \in \Psi$
ii) the pairs $(P, S) \; and \; (P, T)$ are compatible
iii) there exists a sequence $\{x_n\}$ which is asymptotically regular with respect to (P, S) and (P, T)
iv) $S \; and \; T$ are continuous.

Then $P, S \; and \; T$ have unique common fixed point in X.

Proof: Let $\{x_n\}$ be a sequence in X satisfying condition (iii).

By taking $x = x_n \; and \; y = x_m$ in (i), we obtain

$$F_{Px_n \; Px_m \; a}(t) \geq \psi\left(min\left\{F_{Px_n Sx_n \; a}(t), F_{Px_m Sx_m \; a}(t)\right\}\right)$$

On letting $n \to \infty$, using condition(iii), we obtain

$$\lim_{n\to\infty} F_{Px_n \; Px_m \; a}(t) \geq \psi(min\{1,1\}) = \psi(1) = 1$$

This implies

$$\lim_{n\to\infty} F_{Px_n \; Px_m \; a}(t) = 1 \; \forall a \in X.$$

Thus $\{Px_n\}$ is a Cauchy sequence in X. Since X is complete we have
$Px_n \to z$ for some $z \in X$. (2.1.1)

Now $F_{Sx_n \; z \; a}(t) \geq * \left(F_{Sx_n \; z \; Px_n}(t), F_{Sx_n \; Px_n \; a}(t), F_{P \; x_n \; z \; a}(t)\right)$

On letting $n \to \infty$, using condition(iii), (2.1.1) and continuity of $*$, we get

$$\lim_{n\to\infty} F_{Sx_n \; z \; a}(t) \geq * \, (1,1,1) = 1$$

This implies $\lim_{n\to\infty} F_{S \; x_n \; z \; a}(t) = 1 \;\; \forall t > 0$.
$i.e$ $Sx_n \to z.$ (2.1.2)

Now
$F_{Tx_n \; z \; a}(t) \geq * \left(F_{Tx_n \; z \; Px_n}(t), F_{Tx_n \; Px_n \; a}(t), F_{Px_n \; z \; a}(t)\right)$

On letting $n \to \infty$, using condition (iii), (2.1.1) and continuity of $*$, we get

$$\lim_{n\to\infty} F_{Tx_n \; z \; a}(t) \geq * \, (1,1,1) = 1$$

This implies $\lim_{n\to\infty} F_{Tx_n \; z \; a}(t) = 1 \;\; \forall t > 0$
i.e $Tx_n \to z.$ (2.1.3)

Since
$F_{PSx_n \; Sz \; a}(t) \geq * \left(F_{PTx_n \; Sz \; SPx_n}(t), F_{PSx_n \; SPx_n \; a}(t), F_{SPx_n \; Sz \; a}(t)\right)$ (2.1.4)

applying condition (iv) in (2.1.1), we get
$SPx_n \to Sz$ (2.1.5)

On letting $n \to \infty$ in (2.1.4), using condition (ii) and (2.1.5), we get
$\lim_{n\to\infty} F_{PSx_n Sz \; a}(t) = 1 \;\; \forall a \in X \; and \; t > 0.$

This implies
$PS \; x_n \to Sz$ (2.1.6)

From condition (iv) $we \; have \; T$ is continuous, applying this in (2.1.1) we get
$TPx_n \to Tz$ (2.1.7)

Since
$F_{PTx_n Tz \; a}(t) \geq * \left(F_{PTx_n Tz \; TPx_n}(t), F_{PTx_n \; TPx_n \; a}(t), F_{TPx_n Tz \; a}(t)\right)$

On letting $n \to \infty$, using condition (ii) and (2.1.7), we get

$$\lim_{n\to\infty} F_{PTx_n Tz\ a}(t) = 1, \quad \forall a \in X \ \ and\ t > 0.$$

This implies

$PTx_n \to Tz$. (2.1.8)

By letting $x = Sx_n$ and $y = Tx_n$ in (i), we get

$$F_{PSx_n PTx_n\ a}(t) \geq \ \psi\big(min\big\{F_{PSx_n SSx_n\ a}(t), F_{PTx_n STx_n\ a}(t)\big\}\big)$$

On letting $n \to \infty$, using condition(iv), we have

$$Tx_n \to z \ implies\ TTx_n \to Tz\ and\ Sx_n \to z\ implies\ SSx_n \to Sz$$

Using this and also using (2.1.7) and (2.1.8), we get

$$F_{Sz\ Tz\ a}(t) \geq \psi(min\{F_{Sz\ Sz\ a}(t), F_{Tz\ Sz\ a}(t)\})$$

This implies $F_{Sz\ Tz\ a}(t) \geq \psi(F_{Tz\ Sz\ a}(t))$

That is $F_{Sz\ Tz\ a}(t) = 1$

Therefore

$Sz = Tz$ (2.1.9)

Again by taking $x = Tx_n\ and\ y = z$ in (i), we have

$$F_{PTx_n\ Pz\ a}(t) \geq \ \psi\big(min\big\{F_{PTx_n\ \ STx_n\ a}(t), F_{Pz\ Sz\ a}(t)\big\}\big)$$

On letting $n \to \infty$, using (2.1.8) and applying condition (iv) in (2.1.3) implies $STx_n \to Sz$, applying this in the above equation, we get

$$F_{Tz\ Pz\ a}(t)\ \geq \psi(min\{1, F_{Pz\ Tz\ a}(t)\})$$

This implies

$$F_{Tz\ Pz\ a}(t)\ \geq \psi\big(F_{Pz\ Tz\ a}(t)\big)$$

That is

$F_{Tz\ Pz\ a}(t) = 1 \ \ \forall a \in X$

Hence

$Tz = Pz$

Thus

$Pz = Tz = Sz.$ (2.1.10).

