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A FIXED POINT THEOREM OF PRESIC TYPE IN G-METRIC SPACE

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ABSTRACT

In this paper, a unique common fixed point theorem is obtained in settings of generalized metric space by using the concept of Presic fixed point theorem. The result generalizes several well known comparable results in the literature.

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Keywords: *Presic type fixed point theorem, G-metric space.*

1. INTRODUCTION

The notion of G-metric space was introduced by Mustafa and Sims [10], [11] as generalization of metric spaces. The authors discussed topological properties of space and proved analog of Banach contraction principle in context of G-metric spaces [2], [6], [8]. Abbas and Rhoades [1] initiated the study of a common fixed point theorem in generalized metric spaces for non commuting mappings without continuity in G-metric spaces. Mustafa *et al.* [7], [9] and others [3], [4], proved fixed point theorem for mappings satisfying different contractive conditions in G-metric space.

On other hand, considering the convergence of certain sequences S. B. Presic [13] generalized Banach contraction principle as follows:

Theorem 1.1: Let (X, d) be a complete metric space, k a positive integer and $T: X^k \to X$ a mapping satisfying the following contractive type condition

$$d(T(x_1,\ x_2, \dots, x_{k-1}, x_k), T(x_2, x_3 \dots, x_k, x_{k+1}) \le q_1 d(x_1, x_2) + q_2 d(x_2, x_3) + \cdots + q_k d(x_k, x_{k+1}) \qquad (1.1.1)$$

for every $x_1, x_2, \dots, x_k\ x_{k+1}$ in X, where $q_1, q_2, \dots, q_k$ are non negative constants such that $q_1 + q_2 + \cdots + q_k < 1$.

Then there exists a unique point x in X such that $T(x, x, \dots, x) = x$.

Moreover, if $x_1, x_2, x_3, \dots, x_k$ are arbitrary points in X and for $n \in N$,

$$x_{n+k} = T(x_n,\ x_{n+1}, \dots ,\ x_{n+k-1}),$$

then the sequence $\{x_n\}$ is convergent and

$$\lim x_n = T(\lim x_n, \lim x_n, \dots, \lim x_n).$$

Ciric and Presic [5] generalized Theorem 1.1 as follows:

Theorem 1.2: Let (X, d) be a complete metric space, k a positive integer and $T: X^k \to X$ a mapping satisfying the following contractive type condition

$$d(T(x_1,\ x_2, \dots, x_k), T(x_2, x_3 \dots,\ x_{k+1}) \le \lambda \max\{d(x_i, x_{i+1})/1 \le i \le k\} \qquad (1.2.1)$$

for every $x_1, x_2, x_3, \dots, x_k, x_{k+1}$ in X, where $\lambda \in (0,1)$ is constant.

Then there exists a point x in X such that $T(x, x, \dots, x) = x$.

Moreover, if $x_1, x_2, x_3, \dots, x_k$ are arbitrary points in X and for $n \in N$,
$x_{n+k} = T(x_n,\ x_{n+1}, \dots ,\ x_{n+k-1})$ then the sequence $\{x_n\}$ is convergent and $\lim x_n = T(\lim x_n, \lim x_n, \dots, \lim x_n)$.

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If in addition, we suppose that on a diagonal $\Delta \subset X^k$,

$$d(T(u,u,\dots,u),T(v,v,\dots,v)) < d(u,v) \text{ holds for all u, v} \in X \text{, with u} \neq \text{v}, \tag{1.2.2}$$

then x is the unique point in X with $T(x,x,x,\dots,x) = x$.

Definition 1.3: [9] Let X be a nonempty set, and let $G: X \times X \times X \to \mathbb{R}^+$, be a function satisfying:
$(G1)$ $G(x,y,z) = 0$ if $x = y = z$,
$(G2)$ $0 < G(x,x,y)$; for all $x,y \in X$ with $x \neq y$
$(G3)$ $G(x,x,y) \leq G(x,y,z)$ for all $x,y,z \in X$ with $z \neq y$
$(G4)$ $G(x,y,z) = G(x,z,y) = G(y,z,x) = \cdots$ (symmetry in all three variables), and
$(G5)$ $G(x,y,z) \leq G(x,a,a) + G(a,y,z)$ for all $x,y,z,a \in X$ (rectangle inequality).

Then the function G is called a generalized metric or more specifically a G-metric on X, and the pair (X, G) is called a G-metric space.

