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AN SIRS MODEL WITH EMIGRATION RATE AND SIMPLE MASS ACTION INCIDENCE

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ABSTRACT

In this paper, an SIRS model with emigration rate and simple mass action incidence is formulated and studied. Equilibrium and threshold are determined for the system of ordinary differential equations and discussed. For both, disease free and endemic equilibrium point, stability conditions are determined to see whether the disease dies out or approaches an endemic equilibrium state. An example is also furnished which demonstrates validity of main result.

Keywords*: Epidemiological model, Threshold, Hurwitz Criterion, Equilibrium point, Stability.*

1. INTRODUCTION

Epidemic models described by ordinary differential equations have become important tools in analyzing the spread and control of infectious diseases. The first SIR epidemic model was proposed by Kermack and Mckendric [12] in the year 1927. In the previous year's more and more SIRS models have been investigated during the study of epidemic models [1-6]. The SIRS epidemic model has been studied by many authors (see Hethcote [10, 11], Capasso and Serio [7], Mena-Lorca [13]).

If $N(\mathrm{t})$ is the total varying population size as a function of time t, b is the birth rate constant and d is the death rate constant, then

$$\frac{dN}{dt} = (b-d)N,\ N(0) = \mathrm{N}_0 \tag{1.1}$$

is the initial value problem. If $r = b-d$ then solution of (1.1) is $N(\mathrm{t}) = \mathrm{N}_0 e^{rt}$ so $N(\mathrm{t}) = \mathrm{N}_0$ if $r = 0$, $N(\mathrm{t})$ is grows exponentially if $r > 0$ and $N(\mathrm{t})$ is decays exponentially if $r < 0$, where $r = b-d$ is called the net growth rate. This form of population dynamics is called exponential births and deaths.

The number of individuals who are susceptible, infectious and recovered at time t, are denoted by $X(\mathrm{t}), \mathrm{Y}(\mathrm{t})$ and $Z(\mathrm{t})$ respectively and $X(\mathrm{t}) + \mathrm{Y}(\mathrm{t}) + Z(\mathrm{t}) = N(\mathrm{t})$.

The rate at which susceptible becomes infectious is called the incidence in an epidemiological model. If the unit time is days, then the incidence is the number of new infection per day. The number of susceptible who are infected by an infected individual per unit of time, at time t, is proportional to the number of susceptible with the proportional coefficient (transmission coefficient) β, so that the total number of newly infective, at time t, is $\beta X(\mathrm{t})\,\mathrm{Y}(\mathrm{t})$. The number of recovered individuals from the infected compartment per unit time is $\gamma Y(\mathrm{t})$ at time t, where γ is the recovery rate coefficient.

The average number of secondary infections produced by one infected individuals during the mean course of infection (infectious period) in a completely susceptible population is called a basic reproductive number or simply the reproductive number σ. If $\sigma < 1$, then on average, the number of new infection by one infected individual over the mean course of the disease (infectious period) is < 1, which implies that the disease dies out. If $\sigma > 1$, then the number of new infections produced by one infected individual is > 1, which implies that disease persist.

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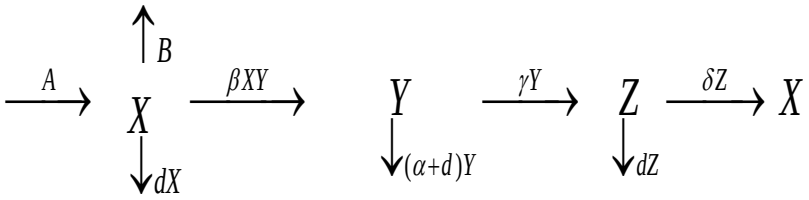
In this paper we consider an SIRS epidemic model with an emigration rate and simple mass action incidence In the next section, we present the model and derive the disease free equilibrium and the endemic equilibrium. In the third section, we prove some theorems for the global stability of the disease free and endemic equilibrium. The fourth section contains an example which demonstrates validity of main result. In the last section, we give conclusion.

