International Journal of Mathematical Archive-6(2), 2015, 90-95
Available online through www.ijma.info ISSN 2229 – 5046

WQ-PRINCIPALLY INJECTIVE MODULES

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(Received On: 02-02-15; Revised & Accepted On: 25-02-15)

ABSTRACT

Let M *be a right* $R-$*module. A right* $R-$*module* N *is called* $WM-$*principally injective if, for each* $s \in S \setminus W(S)$, *any* $R-$*homomorphism from* $s(M)$ *to* N *can be extended to an* $R-$*homomorphism from* M *to* N. M *is called* $WQ-$*principally injective if, it is* $WM-$*principally injective. In this paper, we give some characterizations and properties of* $WQ-$*principally injective modules.*

2010 Mathematics Subject Classification: *13C10, 13C11, 13C60.*

Key words and phrases*:* $WM-$*principally Injective Modules and Endomorphism Rings.*

1. INTRODUCTION

Throughout this paper, R will be an associative ring with identity and all modules are unitary right $R-$modules. For right $R-$modules M and N, $\mathrm{Hom}_R(M,N)$ denotes the set of all $R-$homomorphisms from M to N and $S=\mathrm{End}_R(M)$ denotes the endomorphism ring of M. A submodule X of M is said to be $M-$cyclic submodule of M if it is the image of an element of S. If X is a subset of M the right (resp. left) annihilator of X in R (resp. S) is denoted by $r_R(X)$ (resp. $l_S(X)$). By notations, $N \subset^{\oplus} M$, $N \subset^{e} M$, and $N \ll M$ we mean that N is a direct summand, an essential submodule and a superfluous submodule of M, respectively. We denote the Jacobson radical of M by $J(M)$.

Let R be a ring. A right $R-$module M is called *principally injective* (or $P-$*injective*), if every $R-$homomorphism from a principal right ideal of R to M can be extended to an $R-$homomorphism from R to M. Equivalently, $l_M r_R(a)=Ma$ for all $a \in R$. This notion was introduced by Camillo [2] for commutative rings. In [7], Nicholson and Yousif studied the structure of principally injective rings and gave some applications. Nicholson, Park, and Yousif [8] extended this notion of principally injective rings to the one for modules.

Following [9], a right $R-$module M is called *quasi-principally injective*, if for each $s \in S$, $l_S(\mathrm{Ker}(s))=Ss$. In [10], L.V. Thuyet, and T.C. Quynh, introduced the definition of small principally injective modules, a right $R-$module M is called *small principally injective* (briefly, $SP-$*injective*) if, every $R-$homomorphism from a small and principal right ideal aR to M can be extended to an $R-$homomorphism from R to M. A ring R is called right $SP-$*injective*, if R_R is $SP-$*injective*. A right $R-$module N is called *small principally* $M-$*injective* (briefly, $SP-M-$*injective*) [13] if, every $R-$homomorphism from a small and principal submodule of M to N can be extended to an $R-$homomorphism from M to N. In [5] W. Junchao introduced the definition of $JCP-$injective rings, a ring R is called right $Jcp-$injective if for each $a \in R \setminus Z_r$, any $R-$homomorphism from aR to R can be extended to an $R-$homomorphism from R to R. In this note we introduce the definition of $WQ-$principally injective modules and give some characterizations and properties. Some results on principally quasi-injective modules [8] are extended to these modules.

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2. $WQ-$ PRINCIPALLY INJECTIVE MODULES

Let M be a right $R-$module with $S=\mathrm{End}_R(M)$. Following [8], write

$$W(S)=\left\{s\in S:\mathrm{Ker}(s)\subset^e M\right\}.$$

It is known that $W(S)$ is an ideal of S.

Definition 2.1: Let M be a right $R-$module. A right $R-$module N is called $WM-$*principally injective* if, for each $s\in S\setminus W(S)$, any $R-$homomorphism from $s(M)$ to N can be extended to an $R-$homomorphism from M to N. M is called $WQ-$*principally injective* if, it is $WM-$principally injective.

