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ORDER STATISTICS, LORENZ TRANSFORM AND THE CVAR RISK MEASURE

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ABSTRACT

The class of linear functions of order statistics or L-estimates is considered. Under finite variances and other suitable restrictions, it is known that L-estimates converge in distribution to a normal distribution as the sample size increases to infinity. This result is applied to obtain approximate confidence intervals for the Lorenz transform and the conditional value-at-risk measure using L-estimates in case the data follows an approximate generalised Pareto distribution with finite variance. By infinite variance, the goodness-of-fit of the L-estimate compared to the true Lorenz transform is measured using the expected relative error of approximation.

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Keywords: *order statistics, L-estimate, Lorenz transform, conditional value-at-risk, generalised Pareto, normal distribution, approximate confidence interval*

1. INTRODUCTION

An important class of statistics consists of the linear functions of order statistics, usually called L-estimates (e.g. Rychlik [17]). It appears to have been first extensively studied by Percy Daniell in 1920 (see Stigler [22]). Given the *order statistics* $X_{(1)} \le \ldots \le X_{(n)}$ of a random sample $X = \left(X_1, \ldots, X_n\right)$ of size *n* and a sequence of real numbers $c_1, \ldots, c_n$, the *L-estimate* is the statistics defined by $S_n = \sum_{i=1}^{n} c_i X_{(i)}$ .Well-known examples include the *sample mean* $S_n = \overline{X} = n^{-1} \cdot \sum_{i=1}^{n} X_{(i)}$, the α *-trimmed mean* $S_n = (n - 2[\alpha n])^{-1} \cdot \sum_{i=[\alpha n]+1}^{n-[\alpha n]} X_{(i)}$, where $[x]$ denotes the greatest integer less than or equal to *x*, and *Gini's mean difference* $g = 2[n(n-1)]^{-1} \cdot \sum_{i=1}^{n} (2i - n - 1) X_{(i)}$ (Gini [6], David [4]).

The present work emphasizes some essential statistical properties of the conditional value-at-risk or expected shortfall measure, which has been recognised as an important risk measure in modern risk management (an extensive recent review is Nadarajah *et al.* [15]). For a random variable *X* with distribution function $F(x), x \in R$, and quantile function $Q(u) = \inf\left\{x \middle| F(x) \ge u\right\}$, $u \in (0,1)$, the *conditional value-at-risk* (CVaR) measure to the confidence level $\alpha \in (0,1)$ is defined as follows (e.g. Hürlimann [9], Proposition 2.1):

$$CVaR_\alpha[X] = \frac{1}{1-\alpha}\left\{E[X] - L_\alpha[X]\right\}, \tag{1.1}$$

where $L_\alpha[X] = \int_0^\alpha Q(u)du$ denotes the *Lorenz transform* of *X*. The relationship (1.1) suggests two ways of statistical estimation. By known mean, one uses the obvious L-estimate of the Lorenz transform

$$L_\alpha^{(n)}[X] = n^{-1} \cdot \sum_{i=1}^{[\alpha n]+1} X_{(i)}, \tag{1.2}$$

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to estimate CVaR by $(1-\alpha)^{-1}\cdot\left\{E[X]-L_{\alpha}^{(n)}[X]\right\}$. Alternatively, from (1.2) one obtains through (1.1) immediately the following L-estimate:

$$CVaR_{\alpha}^{(n)}[X]=\left[(1-\alpha)n\right]^{-1}\cdot\sum_{i=[\alpha n]}^{n}X_{(i)}\,. \tag{1.3}$$

In the risk management context, where these L-estimates are calculated using market or simulated data, it is useful to know the asymptotic distribution of these quantities. Consider L-estimates of the form

$$S_n=n^{-1}\cdot\sum_{i=1}^{n}J\left(\frac{i}{n+1}\right)\cdot X_{(i)}\,, \tag{1.4}$$

where $J(u)$, $u\in(0,1)$, is an appropriate weight function. Formulas for the asymptotic mean and variance of such L-estimates have been found since Jung [10]. Under suitable restrictions, in particular finite variance of X_i, $i=1,\dots,n$, it has been known for a long time that L-estimates converge in distribution to a normal distribution as the sample size increases to infinity (e.g. Govindarajulu *et al.* [7], Chernoff *et al.* [3], Moore [14], Shorack [20] and Stigler [21], [3]). A more detailed account of the content follows.

