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CONVERGENCE OF AN ITERATIVE PROCESS IN CAT (0) SPACES

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ABSTRACT

In this paper we give you an idea about strong and Δ-convergence for CR-iterative process of nonexpansive mapping whose domain is a nonempty closed convex subset of a CAT(0) space.

Keywords: *CR-Iterative process, Nonexpansive mappings, Condition (A), Δ-Convergence, Strong convergence.*

1. INTRODUCTION

In 2012, R. Chugh, V. Kumar and S. Kumar [3] establish an iterative process named as CR-iterative process. This iterative process is defined by a sequence $\{x_n\}$:
$x_{n+1} = \alpha_n Ty_n + (1-\alpha_n)\, y_n$,
$y_n = \beta_n Tz_n + (1-\beta_n)Tx_n$,
$z_n = \gamma_n Tx_n + (1-\gamma_n)\, x_n$, where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ are sequences of positive numbers in [0, 1] with $\sum_{n=0}^{\infty} \alpha_n = \infty$.

Now we transform this notion of iteration in the CAT(0) space setting which is as follows:
$x_{n+1} = \alpha_n Ty_n \oplus (1-\alpha_n)\, y_n$,
$y_n = \beta_n Tz_n \oplus (1-\beta_n)Tx_n$, (RS)
$z_n = \gamma_n Tx_n \oplus (1-\gamma_n)\, x_n$, where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ are sequences of positive numbers in [0, 1].

Let us recall some basic concepts for CAT(0) spaces. Let (X, d) be a metric space. A geodesic path joining $x \in X$ to $y \in X$ (or, more briefly, a geodesic from x to y) is a map c from a closed interval $[0, l] \subset R$ to X such that c(0) = x, $c(l) = y$ and $d(c(t),c(t')) = |t- t'|$ for all t, $t' \in [0, l]$. In particular, c is an isometry and $d(x, y) = l$. The image α of c is called a geodesic (or metric) segment joining x and y. When it is unique this geodesic segment is denoted by [x, y]. The space (X, d) is said to be a geodesic space if every two points of X are joined by a geodesic and X is said to be uniquely geodesic if there is exactly one geodesic joining x and y for each x, $y \in X$. A subset $Y \subseteq X$ is said to be convex if Y includes every geodesic segment joining any two of its points. A geodesic triangle $\Delta(x_1, x_2, x_3)$ in a geodesic metric space (X, d) consists of three points x_1, x_2, x_3 in X (the vertices of Δ) and a geodesic segment between each pair of vertices (the edges of Δ). A comparison triangle for the geodesic triangle $\Delta(x_1, x_2, x_3)$ in (X, d) is a triangle $\bar{\Delta}(x_1, x_2, x_3) = \Delta(\overline{x_1}, \overline{x_2}, \overline{x_3})$ in the Euclidean plane $\mathbb{E}^2$ such that

$d_{\mathbb{E}^2}(\overline{x_i}, \overline{x_j}) = d(x_i, x_j)$ for $i, j \in \{1, 2, 3\}$.

A geodesic space is said to be a CAT(0) space if all geodesic triangles satisfy the following comparison axiom: Let Δ be a geodesic triangle in X and let $\bar{\Delta}$ be a comparison triangle for Δ. Then Δ is said to satisfy the CAT(0) inequality if for all x, $y \in \Delta$ and all comparison points $\bar{x}, \bar{y} \in \bar{\Delta}$, $d(x, y) \le d_{\mathbb{E}^2}(\bar{x}, \bar{y})$.

If x, y_1, y_2 are points in a CAT(0) space and if y_0 is the midpoint of the segment $[y_1, y_2]$, then the CAT(0) inequality implies
$d(x, y_0)^2 \le \frac{1}{2} d(x, y_1)^2 + \frac{1}{2} d(x, y_2)^2 - \frac{1}{4} d(y_1, y_2)^2$ (CN)

This is the (CN) inequality of Bruhat and Tits [2]. In fact, a geodesic space is a CAT(0) space if and only if it satisfy (CN) inequality.

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Let X be a CAT(0) space and let C be a nonempty subset of X and $T: C \to X$ be a mapping. Denote F(T) by the set of fixed points of T, i.e., $F(T) = \{x \in C : Tx = x\}$.

Definition 1.1: T is said to be nonexpansive if $d(Tx, Ty) \leq d(x, y)$ for all $x, y \in C$ and that T is quasi-nonexpansive if $F(T) \neq \emptyset$ and $d(Tx, p) \leq d(x, p)$ for all $x \in C$ and $p \in F(T)$.

Definition 1.2 [6]: Let $\{x_n\}$ be a bounded sequence in a CAT(0) space X. For $x \in X$, we set
$r(x,\{x_n\}) = \lim_{n\to\infty} \sup d(x, x_n)$.