Lemma 2.2: Let M and N be right $R-$modules. Then N is $WM-$principally injective if and only if for each $s\in S\setminus W(S)$,

$$\mathrm{Hom}_R(M,N)s=\left\{f\in \mathrm{Hom}_R(M,N):f(\mathrm{Ker}(s))=0\right\}.$$

Proof: Clearly, $\mathrm{Hom}_R(M,N)s\subset\left\{f\in \mathrm{Hom}_R(M,N):f(\mathrm{Ker}(s))=0\right\}$.
Let $f\in \mathrm{Hom}_R(M,N)$ such that $f(\mathrm{Ker}(s))=0.$ Then there exists an $R-$homomorphism

$WQ-$Principally Injective Modules $\varphi:s(M)\to N$ suchthat $\varphi s=f$ by Factor Theorem because $\mathrm{Ker}(s)\subset\mathrm{Ker}(f)$. Since N is $WM-$principally injective, there exists an $R-$homomorphism $t:M\to N$ such that $\varphi=t\iota$ where $\iota:s(M)\to M$ is the inclusion map. Hence $f=ts$ and therefore $f\in \mathrm{Hom}_R(M,N)s.$

Conversely, let $s\in S\setminus W(S)$ and $\varphi:s(M)\to N$ be an $R-$homomorphism. Then $\varphi s\in \mathrm{Hom}_R(M,N)$ and $\varphi s(\mathrm{Ker}(s))=0.$ By assumption, we have $\varphi s=us$ for some $u\in \mathrm{Hom}_R(M,N).$ This shows that N is $WM-$principally injective.

Lemma 2.3: Let $N_i\ (1\le i\le n)$ be $WM-$principally injective modules. Then $\oplus_{i=1}^n N_i$ is $WM-$principally injective.

Proof: It is enough to prove the result for $n=2.$ Let $s\in S\setminus W(S)$ and $\varphi:s(M)\to N_1\oplus N_2$ be an $R-$homomorphism. Since N_1 and N_2 are $WM-$principally injective, there exists $R-$homomorphisms $\varphi_1:M\to N_1$ and $\varphi_2:M\to N_2$ such that $\varphi_1\iota=\pi_1\varphi$ and $\varphi_2\iota=\pi_2\varphi$ where π_1 and π_2 are the projection maps from $N_1\oplus N_2$ to N_1 and N_2, respectively, and $\iota:s(M)\to M$ is the inclusion map. Put
$\hat{\varphi}=\iota_1\varphi_1+\iota_2\varphi_2:M\to N_1\oplus N_2.$
Thus it is clear that $\hat{\varphi}$ extends $\varphi.$

Lemma 2.4: Any direct summand of a $WM-$principally injective modules is again $WM-$principally injective.

Proof: By definition.

Lemma 2.5: If $s\in S\setminus W(S)$ and $s(M)$ is $WM-$principally injective, then $s(M)\subset^{\oplus} M.$

Proof: Since $s(M)$ is $WM-$principally injective, there exists an $R-$homomorphism $\varphi:M\to s(M)$ such that $\varphi\iota=1_{s(M)}$ where $\iota:s(M)\to M$ is the inclusion map. Then by [1, Lemma 5.1], ι is a split monomorphism, therefore $s(M)\subset^{\oplus} M.$

Theorem 2.6: The following conditions are equivalent for a projective module M.

(1) Every $s \in S \setminus W(S)$, $s(M)$ is projective.

(2) Every factor module of a $WM-$principally injective module is $WM-$principally injective.

(3) Every factor module of an injective $R-$module is $WM-$principally injective.

Proof: $(1) \Rightarrow (2)$: Let N be a $WM-$principally injective, X a submodule of N, $s \in S \setminus W(S)$ and let $\varphi : s(M) \to N/X$ be an $R-$homomorphism. Then by (1), there exists an $R-$homomorphism $\hat{\varphi} : s(M) \to N$ such that $\varphi = \eta\hat{\varphi}$ where $\eta : N \to N/X$ is the natural $R-$epimorphism. Since N is $WM-$principally injective, there exists an $R-$homomorphism $t : M \to N$ which is an extension of $\hat{\varphi}$ to M. Then ηt is an extension of φ to M.

$(2) \Rightarrow (3)$: is clear.

$(3) \Rightarrow (1)$: Let $s \in S \setminus W(S)$ and $\alpha : A \to B$ an $R-$epimorphism, and let $\varphi : s(M) \to B$ be an $R-$homomorphism. Embed A in an injective module E [1, 18.6]. Then $B \simeq A/\mathrm{Ker}(\alpha)$ is a submodule of $E/\mathrm{Ker}(\alpha)$ so by hypothesis, φ can be extended to $\hat{\varphi} : M \to E/\mathrm{Ker}(\alpha)$.