Section 2 summarises a main result of Stigler [23] and applies it to the L-estimates (1.2), (1.3). Its use is illustrated with the generalised Pareto distribution in Section 3. Since scaled excesses over high thresholds are in the limit generalised Pareto distributed by the theorem of Pickands [16], and Balkema and de Haan [2], a discussion of the distribution properties of these quantities should be based upon this choice (e.g. McNeil [12], Section 3). By finite variance, the L-estimates (1.2), (1.3) have an asymptotic normal distribution. This allows the construction of approximate confidence intervals for the Lorenz transform and the CVaR measure. To illustrate, we list in tabular form the critical sample size required to estimate these quantities with a fixed precision. In case the variance is infinite, the preceding results do not apply. To measure the goodness of approximation of the true Lorenz transform by the L-estimate (1.2) for the generalised Pareto with infinite variance, we calculate in Section 4 the expected relative error of approximation. To obtain given relative errors, an increasing sample size is required by increasing confidence level.

2. ASYMPTOTIC DISTRIBUTION OF L-ESTIMATES BY FINITE VARIANCE

The simplest main result about the asymptotic normality of L-estimates of the form (1.4) is due to Stigler [23], Theorem 2. The reader interested in more details and up-to-date mathematical treatment is refereed to Serfling [19], Sen [18], and Jureckovc and Sen [11].

Theorem 2.1: Let $X=(X_1,\dots,X_n)$ be a random sample of size *n* such that $E\left[X_i^2\right]<\infty$, $i=1,\dots,n$, and let $S_n=n^{-1}\cdot\sum_{i=1}^{n}J\left(\frac{i}{n+1}\right)\cdot X_{(i)}$ be a L-estimate. If $J(u)$, $u\in(0,1)$, is bounded and continuous almost everywhere at $u=F(x),\ x\in R$, then one has

$$\lim_{n\to\infty}P\left(\frac{S_n-E[S_n]}{\sigma[S_n]}\le x\right)=\Phi(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x}e^{-\frac{1}{2}z^2}dz\,, \tag{2.1}$$

and the asymptotic mean and variance of the L-estimate are given by

$$\mu(J,F)=\lim_{n\to\infty}E[S_n]=\int_0^1 Q(u)J(u)du\,, \tag{2.2}$$

$$\sigma^2(J,F)=\lim_{n\to\infty}n\sigma^2[S_n]=2\cdot\int_{-\infty}^{\infty}J[F(x)]F(x)\cdot\left\{\int_{x<y}J[F(y)][1-F(y)]dy\right\}dx\,. \tag{2.3}$$

Proof: See Stigler [23]. ◊

Let us apply this important result to the L-estimates (1.2) and (1.3) of the Lorenz transform and the CVaR measure.

2.1. ASYMPTOTIC DISTRIBUTION OF THE SAMPLE CVaR

Consider the L-estimate (1.4) with $J(u)=0$ if $u \le \alpha$ and $J(u)=(1-\alpha)^{-1}$ if $u>\alpha$, which yields (1.3). From (2.2) one obtains for the asymptotic mean

$$\mu(J,F)=\frac{1}{1-\alpha}\int_{\alpha}^{1} Q(u)du = CVaR_{\alpha}[X], \tag{2.4}$$

where the last equality follows by Definition (1.1) and Proposition 2.1 in Hürlimann [9]. Therefore, the mean of the CVaR L-estimate is asymptotically unbiased. The asymptotic variance is determined by the variance of the stop-loss random variable $(X-d)_+$ with value-at-risk $d=Q(\alpha)$, namely

