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# MEAN LABELING FOR UNION AS WELL AS PATH UNION OF MEAN GRAPHS <br> V J Kaneria <br> Department of Mathematics, Saurashtra University, Rajkot-360005. 

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#### Abstract

In this paper we have obtained mean labeling for union of finite number of grid graphs and path union of connected mean graphs by path of arbitrary length.


Key words: Grid graph, cycle, path, complete bipartite graph, path union of graphs by path of arbitrary length and mean labeling.

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## 1. INTRODUCTION

Somasundaram and Ponraj [3] have introduced the notion of mean labeling of graphs in 2003. They proved that $P_{n}, C_{n}, K_{2, n}, P_{m} \times P_{n}$ are mean graphs and $K_{n}, K_{1, n}$ are mean graphs iff $n \leq 3$. Also they proved that one point union of a cycle and $K_{1, n}$ for small values of $n$ and the arbitrary super subdivision of path, which is obtained by replacing each edge of a path by $K_{2, m}$ are mean graphs.

A mean graph $G$ will always have vertices with labels $q, q-1$ and 0 , where $q \geq 2$. Also two vertices with labels $q$ and $q-1$ are adjacent in the mean graph $G$. We begin with a simple, undirected and finite graph $G=(V, E)$ with $p=|V|$ vertices and $q=|E|$ edges. For all terminology, notations and basic definitions we follow Harary [2]. First of all we give brief summary of definitions which are useful in this paper.

Definition-1.1: If the vertices of the graph are assigned values subject to certain conditions then it is known as graph labeling.

Definition-1.2: A function $f$ is called mean labeling of a graph $G=(V, E)$ if $f: V \rightarrow\{0,1, \ldots, q\}$ is injective and the induced function $f^{*}: E \rightarrow\{1,2, \ldots, q\}$ defined as $f^{*}(e)=\left\lceil\frac{f(u)+f(v)}{2}\right\rceil$ is bijective for every edge $e=(u, v) \in E$. A graph $G$ is called mean graph if it admits a mean labeling.

Definition-1.3: Let $G$ be a graph and $G_{1}, G_{2}, \ldots, G_{t}, t \geq 2$ be $t$ copies of graph $G$. Then the graph obtained by adding an edge from $G_{i}$ to $G_{i+1}(i=1,2, \ldots, t-1)$ is called path union of graph $G$, We shall denote it by $P\left(G_{1}, G_{2}, \ldots, G_{t}\right)$.

If we replace each graphs $G_{1}, G_{2}, \ldots, G_{t}$ by a graph $G$, i.e. $G_{1}=G=G_{2}=\cdots=G_{t}$, such path union of $t$ copies of $G$, we shall denote it by $P(t \cdot G)$.

Definition-1.4: Let $G_{1}, G_{2}, \ldots, G_{t}$ be connected graphs. Then graph $G=\left\langle G_{1} ; G_{2} ; \ldots ; G_{t}\right\rangle$ obtained by joining two graphs $G_{i}$ and $G_{i+1}$ by a vertex ( $1 \leq i \leq t-1$ ) is called join sum of graphs.

Definition-1.5: Let $G_{1}, G_{2}, \ldots, G_{t}$ be connected graphs. Consider $P_{n_{1}}, P_{n_{2}}, \ldots, P_{n_{t-1}}$ paths on vertices $n_{1}, n_{2}, \ldots, n_{t-1}$ respectively. Then path union of graphs $G_{1}, G_{2}, \ldots, G_{t}$ by path of arbitrary length is denoted by $\left\langle G_{1}, P_{n_{1}}, G_{2}, P_{n_{2}}, \ldots, G_{t-1}, P_{n_{t-1}}, G_{t}\right\rangle$ and such graph obtained by joining two graphs $G_{i}, G_{i+1}$ by a path $P_{n_{i}}(1 \leq i \leq t-1)$.

If we replace each paths $P_{n_{1}}, P_{n_{2}}, \ldots, P_{n_{t-1}}$ by a path $P_{n}$, i.e. $P_{n_{1}}=P_{n}=P_{n_{2}}=\cdots=P_{n_{t-1}}$, such path union of graphs $G_{1}, G_{2}, \ldots, G_{t}$ we shall denote it by $P_{n}\left(G_{1}, G_{2}, \ldots, G_{t}\right)$.

Particularly $P_{1}\left(G_{1}, G_{2}, \ldots, G_{t}\right)=P\left(G_{1}, G_{2}, \ldots, G_{t}\right)$ and $P_{2}\left(G_{1}, G_{2}, \ldots, G_{t}\right)=\left\langle G_{1} ; G_{2} ; \ldots ; G_{t}\right\rangle$.
For detail survey of various graph labeling and bibliographic references we refer Gallian [1]. In this paper we have proved that union of finite number of grid graphs and path union of mean graphs by path of arbitrary length are mean graphs.