Definition 1.4: [9] Let (X, G) be a G-metric space and let $\{x_n\}$ be a sequence of points of X. We say that $\{x_n\}$ is G-convergent to x if $lim_{n,m\to\infty} G(x,x_n,x_m) = 0$; that is, for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $G(x,x_n,x_m) < \epsilon$, for all $\mathrm{n,m} \geq \mathrm{N}$. we refer to x as the limit of the sequence $\{x_n\}$ and write $x_n \xrightarrow{G} x$.

Proposition 1: [9] Let (X, G) be a G-metric space. The following statements are equivalent.

(1) $\{x_n\}$ is G-convergent to x.
(2) $G(x_n,x_n,x) \to 0$, as $n \to \infty$.
(3) $G(x_n,x,x) \to 0$, as $n \to \infty$.

Definition 1.5: [9] Let (X, G) be a G-metric space. A sequence $\{x_n\}$ is called G-Cauchy if given $\epsilon > 0$, there is $N \in N$ such that $G(x_n,x_m,x_l) < \epsilon$, for all $n,m,l \geq N$; That is if $G(x_n,x_m,x_l) \to 0$, as $\mathrm{n,m,l} \to \infty$.

Proposition 2: [9] In a G-metric space (X, G), the following two statements are equivalent.

(1) The sequence $\{x_n\}$ is G- Cauchy.
(2) For every $\epsilon > 0$, there exists $N \in \boldsymbol{N}$ such that $G(x_n,x_m,x_m) < \epsilon$ for all $\mathrm{n,m} \geq \mathrm{N}$.

Definition 1.6: [9] A G-metric space (X, G) is said to be G-complete (or a complete G-metric space) if every G-Cauchy sequence in (X, G) is G-convergent in (X,G).

Definition 1.7: [9] A G-metric space (X, G) is called symmetric if $G(x,y,y) = G(y,x,x)$ for all $x,y \in X$.

Proposition 3: [9] Let (X, G) be a G-metric space. Then the function $G(x,y,z)$ is jointly continuous in all three of its variables.

Proposition 4: [9] Every G-metric space (X, G) defines a metric space (X,d_G) by

$$d_G(x,y) = G(x,y,y) + G(y,x,x)$$

for all $x,y \in X$.

Note that if (X, G) is a symmetric G-metric space, then

$$d_G(x,y) = 2G(x,y,y) \quad \forall x,y \in X.$$

In this paper, we extend and generalize the above Theorem 1.2 into generalized metric space.

2. MAIN THEOREM

Theorem 2.1: Let (X, G) be a complete G metric space, k a positive integer and $T: X^k \to X$ a mapping satisfying the following contractive type condition

$$G(T(x_1,\ x_2,\dots,x_{k-1},x_k),T(x_2,x_3\dots,x_k,x_{k+1}),T(x_3,x_4\dots,x_{k+1},x_{k+2}) \leq \lambda \max\{G(x_i,\ x_{i+1},\ x_{i+2}): 1 \leq i \leq k\}, \tag{2.1.1}$$

where $\lambda \in (0,1)$ is constant and $x_1,x_2,\dots,x_{k+2}$ are arbitrary elements in X. Then there exists a point x in X such that $T(x,x,\dots,x) = x$. Moreover, if $x_1,x_2,x_3,\dots,x_k$ are arbitrary points in X and for $n \in \boldsymbol{N}$,

$$x_{n+k} = T(x_n,\ x_{n+1},\dots,\ x_{n+k-1}),$$

then the sequence $\{x_n\}_{n=1}^{\infty}$ is convergent and

$$\lim x_n = T(\lim x_n, \lim x_n, \dots, \lim x_n).$$

If, in addition we suppose that on diagonal $\Delta \subset X^k$,

$$G(T(u,\dots,u),T(v,\dots,v),T(w,\dots,w)) < G(u,v,w) \tag{2.1.2}$$

holds for all u ,v, w ∈ X, with $u \neq v \neq w$, then x is unique point in X with $T(x,\dots,x) = x$.

Proof: Let $x_1, x_2, \dots, x_k$ be k arbitrary points in X. Using these points define a sequence $\{x_n\}$ as follows:

$$x_{n+k} = T(x_n, x_{n+1}, \dots, x_{n+k-1}) \quad (n = 1, 2, \dots).$$

For simplicity set $\alpha_n = G(x_n, x_{n+1}, x_{n+2})$. We shall prove by induction that for each $n \in \boldsymbol{N}$:

$$\alpha_n \le K\theta^n \quad (\text{where } \theta = \lambda^{1/k}, K = \max\{\tfrac{\alpha_1}{\theta}, \tfrac{\alpha_1}{\theta^2}, \dots, \tfrac{\alpha_k}{\theta^k}\}). \tag{2.1.3}$$