$$\sigma^2(J,F)=Var\left[\left(X-Q(\alpha)\right)_+\right]=\pi^{(2)}\left[Q(\alpha)\right]-\pi^{(1)}\left[Q(\alpha)\right]^2, \tag{2.5}$$

where $\pi^{(k)}(x)=E\left[(X-x)_+^k\right],\quad k=1,2$, denote the stop-loss transforms of degree one and two, and the quantile $Q(\alpha)$ identifies with the usual value-at-risk functional $VaR_{\alpha}[X]$. The formula (2.5) is derived as follows. Using (2.3) and the facts (e.g. Hürlimann [8], Theorem 2.1)

$\pi^{(1)}(x)=\int_x^{\infty}\overline{F}(y)dy$, $\overline{F}(y)=1-F(y)$, $\pi^{(2)}(x)=2\cdot\int_x^{\infty}\pi^{(1)}(y)dy$, one gets

$$\begin{aligned}\sigma^2(J,F)&=2\cdot\int_{Q(\alpha)}^{\infty}\left[\int_x^{\infty}\overline{F}(y)dy\right]F(x)dx=2\cdot\int_{Q(\alpha)}^{\infty}\pi^{(1)}(x)F(x)dx\\&=2\cdot\int_{Q(\alpha)}^{\infty}\pi^{(1)}(x)dx-2\cdot\int_{Q(\alpha)}^{\infty}\pi^{(1)}(x)\overline{F}(x)dx=\pi^{(2)}\left[Q(\alpha)\right]-\pi^{(1)}\left[Q(\alpha)\right]^2,\end{aligned} \tag{2.6}$$

where the value of the last integral follows from the relation (use partial integration)

$$\int_{Q(\alpha)}^{\infty}\pi^{(1)}(x)\overline{F}(x)dx=\pi^{(1)}\left[Q(\alpha)\right]^2-\int_{Q(\alpha)}^{\infty}\pi^{(1)}(x)\overline{F}(x)dx. \tag{2.7}$$

2.2. ASYMPTOTIC DISTRIBUTION OF THE SAMPLE LORENZ TRANSFORM

The L-estimate (1.2) of the Lorenz transform can be written in the form (1.4) with $J(u)=1$ if $u\le\alpha$ and $J(u)=0$ if $u>\alpha$. The asymptotic mean of this estimate equals

$$\mu(J,F)=\int_0^{\alpha}Q(u)du=L_{\alpha}[X], \tag{2.8}$$

which shows that the mean of the Lorenz transform L-estimate is asymptotically unbiased. To determine the asymptotic variance using (2.3), we assume that $X\ge\mu$, as will be the case in our application to the generalised Pareto distribution in Section 3. We show the formula

$$\begin{aligned}\sigma^2(J,F)&=\pi^{(2)}(\mu)-\pi^{(2)}\left[Q(\alpha)\right]-\pi^{(1)}(\mu)^2-\pi^{(1)}\left[Q(\alpha)\right]^2\\&-2\cdot\left[Q(\alpha)-\mu-\pi^{(1)}(\mu)\right]\cdot\pi^{(1)}\left[Q(\alpha)\right].\end{aligned} \tag{2.9}$$

Using (2.3) we have

$$\begin{aligned}\sigma^2(J,F)&=2\cdot\int_{\mu}^{Q(\alpha)}\left[\int_x^{Q(\alpha)}\overline{F}(y)dy\right]F(x)dx=2\cdot\int_{\mu}^{Q(\alpha)}\left\{\pi^{(1)}(x)-\pi^{(1)}\left[Q(\alpha)\right]\right\}F(x)dx\\&=2\cdot\left\{\int_{\mu}^{Q(\alpha)}\pi^{(1)}(x)dx-\int_{\mu}^{Q(\alpha)}\pi^{(1)}(x)\overline{F}(x)dx-\pi^{(1)}\left[Q(\alpha)\right]\cdot\left(\int_{\mu}^{Q(\alpha)}dx-\int_{\mu}^{Q(\alpha)}\overline{F}(x)dx\right)\right\}.\end{aligned}$$