## 2. MAIN RESULTS

Theorem-2.1: $\bigcup_{i=1}^{t}\left(P_{\boldsymbol{n}_{\boldsymbol{i}}} \times P_{\boldsymbol{m}_{\boldsymbol{i}}}\right)$, union of finite number of grid graphs is a mean graph.
Proof: Let $G=\bigcup_{i=1}^{t}\left(P_{\boldsymbol{n}_{\boldsymbol{i}}} \times P_{\boldsymbol{m}_{\boldsymbol{i}}}\right)$. Let $u_{i, j, k}\left(1 \leq j \leq n_{i}, 1 \leq k \leq m_{i}\right)$ be vertices of $i^{\text {th }}$ grid $P_{\boldsymbol{n}_{\boldsymbol{i}}} \times P_{\boldsymbol{m}_{\boldsymbol{i}}}$ in $G$, $\forall i=1,2, \ldots, t$.

We know that each grid $\left(P_{\boldsymbol{n}_{\boldsymbol{i}}} \times P_{\boldsymbol{m}_{\boldsymbol{i}}}\right.$ ) (where $n_{i} \leq m_{i}$ ) is a mean graph on $p_{i}=n_{i} m_{i}$ vertices and $q_{i}=2 m_{i} n_{i}-$ $\left(m_{i}+n_{i}\right)$ edges with following vertex labeling functions $f_{i}: V\left(P_{\boldsymbol{n}_{\boldsymbol{i}}} \times P_{m_{i}}\right) \rightarrow\left\{0,1, \ldots, q_{i}\right\}$ defined by,

$$
\begin{array}{rlrl}
f_{i}\left(u_{i, 1, k}\right) & =k^{2}-1, & & \text { when } 1 \leq k \leq n_{i}-1 \\
& =f_{i}\left(u_{1,1, n_{i}-1}\right)+\left(2 n_{i}-1\right)\left(k-n_{i}+1\right), & & \text { when } n_{i} \leq k \leq m_{i} ; \\
& & \forall j=1,2, \ldots, n_{i} ; \\
f_{i}\left(u_{i, j, m_{i}}\right) & =q_{i}-\left(n_{i}-j\right)^{2}, & \forall k=m_{i}-1, m_{i}-2, \ldots, 1, \forall j=2,3, \ldots, n_{i} ; \forall i=1,2, \ldots, t .
\end{array}
$$

In graph $G$, we see that the number of vertices $|V(G)|=P=\sum_{i=1}^{t} m_{i} n_{i}$ and the edges $|E(G)|=Q=\sum_{i=1}^{t} q_{i}$. we define the labeling function $g: V(G) \rightarrow\{0,1, \ldots, Q\}$ as follows:
$g\left(u_{1,2,1}\right)=f_{1}\left(u_{1,2,1}\right)+\left(Q-q_{1}+1\right)$;
$g\left(u_{1,2, k}\right)=f_{1}\left(u_{1,2, k}\right)+\left(Q-q_{1}\right), \quad \forall k=2,3, \ldots, m_{1} ;$
$g\left(u_{1, j, k}\right)=f_{1}\left(u_{1, j, k}\right)+\left(Q-q_{1}\right), \quad \forall j=1,3, \ldots, n_{1}, \forall k=1,2, \ldots, m_{1}$;
$g\left(u_{i, n_{i}, m_{i}}\right)=f_{i}\left(u_{i, n_{i}, m_{i}}\right)+\sum_{l=i+1}^{t} q_{l}+1$,
$\forall i=2,3, \ldots, t-1$;
$g\left(u_{i, 2,1}\right)=f_{i}\left(u_{i, 2,1}\right)+1+\sum_{l=i+1}^{t} q_{l}$,
$\forall i=2,3, \ldots, t-1$;
Case-I: $n_{i} \neq 2$

$$
\begin{array}{lll}
g\left(u_{i, n_{i}, k}\right)=f_{i}\left(u_{i, n_{i}, k}\right)+\sum_{l=i+1}^{t} q_{l}, & \forall k=1,2, \ldots, m_{i}-1, & \forall i=2,3, \ldots, t-1 \\
g\left(u_{i, 2, k}\right)=f_{i}\left(u_{i, 2, k}\right)+\sum_{l=i+1}^{t} q_{l}, & \forall k=2,3, \ldots, m_{i}, & \forall i=2,3, \ldots, t-1
\end{array}
$$