The asymptotic radius $r(\{x_n\})$ of $\{x_n\}$ is given by $r(\{x_n\}) = \inf\{ r(x, \{x_n\}): x \in X\}$ and the asymptotic center $A(\{x_n\})$ of $\{x_n\}$ is the set $A(\{x_n\}) = \{x \in X: r(x, \{x_n\}) = r(\{x_n\}) \}$.

Remark 1.3: $A(\{x_n\})$ consists of exactly one point in CAT(0) spaces (see, e.g., [5], Proposition 7).

Definition 1.4 [6]: A sequence $\{x_n\}$ in a CAT(0) space X is said to Δ-converge to $x \in X$ if x is the unique asymptotic center of $\{u_n\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$. In this case we write $\Delta\text{-}\lim x_n = x$ and call x the Δ-limit of $\{x_n\}$.

Senter and Dotson [8] introduced the condition (A) which is as follows:

Definition 1.5: A mapping $T: C\to C$ is said to satisfy the condition (A) if there exists a non-decreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$, $f(r) > 0$ for all $r \in (0, \infty)$ such that $d(x, Tx) \geq f(d(x, F(T)))$ for all $x \in C$.

Lemma 1.6 [6]: Let X be a CAT(0) space. Then
$d((1- t) x \oplus t y, z) \leq (1- t) d(x, z) + t d(y, z)$ for all $x, y, z \in X$ and $t \in [0, 1]$. (1.1)

Lemma 1.7 [6]: Let (X, d) be a CAT(0) space. Then
$d((1-t) x \oplus t y, z)^2 \leq (1-t) d(x, z)^2 + t d(y, z)^2 - t (1-t) d(x, y)^2$ (1.2)
for all $t \in [0, 1]$ and $x, y, z \in X$.

Lemma 1.8 [7]: Every bounded sequence in a complete CAT(0) space always has a Δ-convergent subsequence.

Lemma 1.9 [4]: If C is a closed convex subset of a complete CAT(0) space and if $\{x_n\}$ is a bounded sequence in C then the asymptotic center of $\{x_n\}$ is in C.

Lemma 1.10 [6]: If C is a closed convex subset of a complete CAT(0) space and if $T: C\to X$ is a nonexpansive mapping then the conditions, $\{x_n\}$ Δ-converges to x and $d(x_n , T (x_n)) \to 0$, imply $x \in C$ and $T(x) = x$.

2. MAIN RESULTS

Lemma 2.1: Let C be a nonempty closed convex subset of a CAT(0) space X. Let T be a nonexpansive mapping of C. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ be such that $0 < a \leq \alpha_n, \beta_n, \gamma_n \leq b < 1$ for all $n \in N$ and for some a, b. Let $\{x_n\}$ be defined by the iteration process (RS). Then $\lim_{n\to\infty} d(x_n, p)$ exists for all $p \in F(T)$ and $\lim_{n\to\infty} d(x_n, Tx_n) = 0$.

Proof: Let $p \in F(T)$. Then

$$\begin{aligned}
d(x_{n+1}, p) &= d(\alpha_n Ty_n \oplus (1-\alpha_n) y_n, p)\\
&\leq \alpha_n\, d(Ty_n, p) + (1-\alpha_n)\, d(y_n, p)\\
&\leq \alpha_n\, d(y_n, p) + (1-\alpha_n)\, d(y_n, p) = d(y_n, p)\\
&= d(\beta_n Tz_n \oplus (1-\beta_n)Tx_n, p)\\
&\leq \beta_n\, d(Tz_n, p) + (1-\beta_n)\, d(Tx_n, p)\\
&\leq \beta_n\, d(z_n, p) + (1-\beta_n)\, d(x_n, p)\\
&= \beta_n\, d(\gamma_n Tx_n \oplus (1-\gamma_n)\, x_n, p) + (1-\beta_n)\, d(x_n, p)\\
&\leq \beta_n[\gamma_n\, d(Tx_n, p) + (1-\gamma_n)\, d(x_n, p)] + (1-\beta_n)\, d(x_n, p)\\
&\leq d(x_n, p)
\end{aligned}$$

This implies that $\{d(x_n, p)\}$ is decreasing and bounded. Thus $\lim_{n\to\infty} d(x_n, p)$ exists and let
$\lim_{n\to\infty} d(x_n, p) = c$. (2.1)

Now we prove that $\lim_{n\to\infty} d(y_n, p) = c$ and $\lim_{n\to\infty} d(z_n, p) = c$

Since $d(x_{n+1}, p) \leq d(y_n, p)$. This implies $\lim_{n\to\infty} d(x_{n+1}, p) \leq \lim_{n\to\infty} d(y_n, p)$
or $c \leq \lim_{n\to\infty} d(y_n, p)$ (2.2)

But $d(y_n, p) \leq d(x_n, p)$.