According to definition of K we see that (2.1.3) is true for n = 1, 2, …, k. Now let follow k inequalities:

$$\alpha_n \le K\theta^n,\ \alpha_{n+1} \le K\theta^{n+1}, \dots, \alpha_{n+k-1} \le K\theta^{n+k-1}$$

be the induction hypotheses. Then we have:

$$\begin{aligned}
\alpha_{n+k} &= G((x_{n+k}, x_{n+k+1}, x_{n+k+2}) \\
&= G(T(x_n, x_{n+1}, \dots, x_{n+k-1}), T(x_{n+1}, x_{n+2}, \dots, x_{n+k}), T(x_{n+2}, x_{n+3}, \dots, x_{n+k+1}) \\
&\le \lambda \max\{\alpha_n, \alpha_{n+1}, \dots, \alpha_{n+k-1}\} \qquad \text{(by (2.1.1) and definition of } \alpha_i) \\
&\le \lambda \max\{K\theta^n, K\theta^{n+1}, \dots, K\theta^{n+k-1}\} \qquad \text{(by the induction hypotheses)} \\
&= \lambda K\theta^n \text{ as } (0 \le \theta < 1) \\
&= K\theta^{n+k} \text{ as } (\theta = \lambda^{1/k})
\end{aligned}$$

and inductive proof of (2.1.3) is complete. Next using (2.1.3) for any n, p, l ∈ N we have the following argument:

$$\begin{aligned}
G(x_n, x_p, x_l) &\le G(x_n, x_{n+1}, x_{n+1}) + G(x_{n+1}, x_{n+2}, x_{n+2}) + \dots + G(x_{l-1}, x_l, x_l) \\
&\le G(x_n, x_{n+1}, x_{n+2}) + G(x_{n+1}, x_{n+2}, x_{n+3}) + \dots + G(x_{l-2}, x_{l-1}, x_l) \\
&\le K\theta^n + K\theta^{n+1} + \dots + K\theta^{l-2} \\
&\le K\theta^n(1 + \theta + \theta^2 + \dots) \\
&\le K\theta^n/(1-\theta) \to 0 \text{ as } n \to \infty
\end{aligned}$$

by which we conclude that $\{x_n\}$ is a Cauchy sequence. Since X is a complete space, there exists x in X such that

$$x = \lim_{n\to\infty} x_n$$

Then for any integer n we have

$$\begin{aligned}
G(x, T(x, \dots, x), T(x, \dots, x)) &\le G(x, x_{n+k}, x_{n+k}) + G(x_{n+k}, T(x, \dots, x), T(x, \dots, x)) \\
&= G(x, x_{n+k}, x_{n+k}) + G(T(x_n, \dots, x_{n+k-1}), T(x, \dots, x), T(x, \dots, x)) \\
&\le G(x, x_{n+k}, x_{n+k}) + G(T(x, \dots, x), T(x, \dots, x, x_n), T(x, \dots, x, x_n)) \\
&\quad + G\big(T(x, \dots, x, x_n), T(x, \dots, x, x_n, x_{n+1}), T(x, \dots, x, x_n, x_{n+1})\big) \\
&\quad + \dots + G\big(T(x, \dots, x_{n+k-2}), T(x_n, \dots, x_{n+k-1}), T(x_n, \dots, x_{n+k-1})\big) \\
&\le G(x, x_{n+k}, x_{n+k}) + \lambda G(x, x_n, x_n) + \lambda \max\{G(x, x_n, x_n), G(x_n, x_{n+1}, x_{n+1})\} \\
&\quad + \dots + \lambda \max\{G(x, x_n, x_n), G(x_n, x_{n+1}, x_{n+1}), \dots, G(x_{n+k-2}, x_{n+k-1}, x_{n+k-1})\}
\end{aligned}$$

Taking limit when n tends to infinity we obtain $G(x, T(x, \dots, x), T(x, \dots, x)) \le 0$, which implies $T(x, \dots, x) = x$. Thus we proved that

$$\lim x_n = T(\lim x_n, \lim x_n, \dots, \lim x_n)$$

Now suppose (2.1.2) holds. To prove uniqueness of the fixed point, let us assume that for some $y, z \in X$, $y \neq x \neq z$, we have $T(y, y, \dots, y) = y$ and $T(z, z, \dots, z) = z$. Then by (2.1.2)

$$G(x, y, z) = G\big(T(x, \dots, x), T(y, \dots, y), T(z, \dots, z)\big) < G(x, y, z)$$

which is a contradiction. So x is the unique point in X such that $T(x, \dots, x) = x$.

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