Case-II: $n_{i}=2$

$$
\begin{array}{ll}
g\left(u_{i, 2, k}\right)=f_{i}\left(u_{i, 2, k}\right)+\sum_{l=i+1}^{t} q_{l}, & \forall k=2,3, \ldots, m_{i}-1, \quad \forall i=2,3, \ldots, t-1 ; \\
g\left(u_{i, j, k}\right)=f_{i}\left(u_{i, j, k}\right)+\sum_{l=i+1}^{t} q_{l}, & \forall j=1,3,4, \ldots, n_{i}-1, \forall k=1,2, \ldots, m_{i}, \quad \forall i=2,3, \ldots, t-1 ; \\
g\left(u_{t, n_{t}, m_{t}}\right)=f_{t}\left(u_{t, n_{t}, m_{t}}\right)+1 ; & \\
g\left(u_{t, n_{t}, k}\right)=f_{t}\left(u_{t, n_{t}, k}\right), & \forall k=1,2, \ldots, m_{t}-1 ; \\
g\left(u_{t, j, k}\right)=f_{t}\left(u_{t, j, k}\right), & \forall j=1,2, \ldots, m_{t}-1, \quad \forall k=1,2, \ldots, m_{t} .
\end{array}
$$

Above labeling pattern give rise mean labeling to the graph $G$ and so it is a mean graph.
Illustration-2.2: $\left(P_{3} \times P_{3}\right) \cup\left(P_{4} \times P_{2}\right) \cup\left(P_{5} \times P_{3}\right) \cup\left(P_{2} \times P_{5}\right)$ and its mean labeling shown in figure -1 .
Here $Q=\sum_{l=1}^{4} q_{l}=12+10+22+13=57, \sum_{l=3}^{4} q_{l}=35, \sum_{l=4}^{4} q_{l}=13$, and $Q-q_{1}=45$.


Figure-1: union of 4 grid graphs and its mean labeling. .
Theorem-2.3: Let $G_{i}(1 \leq i \leq t)$ be $t$ connected mean graphs on $\left|V\left(G_{i}\right)\right|=p_{i}$ vertices and $\left|E\left(G_{i}\right)\right|=q_{i}$ edges, $\forall i=1,2, \ldots, t$. Then path union of graphs $G_{i}(1 \leq i \leq t)$ by paths of arbitrary lengths is also a mean graph.

Proof: Let $G=\left\langle G_{1}, P_{n_{1}}, G_{2}, P_{n_{2}}, \ldots, G_{t-1}, P_{n_{t-1}}, G_{t}\right\rangle$ be the path union of graphs $G_{i}(1 \leq i \leq t)$ by path of arbitrary length. Let $V\left(G_{i}\right)=\left\{v_{i, j} / j=1,2, \ldots, p_{i}\right\}, E\left(G_{i}\right)$ be the edges set for $G_{i}$, where $\left|E\left(G_{i}\right)\right|=q_{i}, \forall i=1,2, \ldots, t$. Let $V\left(P_{n_{i}}\right)=\left\{w_{i, k} / k=1,2, \ldots, n_{i}\right\}, E\left(P_{n_{i}}\right)=\left\{\left(w_{i, k}, w_{i, k+1}\right) / k=1,2, \ldots, n_{i}-1\right\}, \forall i=1,2, \ldots, t$.

Since a mean graph $G_{i}(1 \leq i \leq t)$ will always have vertices with labels $q_{i}, q_{i-1}$ and 0 , w.l.o.g. we may assume that vertex label of $v_{i, 1}$ is $q_{i}$ and vertex label of $v_{i, p_{i}}$ is $0, \forall i=1,2, \ldots, t$. Now take $v_{i, p_{i}}=w_{i, 0}$ and $w_{i, n_{i}}=v_{i+1,1}$ for each $i=1,2, \ldots, t-1$ to form the graph $G=\left\langle G_{1}, P_{n_{1}}, G_{2}, P_{n_{2}}, \ldots, G_{t-1}, P_{n_{t-1}}, G_{t}\right\rangle$ arbitrary path union of graphs.