This implies $\lim_{n\to\infty} \sup d(y_n, p) \leq c$ (2.3)

From (2.2) and (2.3), we get $\lim_{n\to\infty} d(y_n, p) = c$. (2.4)

Similarly we can get $\lim_{n\to\infty} d(z_n, p) = c$. (2.5)

$$\begin{aligned} \text{Now } d(z_n, p)^2 &= d(\gamma_n Tx_n \oplus (1-\gamma_n)\, x_n, p)^2 \\ &\leq \gamma_n\, d(Tx_n, p)^2 + (1-\gamma_n)\, d(x_n, p)^2 - \gamma_n(1-\gamma_n)\, d(Tx_n, x_n)^2 \\ &\leq d(x_n, p)^2 - \gamma_n(1-\gamma_n)\, d(Tx_n, x_n)^2 \end{aligned}$$

Thus $\gamma_n(1-\gamma_n) d(Tx_n, x_n)^2 \leq d(x_n, p)^2 - d(z_n, p)^2$ so that

or $d(Tx_n, x_n)^2 \leq \frac{1}{\gamma_n(1-\gamma_n)} [d(x_n, p)^2 - d(z_n, p)^2]$

or $d(Tx_n, x_n)^2 \leq \frac{1}{a(1-b)} [d(x_n, p)^2 - d(z_n, p)^2]$

By (2.1) and (2.5), $\lim_{n\to\infty} \sup d(Tx_n, x_n) \leq 0$ and hence $\lim_{n\to\infty} d(Tx_n, x_n) = 0$.

Theorem 2.2: Let X, C, T, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{x_n\}$ be as in Lemma 2.1, then $\{x_n\}$ Δ-converges to a fixed point of T.

Proof: Lemma 2.1 guarantees that the sequence $\{x_n\}$ is bounded and $\lim_{n\to\infty} d(Tx_n, x_n) = 0$.

We now let $\omega_w(x_n) = \cup A(\{u_n\})$ where the union is taken over all subsequences $\{u_n\}$ of $\{x_n\}$.

We claim that $\omega_w(x_n) \subset F(T)$.

Let $u \in \omega_w(x_n)$, then there exists a subsequence $\{u_n\}$ of $\{x_n\}$ such that $A(\{u_n\}) = \{u\}$. By Lemmas 1.8, 1.9 there exists a subsequence $\{v_n\}$ of $\{u_n\}$ such that Δ- $\lim_n v_n = v \in C$.

Since $\lim_n d(v_n, T v_n) = 0$, then $v \in F(T)$ by Lemma 1.10. We claim that u = v. Suppose not, by the uniqueness of asymptotic centers,

$$\begin{aligned} \lim_n \sup d(v_n, v) &< \lim_n \sup d(v_n, u) \\ &\leq \lim_n \sup d(u_n, u) \\ &< \lim_n \sup d(u_n, v) \\ &= \lim_n \sup d(x_n, v) \\ &= \lim_n \sup d(v_n, v), \end{aligned}$$

which is a contradiction and hence $u = v \in F(T)$.

To show that $\{x_n\}$ Δ-converges to a fixed point of T, it suffices to show that $\omega_w(x_n)$ consists of exactly one point. Let $\{u_n\}$ be a subsequence of $\{x_n\}$. By Lemmas 1.8, 1.9 there exists a subsequence $\{v_n\}$ of $\{u_n\}$ such that Δ-$\lim_n v_n = v \in C$. Let $A(\{u_n\}) = \{u\}$ and $A(\{x_n\}) = \{x\}$. We have seen that u = v and $v \in F(T)$. We can complete the proof by showing that x = v. If not, since $\{d(x_n, v)\}$ is convergent, then by the uniqueness of asymptotic centers,

$$\begin{aligned} \lim_n \sup d(v_n, v) &< \lim_n \sup d(v_n, x) \\ &\leq \lim_n \sup d(x_n, x) \\ &< \lim_n \sup d(x_n, v) \\ &= \lim_n \sup d(v_n, v), \end{aligned}$$

which is a contradiction and hence the conclusion follows.

Theorem 2.3: Let C be a nonempty closed convex subset of a complete CAT(0) space X and let T: C $\to$ C be a nonexpansive mapping with $F(T) \neq \emptyset$. Let $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ be sequences in [0, 1]. From arbitrary $x_1 \in C$, define the sequence $\{x_n\}$ by the recursion (RS). Then $\{x_n\}$ converges strongly to a fixed point of T if and only if $\lim \inf_{n\to\infty} d(x_n, F(T)) = 0$, where $d(x, F(T)) = \inf\{d(x, p): p \in F(T)\}$.

Proof: The Necessary condition is quite obvious.