The formula (2.9) follows by noting that

$$\int_{\mu}^{Q(\alpha)} \pi^{(1)}(x)dx = \frac{1}{2}\left\{\pi^{(2)}(\mu) - \pi^{(2)}\left[Q(\alpha)\right]\right\}, \quad \int_{\mu}^{Q(\alpha)} \pi^{(1)}(x)\overline{F}(x)dx = \frac{1}{2}\left\{\pi^{(1)}(\mu)^2 - \pi^{(1)}\left[Q(\alpha)\right]^2\right\},$$

$$\int_{\mu}^{Q(\alpha)} \overline{F}(x)dx = \pi^{(1)}(\mu) - \pi^{(1)}\left[Q(\alpha)\right].$$

3. GENERALISED PARETO WITH FINITE VARIANCE

As limiting distribution of scaled excesses over high thresholds, the generalised Pareto distribution (GPD) is an appropriate parametric distribution for use in financial risk management (e.g. Embrechts *et al.* [5], McNeil *et al.* [13]). Its survival function is described by

$$\overline{F}(x) = \left(1 + \xi \frac{x-\mu}{\sigma}\right)^{-\frac{1}{\xi}}, \quad \xi > 0, \quad x \geq \mu > 0, \quad \sigma > 0. \tag{3.1}$$

The *r*-th moment exists only if $\xi < r^{-1}$. Under the assumption $\xi < \frac{1}{2}$ one has

$$E[X] = \mu + \frac{\sigma}{1-\xi} < \infty, \quad Var[X] = \frac{\sigma^2}{(1-\xi)^2(1-2\xi)} < \infty. \tag{3.2}$$

Through calculation one gets the stop-loss transforms of degree one and two

$$\pi^{(1)}(x) = \int_x^{\infty} \overline{F}(y)dy = \frac{\sigma}{1-\xi}\overline{F}(x)^{1-\xi}, \tag{3.3}$$

$$\pi^{(2)}(x) = 2 \cdot \int_x^{\infty} \pi^{(1)}(y)dy = \frac{2\sigma^2}{(1-\xi)(1-2\xi)}\overline{F}(x)^{1-2\xi}. \tag{3.4}$$

Inserted in (2.6) using that $\overline{F}\left[Q(\alpha)\right] = 1-\alpha$ one obtains

$$\sigma^2(J,F) = \frac{\sigma^2(1-\alpha)^{1-2\xi}}{(1-\xi)^2}\left\{\frac{2(1-\xi)}{1-2\xi} - (1-\alpha)\right\}. \tag{3.5}$$

Since the L-estimate (1.3) has an asymptotic normal distribution, an *approximate* ε *-confidence interval* for the conditional value-at-risk from a GPD with finite variance reads

$$\left[CVaR_{\alpha}^{(n)}[X] - \frac{\sigma(J,F)}{\sqrt{n}}Z_{\varepsilon}, CVaR_{\alpha}^{(n)}[X] + \frac{\sigma(J,F)}{\sqrt{n}}Z_{\varepsilon}\right], \tag{3.6}$$

where $Z_{\varepsilon} = \Phi^{-1}(1-\varepsilon/2)$ is the $(1-\varepsilon/2)$-quantile of the standard normal distribution. This interval has the *precision*

$$\Delta = 2 \cdot \frac{\sigma(J,F)}{\sqrt{n}} \cdot Z_{\varepsilon}. \tag{3.7}$$

Table 3.1 lists the critical sample size $n = 4\left[\frac{\sigma(J,F)Z_{\varepsilon}}{\Delta}\right]^2$ required to estimate $CVaR_{\alpha}[X]$ using (3.6) with fixed precision $\Delta = 5\%, \quad \varepsilon = 5\%, \quad \sigma = 1$, but by varying $\xi < \frac{1}{2}$ and α.