Since M is projective, $\hat{\varphi}$ can be lifted to $\beta : M \to E$. It is clear that $\beta(s(M)) \subset A$. Therefore we have lifted φ.

Recall that a right $R-$module M is call $(C2)$ [6] if, every submodule of M that is isomorphic to a direct summand of M is itself a direct summand of M. M is call $(C3)$ if, whenever N and K are direct summands of M with $N \cap K = 0$ then $N \oplus K$ is also a direct summand of M.

Lemma 2.7: Let M be a principal, $WQ-$principally injective module. Then M satisfies the conditions $(C2)$ and $(C3)$.

Proof:
$(C2)$: Let $M = mR$, $m \in M$ and $nR \simeq e(mR)$ where $n \in M$ and $e^2 = e \in S$. Then $e(mR)$ is $WM-$principally injective by Lemma 2.4 and hence nR is also $WM-$principally injective. Since $nR \simeq e(mR)$, there exists an isomorphism σ such that $nR = \sigma e(mR)$. Since $\sigma e \in S \setminus W(S)$, then $nR \subset^{\oplus} M$ Lemma 2.5.

$(C3)$: Let $M = N \oplus K$. Write $N = e(M)$ and $K = f(M)$ where e, f are idempotents in S. Then
$e(M) \oplus f(M) = e(M) \oplus (1-e)f(M)$. Since $(1-e)f(M) \simeq f(M)$, $(1-e)f(M) = g(M)$ for some $g^2 = g \in S$ by $(C2)$. Let $h = e + g - ge$, then $h^2 = h$ and $e(M) \oplus f(M) = h(M)$.
This prove $(C3)$.

Theorem 2.8: Let M be a $WM-$principally injective, quasi-projective module and $s \in S \setminus W(S)$. Then the following conditions are equivalent.

(1) $s(M)$ is a direct summand of M.

(2) $s(M)$ is $M-$projective.

(3) $s(M)$ is $WM-$principally injective.

Proof:
$(1) \Rightarrow (2)$: It follows from the projectivity of M.

$(2) \Rightarrow (3)$: Since the sequence $0 \to \mathrm{Ker}(s) \to M \to s(M) \to 0$ splits, $s(M)$ is isomorphic to a direct summand of M. Hence $s(M)$ is a direct summand of M by $(C2)$, so it is $WM-$principally injective by Lemma 2.4.

$(3) \Rightarrow (1)$: It follows from Lemma 2.5.

Theorem 2.9: Let M be a right $R-$module. Then the following conditions are equivalent.

(1) M is $WQ-$principally injective.

(2) $l_S(\mathrm{Ker}(s)) = Ss$ for each $s \in S \setminus W(S)$.

(3) $\mathrm{Ker}(s) \subset \mathrm{Ker}(t)$, $s, t \in S$ and $s \in S \setminus W(S)$ implies that $St \subset Ss$.

(4) $l_S(\mathrm{Im}(t) \cap \mathrm{Ker}(s)) = l_S(\mathrm{Im}(t)) + Ss$ for $s, t \in S$ with $st \in S \setminus W(S)$.

Proof: $(1) \Rightarrow (2)$: Clearly, $Ss \subset l_S(\mathrm{Ker}(s))$ for all $s \in S \setminus W(S)$. Let $t \in l_S(\mathrm{Ker}(s))$ and define $\varphi : s(M) \to M$ by $\varphi(s(m)) = t(m)$ for every $m \in M$. Then φ is well-defined because $\mathrm{Ker}(s) \subset \mathrm{Ker}(t)$. By (1), there exists an $R-$homomorphism $\hat{\varphi} : M \to M$ such that $\varphi = \hat{\varphi}\iota$ where $\iota : s(M) \to M$ is the inclusion map. Hence $t = \varphi s = \hat{\varphi} s \in Ss$.

$(2) \Rightarrow (3)$: If $\mathrm{Ker}(s) \subset \mathrm{Ker}(t)$, $s, t \in S$ with $s \in S \setminus W(S)$, then $l_S(\mathrm{Ker}(t)) \subset l_S(\mathrm{Ker}(s))$. Since $St \subset l_S(\mathrm{Ker}(t))$ and by (2), $l_S(\mathrm{Ker}(s)) = Ss$, so we have $St \subset Ss$.