Conversely, suppose that $\liminf_{n\to\infty} d(x_n, F(T)) = 0$. As proved in Lemma 2.1, we have $d(x_{n+1}, p) \leq d(x_n, p)$ for all $p \in F(T)$.

This implies that $d(x_{n+1}, F(T)) \leq d(x_n, F(T))$ so that $\lim_{n\to\infty} d(x_n, F(T))$ exists. Thus by hypothesis $\lim_{n\to\infty} d(x_n, F(T)) = 0$.

Now we show that $\{x_n\}$ is a Cauchy sequence in C. Let $\varepsilon > 0$ be arbitrarily chosen. Since $\lim_{n\to\infty} d(x_n, F(T)) = 0$, there exists a positive integer n_0 such that $d(x_n, F(T)) < \varepsilon/4$ for all $n \geq n_0$.

In particular, $\inf\{ d(x_{n_0}, p) : p \in F(T) \} < \varepsilon/4$. Thus there must exist $p^* \in F(T)$ such that $d(x_{n_0}, p^*) < \varepsilon/2$.

Now for all $m, n \geq n_0$, we have

$$\begin{aligned} d(x_{n+m}, x_n) &\leq d(x_{n+m}, p^*) + d(p^*, x_n) \\ &\leq 2\, d(x_{n_0}, p^*) \\ &< 2(\varepsilon/2) = \varepsilon. \end{aligned}$$

Hence $\{x_n\}$ is a Cauchy sequence in a closed subset C of a complete CAT(0) space and so it must converge to a point q in C and $\lim_{n\to\infty} d(x_n, F(T)) = 0$ gives that $d(q, F(T)) = 0$ and closedness of F(T) forces q to be in F(T).

Now we give the strong convergence result of CR-iterative process for the mapping satisfying Condition (A).

Theorem 2.4: Let C be a nonempty closed convex subset of a complete CAT(0) space X and let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ be sequences in [0, 1]. From arbitrary $x_1 \in C$, define the sequence $\{x_n\}$ by the recursion (RS). Let $T: C \to C$ satisfy the condition (A). Then $\{x_n\}$ converges strongly to a fixed point of T.

Proof: From Lemma 2.1, $\lim_{n\to\infty} d(x_n, p)$ exists for all $p \in F(T)$.

Let $\lim_{n\to\infty} d(x_n, p) = c$ where $c \geq 0$. If $c = 0$ then there is nothing to prove. Suppose that $c > 0$.

Now $d(x_{n+1}, p) \leq d(x_n, p)$ gives $\inf_{p\in F(T)} d(x_{n+1}, p) \leq \inf_{p\in F(T)} d(x_n, p)$, which implies that $d(x_{n+1}, F(T)) \leq d(x_n, F(T))$

so that $\lim_{n\to\infty} d(x_n, F(T))$ exists. Using the condition (A),

$$\lim_{n\to\infty} f(d(x_n, F(T))) \leq \lim_{n\to\infty} d(Tx_n, x_n) = 0.$$

Thus we get $\lim_{n\to\infty} f(d(x_n, F(T))) = 0$. Since f is a non-decreasing function and $f(0) = 0$, it follows tha $\lim_{n\to\infty} d(x_n, F(T)) = 0$. The conclusion now follows from the above Theorem 2.3.

REFERENCES

1. M. Bridson and A. Haefliger, Metric Spaces of Non-Positive Curvature, *Springer-Verlag, Berlin, Heidelberg* (1999).
2. F. Bruhat and J. Tits, Groupes reductifs sur un corps local. I. Donnees radicielles values, *Inst. Hautes Etudes Sci. Publ. Math*. 41 (1972) 5-251.
3. R. Chugh, V. Kumar and S. Kumar, Strong convergence of a new three step iterative scheme in Banach spaces, *American Journal of Computational Mathematics* 2(4) (2012) 345-357.
4. S. Dhompongsa, W. A. Kirk and B. Panyanak, Non-expansive set-valued mappings in metric and Banach spaces, *Journal of Nonlinear and Convex Analysis* 8 (2007) 35-45.
5. S. Dhompongsa, W. A. Kirk and B. Sims, Fixed points of uniformly lipschitzian mappings, *Nonlinear Analysis: TMA* 65 (2006) 762-772.
6. S. Dhompongsa and B. Panyanak, On Δ-convergence theorems in CAT(0) spaces, *Computer and Mathematics with Applications* 56 (2008) 2572-2579.
7. W. A. Kirk and B. Panyanak, A concept of convergence in geodesic spaces, *Nonlinear Analysis: TMA* 68 (2008) 3689-3696.
8. H. F. Senter and W. G. Dotson, Approximating fixed points of non-expansive mappings, *Proc. Amer. Math. Soc*. 44 (1974) 375-380.

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