Now by taking $x = x_n$ and $y = z$ in(i), we get

$$F_{Px_n Pz\ a}(t) \geq \psi\big(min\big\{F_{Px_n Sx_n\ a}(t), F_{Pz\ Tz\ a}(t)\big\}\big)$$

On letting $n \to \infty$, using (2.1.1), (2.1.2) and (2.1.10), we get

$$F_{z\ Pz\ a}(t) \geq \psi(min\{1,1\}) = \psi(1) = 1$$

This implies $F_{z\ Pz\ a}(t) = 1 \ \ \forall t > 0$

Thus $z = Pz$.

Hence z is a common fixed point of $P, S\ and\ T$.

Let x be a common fixed point of $P, S\ and\ T$, then from(i), we have

$$F_{Px\ Pz\ a}(t) \geq \psi(min\{F_{Px\ Sx\ a}(t), F_{Pz\ Sz\ a}(t)\})$$

This implies

$$F_{x\ z\ a}(t) \geq \psi(min\{F_{x\ x\ a}(t), F_{z\ z\ a}(t)\})$$

That is$F_{x\ z\ a}(t) \geq \psi(min\{1,1\}) = \psi(1) = 1$ and thus $F_{x\ z\ a}(t) = 1$ hence $x = z$.

Therefore z is the unique common fixed point of P, S and T.

Now, we state our second main result which uses the concept of joint reciprocal continuity.

2.2 Theorem: Let P, S and T be self-mappings of a complete 2-Menger space $(X, F, *)$, where $*$ is a continuous 2 -t-norm, satisfying the following conditions:-

i) $F_{Px\ Py\ a}(t) \geq \psi\left(min\left\{F_{Px\ Sx\ a}(t), F_{Py\ Sy\ a}(t)\right\}\right),\ \forall x, y, a \in X$ and for $some$ $\psi \in \Psi$
ii) S and T are continuous.
iii) (S, T) is jointly reciprocally continuous with respect P in X.
Then P, S and T have unique common fixed point in X.

Proof: From condition (iii) there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Px_n = \lim_{n\to\infty} Tx_n = z \quad for\ some\ z \in X \tag{2.2.1}$$

and $$\lim_{n\to\infty} F_{PSx_n SPx_n\ a}(t) = 1 = \lim_{n\to\infty} F_{PTx_n TPx_n\ a}(t)\ \ \forall a \in X \tag{2.2.2}$$

Applying condition (ii) in equation (2.2.1) and using this in the equation (2.2.2), we get

$$PSx_n \to Sz \text{ and } PTx_n \to Tz \tag{2.2.3}$$

By taking $x = Sx_n, y = Tx_n$ in (i) and on letting $n \to \infty$, using $(ii), (iii)$,(2.2.2) and (2.2.3),we get $Sz = Tz$

Similarly by taking $x = Tx_n$ and $y = z$ in (i) and on letting $n \to \infty$, we get $Pz = Tz$.

Therefore $Sz = Pz = Tz$.

By taking $x = x_n$ and $y = z$ in (i) and on letting $n \to \infty$, we get $Pz = z$. Therefore z is a fixed point of P.

Hence z is a common fixed point of P, S and T . Suppose x is a common fixed point of P, S and T . Then it can be easily proved that $x = z$.

Hence z is the unique common fixed point of P, S and T.

2.3 Example: Let $(X, F, *)$ be a complete 2-Menger space as defined in example (1.14) and $\psi \in \Psi$ be as defined in example (1.16). Let P, S and T be self-maps on X such that $P(x) = x_0, x_0 \in X$ and $S = T = I$. Then P, S and T satisfy all the hypothesis of theorem 2.1 and Theorem 2.2 and x_0 is the unique common fixed point of P, S and T in X.

REFERENCES

1. Badshah, V.H. and Gopal Meena: Common fixed point for compatible mappings on 2-Metric spaces, Journal of Indian Acad.Math.,Vol31 (2009) No. 1, 23-30 .
2. Chang, S.S.: On some fixed point theorems in probabilistic metric space and its application, Z.Wahr, Verw, Gebiete, 63(1983), 463-474.
3. Gähler, S.: 2-Metrische Räume und ihre topologische strucktur. Math. Nachr. 26(1963), 115-148.
4. Golet, I.: Probabilistic 2-metric spaces. Seminarulde Teoria probabilistatlier si aplicatii Univ. Timisoara. Nr 83(1988a) ,1-15.
5. Golet, I.: A fixed point theorem in probabilistic 2-metric spaces. Sem. Math. Phys. Inst. Polit. Timisoara, 1988b, 21-26.
6. Golet, I.: Fixed point theorems for multivalued mappings in probabilistic 2-metric spaces. An.Stiint.Univ.Ovidius Constanta Ser.Mat. 3(1995), 1, 44-51.
7. Hadžić,O.: On common fixed point theorem in 2-metric spaces, Univ.u.Novom sadu, Zb.Rad. Prirod-Mat. Fak, Ser.Mat. 12(1982), 7-18.
8. Iseki, K.: Fixed point theorems in 2-metric spaces Math.Sem.Notes, 3(1975), 133-136.
9. Rhoades, B.E.: Contraction type mappings on a 2-metric space, Math.Nachr.91 (1975), 151-155.
10. Sehgal,V. and Bharucha-Reid, A.: Fixed points of contraction mappings on PM-spaces. Math.System. Theory., 6(1972), 97-102.

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