Table-3.1: Critical sample size by fixed precision for CVaR estimation

α / ξ	95%	99%	99.9%
0.1	1’520	427	68
0.2	4’165	1’610	406
0.3	13’057	6’938	2’769
0.4	55’799	40’714	25’727
0.45	164’893	140’893	112’007
0.47	322’052	293’071	255’384
0.49	1'133’964	1'098’907	1'049’633

The asymptotic variance of the L-estimate (1.2) is obtained from (2.9). We need the quantile function of the GPD, that is

$$Q(u) = \mu + \frac{\sigma}{\xi}\left[(1-u)^{-\xi} - 1\right], \quad u \in (0,1), \tag{3.8}$$

which is obtained from (3.1). Inserting this and the stop-loss transform formulas (3.3) and (3.4) into (2.9) one obtains

$$\sigma^2(J,F) = \sigma^2 \cdot \left\{ \frac{1}{(1-\xi)^2(1-2\xi)} + \frac{2(1-\alpha)^{1-\xi}}{\xi(1-\xi)^2} - \frac{2(1-\alpha)^{1-2\xi}}{\xi(1-2\xi)} - \frac{(1-\alpha)^{2(1-\xi)}}{(1-\xi)^2} \right\}. \tag{3.9}$$

Again, since the L-estimate (1.2) has an asymptotic normal distribution, an *approximate* ε*-confidence interval* for the Lorenz transform from a GPD with finite variance reads

$$\left[L_\alpha^{(n)}[X] - \frac{\sigma(J,F)}{\sqrt{n}} Z_\varepsilon, L_\alpha^{(n)}[X] + \frac{\sigma(J,F)}{\sqrt{n}} Z_\varepsilon \right]. \tag{3.10}$$

Table 3.2 lists the critical sample size required to estimate $L_\alpha[X]$ using (3.6) with fixed precision $\Delta = 5\%, \quad \varepsilon = 5\%, \quad \sigma = 1$, but by varying $\xi < \frac{1}{2}$ and α.

Table 3.2: Critical sample size by fixed precision for Lorenz transform estimation

α / ξ	95%	99%	99.9%
0.1	5’702	8’029	9’176
0.2	7’692	11’949	14’765
0.3	10’534	18’433	25’559
0.4	14’642	29’512	48’117
0.45	17’359	37’871	68’287
0.47	18’601	41’953	79’057
0.49	19’944	46’544	91’862

Some general comments concerning Tables 3.1 and 3.2 and their comparison are in order. As $\xi < \frac{1}{2}$ comes closer to the value $\frac{1}{2}$, an increasing critical sample size is required. For small $\xi < \frac{1}{4}$, one should use the confidence interval (3.6) for conditional value-at-risk. By known mean and $\xi \geq \frac{1}{4}$, it is preferable to use the confidence interval (3.10) and transform it using the formula (1.1).

4. GENERALISED PARETO WITH INFINITE VARIANCE

In case the variance is infinite, the results of Sections 2 and 3 do not apply. This occurs for the GPD with parameter $\xi \in \left[\frac{1}{2}, 1\right)$, for which the mean is however finite. A theoretical justification of this insurance risk model is found in Aebi *et al.* [1]. A straightforward calculation using (3.8) yields the Lorenz transform

$$L_\alpha[X] = \int_0^\alpha Q(u)du = \left(\mu - \frac{\sigma}{\xi}\right)\alpha + \frac{\sigma}{\xi}\left[\frac{1-(1-\alpha)^{1-\xi}}{1-\xi}\right]. \tag{4.1}$$

To measure the goodness of approximation of the Lorenz transform by the L-estimate (1.2) for the GPD with infinite variance, let us calculate the *expected relative error of approximation* defined by

$$E_\alpha^{(n)} = \frac{E\left[L_\alpha^{(n)}[X]\right] - L_\alpha[X]}{L_\alpha[X]}. \tag{4.2}$$