$(3) \Rightarrow (4)$: Clearly, $l_S(\mathrm{Im}(t)) + Ss \subset l_S(\mathrm{Im}(t) \cap \mathrm{Ker}(s))$.
Let $s, t \in S$ with $st \in S \setminus W(S)$ and let $\varphi \in l_S(\mathrm{Im}(t) \cap \mathrm{Ker}(s))$. Then $\mathrm{Ker}(st) \subset \mathrm{Ker}(\varphi t)$, and so $S\varphi t \subset Sst$ by (3) because $st \in S \setminus W(S)$. Thus $\varphi t = \hat{\varphi} st$, $\hat{\varphi} \in S$ so $(\varphi - \hat{\varphi} s) \in l_S(\mathrm{Im}(t))$. It follows that $\varphi \in l_S(\mathrm{Im}(t)) + Ss$.

$(4) \Rightarrow (1)$: Let $s \in S \setminus W(S)$ and $\varphi : s(M) \to M$ be an $R-$homomorphism.
Then $\varphi s \in l_S(\mathrm{Ker}(\varphi s)) \subset l_S(\mathrm{Ker}(s)) = l_S(\mathrm{Ker}(s) \cap \mathrm{Im} 1) = l_S(\mathrm{Im} 1) + Ss = Ss$ by (4) because $s1 \in S \setminus W(S)$. Thus there exists an $R-$homomorphism $\hat{\varphi} \in S$ is an extension of φ to M.

The following theorem is a generalization of [9, Theorem 2.8]

Theorem 2.10: Let M be a $WQ-$principally injective module and $s, t \in S$ with $s \in S \setminus W(S)$.

(1) If $s(M)$ embeds into $t(M)$, then Ss is an image of St.

(2) If $t(M)$ is an image of $s(M)$, then St can be embedded into Ss.

(3) If $s(M) \simeq t(M)$, then $Ss \simeq St$.

Proof: (1) Let $f : s(M) \to t(M)$ be an $R-$monomorphism. Since M is $WQ-$principally injective, there exists an $R-$homomorphism $\hat{f} : M \to M$ such that $\hat{f}\iota_1 = \iota_2 f$ where $\iota_1 : s(M) \to M$ and $\iota_2 : t(M) \to M$ are the inclusion maps. Let $\sigma : St \to Ss$ defined by $\sigma(ut) = u\hat{f}s$ for every $u \in S$. Since $\hat{f}s(M) \subset t(M)$, σ is well-defined. It is clear that σ is an $S-$homomorphism. Since f is monic, $\mathrm{Ker}(s) = \mathrm{Ker}(fs)$ so $fs \in S \setminus W(S)$ and hence by

Theorem 2.9: $Ss \subset Sfs$. Then $s \in Sfs \subset \sigma(St)$.

(2) By the same notations as in (1), let $f : s(M) \to t(M)$ be an $R-$epimorphism.
Since M is $WQ-$principally injective, there exists an $R-$homomorphism $\hat{f} : M \to M$ such that $\hat{f}\iota_1 = \iota_2 f$. Let $\sigma : St \to Ss$ defined by $\sigma(ut) = u\hat{f}s$ for every $u \in S$. It is clear that σ is an $S-$homomorphism. If $ut \in \mathrm{Ker}(\sigma)$, then $0 = \sigma(ut) = u\hat{f}s = ufs = ut$.

(3) Follows from (1) and (2)

If M is any right $R-$module, write

$$\hat{\Delta} = \{ s \in S : \ \mathrm{Ker}(1 + ts) = 0 \text{ for all } t \in S \}.$$

Since $\mathrm{Ker}(s)\cap \mathrm{Ker}(1+ts)=0,$ $W(S)\subset \hat{\Delta}.$ It is well-known that, for a quasi-continuous module $M,$ M is continuous if and only if $S/W(S)$ is regular and $W(S)=J(S)$

[6, Proposition 3.15]. We now investigate when $W(S)=J(S).$

Theorem 2.11: Let M be $WQ-$principally injective.