2. THE BASIC MATHEMATICAL MODEL

The epidemiological model formulated here has population dynamics corresponding to difference between immigration and emigration with deaths. In the transfer diagram,



Where the number of susceptible, infective and recovered individuals as a function of time t are $X(t), Y(t)$ and $Z(t)$, respectively and the total population size $N(t)$. The parameters in the model are:

$A =$ constant immigration rate
$B =$ constant emigration rate
$d =$ natural death rate constant
$\beta =$ transmission coefficient
$\alpha =$ disease-related death rate constant
$\gamma =$ recovery rate constant
$\delta =$ loss of immunity rate constant.

We assume that d, α and δ are nonnegative and that A, B, β, γ and $\delta + d$ are positive.

The autonomous differential equations corresponding to the transfer diagram are:

$$\begin{aligned}
\frac{dX}{dt} &= A - B - \beta XY - dX + \delta Z \\
\frac{dY}{dt} &= \beta XY - (\gamma + \alpha + d)Y \\
\frac{dZ}{dt} &= \gamma Y - (\delta + d)Z \\
\frac{dN}{dx} &= A - B - dN - \alpha Y
\end{aligned} \tag{2.1}$$

Where $N = X + Y + Z$. In the absence of disease i.e. $\alpha = 0$ the population size approaches the constant size $(A-B)/d$, if $A > B$. For simple mass action incidence the contact number i.e. basic reproductive number typically is the productive of β, a population size and an average infectious period so that the contact number is

$$\sigma = \beta\left(\frac{A-B}{d}\right)\Big/(\gamma + \alpha + d). \tag{2.2}$$

For the system (2.1) the first octant in XYZ space is positively invariant. Because $\frac{dN}{dt} < 0$ for $N > (A-B)/d$, all paths in the first octant approach, enter or stay inside the subset $T = \{(X, Y, Z) : X + Y + Z \le (A-B)/d\}$. The continuity of the right side of (2.1)and its derivatives implies that unique solutions exits on a maximal time interval. Since solutions approach, enter or stay in T, they are eventually bounded and hence exist for all positive time [8]. We first consider the existence of equilibrium of system (2.1).

For any values of parameter, model (2.1) always has a disease-free equilibrium $P_o = \left((A-B)/d, 0, 0\right)$. To find the positive equilibrium, set

$$\begin{aligned} &A - B - \beta XY - dX + \delta Z = 0 \\ &\beta XY - (\gamma + \alpha + d)Y = 0 \\ &\gamma Y - (\delta + d)Z = 0 \\ &A - B - dN - \alpha Y = 0 \end{aligned} \tag{2.3}$$

3. MAIN RESULTS

Theorem 3.1: From the system (2.2) it follows that

(i) if $\sigma \le 1,$ then there is no positive equilibrium;

(ii) if $\sigma > 1,$ then there is a unique positive equilibrium $P_e = \left(X_e, Y_e, Z_e\right)$ of the system (2.1), called the "endemic equilibrium", given by

$$\begin{aligned} X_e &= \frac{\gamma + \alpha + d}{\beta} \\ Y_e &= \frac{A - B - dX_e}{\alpha + d\left(1 + \gamma/\delta + d\right)} \\ Z_e &= \frac{\gamma Y_e}{\delta + d} \\ N_e &= \frac{(A-B)\left(1 + \gamma/\delta + d\right) + \alpha X_e}{\alpha + d\left(1 + \gamma/\delta + d\right)} \end{aligned} \tag{3.1}$$

Theorem 3.2: The equilibrium $P_o = \left((A-B)/d, 0, 0\right)$ is locally asymptotically stable if $\sigma \le 1$ and P_o is saddle point if $\sigma > 1.$

Proof: The Jacobian of system (2.1) at P_o is

$$J(P_o) = \begin{pmatrix} -d & \dfrac{-\beta(A-B)}{d} & \delta \\ 0 & \dfrac{\beta(A-B)}{d} - (\gamma + \alpha + d) & 0 \\ 0 & \gamma & -(\delta + d) \end{pmatrix}$$

The characteristic equation is

$$(d+t)(\delta + d + t)\left[\frac{\beta(A-B)}{d} - (\gamma + \alpha + d) - t\right] = 0 \tag{3.2}$$

The roots of (3.2) are

$$-d, -(\delta + d) \text{ and } \frac{\beta(A-B)}{d} - (\gamma + \alpha + d)$$

The first two roots having negative real parts and third root $\dfrac{\beta(A-B)}{d} - (\gamma + \alpha + d)$ will have negative real part if $\sigma \le 1.$ Thus all roots of (3.2) have negative real parts so P_o is locally asymptotically stable if $\sigma \le 1$ and the root $\dfrac{\beta(A-B)}{d} - (\gamma + \alpha + d)$ will have positive real part if $\sigma > 1$ so P_o is saddle point.