Using the explicit formulas for the distribution functions of order statistics and making a transformation of variables one obtains

$$\begin{aligned} E\left[L_\alpha^{(n)}[X]\right] &= n^{-1} \cdot \sum_{i=1}^{[\alpha n]+1} I_i, \quad I_i = \int_0^1 Q(u) N_i(u)du, \\ N_i(u) &= n\binom{n-1}{i-1} u^{i-1}(1-u)^{n-i}, \quad i = 1,\dots,n. \end{aligned} \tag{4.3}$$

Note that the N_i 's is the Bernstein basis of the polynomials of degree not exceeding $n-1$. Since $n-i+1-\xi > 0$ by assumption, one obtains

$$\begin{aligned} N_i(u) &= n\binom{n-1}{i-1} \cdot \int_0^1 \left\{ \mu + \frac{\sigma}{\xi}\left[(1-u)^{-\xi} - 1\right] \right\} u^{i-1}(1-u)^{n-i} du \\ &= n\binom{n-1}{i-1} \left\{ \left(\mu - \frac{\sigma}{\xi}\right) B(i, n-i+1) + \frac{\sigma}{\xi} B(i, n-i+1-\xi) \right\}, \quad i = 1,\dots,n, \end{aligned} \tag{4.4}$$

where $B(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1}dx = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$ is a beta coefficient. Observe that

$$n\binom{n-1}{i-1} B(i, n-i+1) = n\binom{n-1}{i-1} \frac{(i-1)!(n-i)!}{n!} = 1, \tag{4.5}$$

and, using the recurrence relation $\Gamma(x+1) = x\Gamma(x)$, one gets

$$\begin{aligned} n\binom{n-1}{i-1} B(i, n-i+1-\xi) &= \frac{\Gamma(n+1)}{\Gamma(n+1-\xi)} \cdot \frac{\Gamma(n-i+1-\xi)}{\Gamma(n-i)} \\ &= \frac{\prod_{k=0}^{n-1}(k+1)}{\Gamma(1-\xi)\prod_{k=0}^{n-1}(k+1-\xi)} \cdot \frac{\Gamma(1-\xi) \cdot \prod_{k=0}^{n-i-1}(k+1-\xi)}{\prod_{k=0}^{n-i-1}(k+1)} = \prod_{k=n-i}^{n-1}\left(\frac{k+1}{k+1-\xi}\right). \end{aligned} \tag{4.6}$$

Inserting above, the mean of the L-estimate (1.2) equals

$$E\left[L_\alpha^{(n)}[X]\right] = \left(\mu - \frac{\sigma}{\xi}\right) \cdot \frac{[\alpha n]+1}{n} + \frac{\sigma}{\xi} \cdot \frac{1}{n} \cdot \sum_{i=1}^{[\alpha n]+1} C_i^n(\xi), \quad C_i^n(\xi) = \prod_{k=n-i}^{n-1}\left(\frac{k+1}{k+1-\xi}\right). \tag{4.7}$$

For the typical parameter values $(\mu, \sigma, \xi) = (10, 7, \frac{1}{2})$ (e.g. McNeil [12], p. 129), the Table 4.1 displays the expected relative error of approximation by varying α and the sample size.

Table-4.1: Expected relative error of approximation for the sample Lorenz transform in %

α / n	95%	99%	99.9%
50	5.262	12.994	3.813
100	4.382	12.994	3.813
200	2.116	4.822	3.813
300	1.395	2.998	3.813
400	1.040	2.180	3.813
500	0.830	1.713	3.813
1000	0.412	0.829	3.813
2000	0.205	0.408	1.421

We note that the sample Lorenz transform overestimates in average the theoretical Lorenz transform. This means that the corresponding sample CVaR in the text after formula (1.2) underestimates in general the theoretical value. To obtain relative errors of an order less than a given percentage, an increasing sample size is required for increasing values of α .