(1) $J(S)=\hat{\Delta}$

(2) If S is local, then $J(S)=\{s\in S:\mathrm{Ker}(s)\neq 0\}.$

(3) If $S/W(S)$ is regular, then $W(S)=J(S).$

(4) If M is uniform, then $Z(S_S)\subset J(S).$

Proof:

(1) For any $s\in J(S)$ and $t\in S,$ $g(1+ts)=1_M$ for some $g\in S.$ Then $\mathrm{Ker}(1+ts)=0,$ and hence $J(S)\subset \hat{\Delta}.$ On the other hand, if $\mathrm{Ker}(1+ts)=0,$ then $(1+ts)\in S\setminus W(S)$ so by Theorem 2.9, $S=l_S(\mathrm{Ker}(1+ts))=S(1+ts).$ Hence $g(1+ts)=1_M$ for some $g\in S.$ This shows that $s\in J(S).$

(2) Since S is local, $Ss\neq S$ for any $s\in J(S).$ If $\mathrm{Ker}(s)=0,$ then $s\in S\setminus W(S)$ and so $\alpha:s(M)\to M$ given by $\alpha(s(m))=m$ for any $m\in M$ is an $R-$homomorphism. Since M be $WQ-$principally injective, let $\beta\in S$ be an extension of α to $M.$ It follows that $\beta s=1_M$ so $Ss=S,$ which is a contradiction. This shows that $J(S)\subset\{s\in S : \mathrm{Ker}(s)\neq 0\}.$
The other inclusion is clear.

(3) Clearly, $W(S)\subset J(S).$ If $s\in J(S),$ then $(1-s\alpha)s=s-s\alpha s\in W(S)$ for some $\alpha\in S.$ Since $1-s\alpha$ has a left inverse, $s\in W(S).$ This shows that $J(S)\subset W(S).$

(4) Let $s\in Z(S_S).$ Then $\mathrm{Ker}(s)\neq 0.$ For any $t\in S$ we have $\mathrm{Ker}(s)\cap\mathrm{Ker}(1+ts)=0,$ then $\mathrm{Ker}(1+ts)=0.$ Hence $s\in J(S)$ by (1).

A right $R-$module M is called a *self-generator* [11] if it is generates all its submodules. A right $R-$module N is called $M-$*small principally injective* [14] if, for each $s\in S$ with $s(M)\ll M,$ any $R-$homomorphism from $s(M)$ to N can be extended to an $R-$homomorphism from M to $N.$ M is called quasi-small $P-$*injective*, if it is $M-$*Small principally injective*.

Lemma 2.12: [14, Proposition 3.2.3] Let M be a principal module which is a self-generator. If M is quasi-small $P-$injective, then S is a right $SP-$injective ring.

Theorem 2.13: Let M be a principal module which is a self-generator. Then M is quasi-principally injective if and only if M is $WQ-$principally injective and quasi- small $P-$injective.

Proof: Let M be a $WQ-$principally injective module which is a quasi- small $P-$injective module. Then $W(S)\subset J(S)$ by Theorem 2.9. Let $s(M)$ be an $M-$cyclic of $M.$ If $s\in S\setminus W(S),$ then $l_S(\mathrm{Ker}(s))=Ss$ by Theorem 2.9. If $s\in W(S),$ then $s\in J(S).$ We claim that $l_S(\mathrm{Ker}(s))=Ss.$ Clearly $Ss\subset l_S(\mathrm{Ker}(s)).$ Let $\alpha\in l_S(\mathrm{Ker}(s)).$ Define $\varphi:sS\to S$ by $\varphi(s\beta)=\alpha\beta$ for every $\beta\in S.$ Since $\mathrm{Ker}(s)\subset\mathrm{Ker}(\alpha),$ φ is well-defined. It is clear that φ is an $S-$homomorphism. Since S is a right $SP-$injective ring, there exists an $S-$homomorphism $\hat{\varphi}:S\to S$ such that $\varphi=\iota\hat{\varphi}$ where $\iota:sS\to S$ is the inclusion map. Hence $\alpha=\varphi s=\hat{\varphi}s=\hat{\varphi}(1)s\in Ss,$ and so $l_S(\mathrm{Ker}(s))\subset Ss.$

The author is grateful to Prof. S. Wongwai for many helpful comments and suggestions. The author also wishes to thank an anonymous referee for his or her suggestions which led to substantial improvements of this paper. This research is supported by the Rajabhat Rajanagarindra University.

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Source of support: Nil, Conflict of interest: None Declared