Theorem 3.3: The equilibrium $P_o = ((A-B)/d, 0, 0)$ is globally asymptotically stable if $\sigma \leq 1.$

Proof: Since the set $T = \{(X, Y, Z) : X + Y + Z \leq (A-B)/d\}$ is attractive and positive invariant.

To prove that all paths in T approach $P_o = ((A-B)/d, 0, 0)$ for $\sigma \leq 1,$ define the Liapunov function $L = Y$ in T with

$$\frac{dL}{dt} = \frac{dY}{dt} = [\beta X - (\gamma + \alpha + d)]Y \leq 0. \tag{3.3}$$

The Lasalle-Liapunov theory [9] implies that all paths in T approach the largest positively invariant subset of the set T where $\frac{dL}{dt} = 0.$

Here $\frac{dL}{dt} = 0$ only if $Y = 0$ or $(X, Y, Z) = P_o.$ The positively invariant subset of the plane $Y = 0$ is the point P_o so P_o is globally asymptotically stable for $\sigma \leq 1.$

Theorem 3.4: The equilibrium $P_e = (X_e, Y_e, Z_e)$ is locally asymptotically stable if $\sigma > 1.$

Proof: The Jacobian of system (2.1) at P_e is

$$J(P_e) = \begin{pmatrix} -d - \beta Y_e & -\beta X_e & \delta \\ \beta Y_e & \beta X_e - (\gamma + \alpha + d) & 0 \\ 0 & \gamma & -(\delta + d) \end{pmatrix}$$

The cubic characteristic equation is

$$t^3 + a_1 t^2 + a_2 t + a_3 = 0,$$

$$a_1 = \delta + 2d + \beta Y_e > 0,$$

$$a_2 = (\delta + d)(d + \beta Y_e) + \beta^2 X_e Y_e > 0,$$

$$a_3 = \beta^2 X_e Y_e (\delta + d) - \delta\beta\gamma Y_e > 0.$$

Thus all roots have negative real parts iff $a_1 a_2 - a_3 > 0$ by the Routh-Hurwiz criteria [14]. Thus $a_1 a_2 - a_3$ is positive if $\sigma > 1$ equilibrium $P_e = (X_e, Y_e, Z_e)$ is locally asymptotically stable if $\sigma > 1.$

4. EXAMPLE

In this section, we give an example to demonstrate the results obtained in the previous sections.

We take the parameters of the system as $d = 2.33,\ A = 6.4,\ B = 2.9,\ \delta = 1, \alpha = 0.28, \beta = 1.20,\ \gamma = 0.49$ Then $P_O = (1.5021, 0, 0)$ and $\sigma = 0.5814 < 1.$ Therefore, by theorem 2.3, P_O is a global asymptotically stable in the first octant.

Now we take the parameter of the system as $d = 0.36\ \ A = 6.4,\ B = 2.9,\ \delta = 1, \alpha = 2, \beta = 1.20,\ \gamma = 0.18.$ Then $P_e = (2.1166, 1.1372, 0.1505)$ and $\sigma = 4.5931 > 1.$ Therefore, by theorem 3.4 P_e is a locally asymptotically stable in the interior of the first octant.

5. CONCLUSION

In this paper, we have studied an SIRS model with emigration rate and simple mass action incidence. Our main results shows that when $\sigma \leq 1,$ the disease-free equilibrium P_O is globally asymptotically stable. When $\sigma > 1,$ the endemic equilibrium $P_e = (S_e, I_e, R_e)$ exists and is locally asymptotically stable.

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