Suppose $G_{i}(1 \leq i \leq t)$ be mean graphs with following vertex labeling function $f_{i}: V\left(G_{i}\right) \longrightarrow\left\{0,1, \ldots, q_{i}\right\}$ defined by $f_{i}\left(v_{i, 1}\right)=q_{i}$ and $f_{i}\left(v_{i, p_{i}}\right)=0, \forall i=1,2, \ldots, t$ as we mentioned earlier. Obviously $P_{n_{i}}(1 \leq i \leq t-1)$ is mean graphs with vertex labeling function $g_{i}: V\left(P_{n_{i}}\right) \rightarrow\left\{0,1, \ldots, n_{i}\right\}$ defined by,

$$
g_{i}\left(w_{i, k}\right)=n_{i}-k, \quad \forall k=1,2, \ldots, n_{i}, \quad \forall i=1,2, \ldots t-1
$$

In graph $G$, we see that the number of vertices $P=|V(G)|=\sum_{i=1}^{t} p_{i}+\sum_{i=1}^{t-1}\left(n_{i}-2\right)$ and the edges $Q=|E(G)|=\sum_{i=1}^{t} q_{i}+\sum_{i=1}^{t-1}\left(n_{i}-1\right)$. We define the labeling function $g: V(G) \longrightarrow\{0,1, \ldots, Q\}$ as follows:

$$
\begin{array}{ll}
g\left(v_{1, j}\right)=f_{1}\left(v_{1, j}\right)+\left(Q-q_{1}\right), & \forall j=1,2, \ldots, p_{1} ; \\
g\left(w_{1, k}\right)=g_{1}\left(w_{1, k}\right)+\left(Q-q_{1}-n_{1}\right), & \forall k=2,3, \ldots, n_{1} \\
g\left(v_{i, j}\right)=f_{i}\left(v_{i, j}\right)+\left(Q-\sum_{l=1}^{i} q_{l}-\sum_{l=1}^{i-1} n_{l}\right), & \forall j=1,2, \ldots, p_{1}, \\
g\left(w_{i, k}\right)=g_{i}\left(w_{i, k}\right)+\left(Q-\sum_{l=1}^{i}\left(q_{l}+n_{l}\right)\right), & \forall k=2,3, \ldots, n_{i} \\
g\left(v_{t, i}\right)=f_{t}\left(v_{t, i}\right), & \forall i=1,2, \ldots, p_{t} .
\end{array}
$$

$$
\forall j=1,2, \ldots, p_{1}
$$

$$
\forall k=2,3, \ldots, n_{1}-1 ;
$$

$$
\forall j=1,2, \ldots, p_{1}, \quad \forall i=2,3, \ldots, t-1 ;
$$

$$
g\left(w_{i, k}\right)=g_{i}\left(w_{i, k}\right)+\left(Q-\sum_{l=1}^{i}\left(q_{l}+n_{l}\right)\right), \quad \forall k=2,3, \ldots, n_{i}-1, \forall i=2,3, \ldots, t-1
$$

Above labeling pattern give rise mean labeling to the graph $G$ and so path union of graphs $G_{i}(1 \leq i \leq t)$ by paths of arbitrary lengths is a mean graph.

Illustration-2.4: $\left\langle P_{5} \times P_{3}, P_{4}, C_{6}, P_{7}, K_{2,4}, P_{3}, P_{4} \times P_{2}\right\rangle$ and its mean labeling shown in figure-2.
Here $Q=22+4+6+7+8+3+10=60$.


Figure-2: path union of graphs and its mean labeling.
Corollary-2.5: Let $G_{i}(1 \leq i \leq t)$ be connected mean graphs on $p_{i}$ vertices and $q_{i}$ edges $(1 \leq i \leq t)$. Then $P_{n}\left(G_{1}, G_{2}, \ldots, G_{t}\right)$ is a mean graph.

Corollary-2.6: Let $G_{i}(1 \leq i \leq t)$ be connected mean graphs. Then $P\left(G_{1}, G_{2}, \ldots, G_{t}\right)$ and $\left\langle G_{1}, G_{2}, \ldots, G_{t}\right\rangle$ are mean graphs.

Corollary-2.7: Let $G$ be a connected mean graph. Then path union $P(t \cdot G)$ is also a mean graph.

## 3 CONCLUDING REMARKS

Here we have discussed mean labeling for disconnected graph $\cup_{i=1}^{t}\left(P_{n_{i}} \times P_{m_{i}}\right)$, path union of mean graphs by path of arbitrary length, $P_{n}\left(G_{1}, G_{2}, \ldots, G_{t}\right), P\left(G_{1}, G_{2}, \ldots, G_{t}\right),\left\langle G_{1}, G_{2}, \ldots, G_{t}\right\rangle$ (where $G_{1}, G_{2}, \ldots, G_{t}$ are mean graphs) and $P(t \cdot G)$ (where $G$ is a mean graph). These results contribute some new topics to the families of mean labeled graphs. The labeling pattern is demonstrated by means of illustrations.

We raise to open question to get graceful labeling function for the disconnected graph $\bigcup_{i=\mathbf{1}}^{t}\left(P_{\boldsymbol{n}_{\boldsymbol{i}}} \times P_{\boldsymbol{m}_{\boldsymbol{i}}}\right)$ and path union of graceful graphs by path of arbitrary length.

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