REFERENCES

1. Aebi, M., Embrechts, P. and Th. Mikosch. A large claim index. *Bulletin of the Swiss Association of Actuaries* (1992), 143-156.
2. Balkema, A. and L. de Haan (1974). Residual life time at great age. *Annals of Probability* 2 (1974), 792-804.
3. Chernoff, H., Gastwirth, J.L. and M.V. Johns. Asymptotic distribution of linear combinations of functions of order statistics with applications to estimation. *Annals of Mathematical Statistics* 38 (1967), 52-72.
4. David, H.A. (1981). Order Statistics (2nd ed.). *John Wiley, New York*, 1981.
5. Embrechts, P., Klüppelberg, C. and Th. Mikosch. Modelling Extremal Events for Insurance and Finance. Applications of Mathematics – Stochastic Modelling and Applied Probability, vol. 33. *Springer, New York*, 1997.
6. Gini, C. (1912). Variabilità é mutabilita, contributo allo studio delle distribuzioni e delle relazioni statistiche. *Studi Economico-Giuridici della R. Università di Cagliari* 3 (1912), part 2, i-iii, 3-159.
7. Govindarajulu, Z., LeCam, L. and M. Raghavachari. Generalisations of theorems of Chernoff and Savage on the asymptotic normality of test statistics. *Fifth Berkeley Symp. Math. Statist. Prob.* 1 (1965), 609-638.
8. Hürlimann, W. Higher-degree stop-loss transforms and stochastic orders (II) applications. *Blätter der Deutschen Gesellschaft für Vers.math.* 24(3) (2000), 465-476.
9. Hürlimann, W. Conditional value-at-risk bounds for compound Poisson risks and a normal approximation. *Journal of Applied Mathematics* 3(3) (2003), 141-154.
10. Jung, J. On linear estimates defined by a continuous weight function. *Arkiv fur Matematik* 3 (1955), 199-209.
11. Jureckova, J. and P.K. Sen. Robust Statistical Procedures: Asymptotics and Interrelations. *John Wiley, New York*, 1996.
12. McNeil, A.J. Estimating the tails of loss severity distributions using extreme value theory. *ASTIN Bulletin* 27 (1997), 117-137.
13. McNeil, A.J., Frey, R. and P. Embrechts. Quantitative Risk Management: Concepts, Techniques and Tools. *Princeton Series in Finance, Princeton, New Jersey*, 2005.
14. Moore, D.S. An elementary proof of asymptotic normality of linear functions of order statistics. *Annals of Mathematical Statistics* 39 (1968), 263-265.
15. Nadarajah, S., Zhang, B. and S. Chan (2014). Estimation methods for expected shortfall. *Quantitative Finance* 14(2) (2014), 271-291.
16. Pickands, J. Statistical inference using extreme order statistics. *The Annals of Statistics* 3 (1975), 119-131.
17. Rychlik, T. Bounds for expectations of L-estimates. In: Balakrishnan, N. and C.R. Rao (Eds.). Order Statistics : Theory and Methods. Handbook of Statistics, vol. 16. *Elsevier Science B.V., Amsterdam*, 1998.
18. Sen, P.K. Sequential Nonparametric: Invariance Principles and Statistical Inference. *John Wiley, New York*, 1981.
19. Serfling, R.J. Approximation Theorems of Mathematical Statistics. *John Wiley, New York*, 1981.
20. Shorack, G.R. Asymptotic normality of linear combinations of functions of order statistics. *Annals of Mathematical Statistics* 40 (1969), 2041-2050.
21. Stigler, S.M. Linear functions of order statistics. *Annals of Mathematical Statistics* 40 (1969), 770-788.
22. Stigler, S.M. Simon Newcomb, Percy Daniell, and the history of robust estimation 1885-1920. *Journal of the American Statistical Association* 68 (1973), 872-879.
23. Stigler, S.M. Linear functions of order statistics with smooth weight functions. *The Annals of Statistics* 2 (1974), 676-693.

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