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SOME RESULTS OF GENERALIZED (θ,θ) −DERIVATIONS ON PRIME AND SEMIPRIME RINGS

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ABSTRACT

Let R be an associative ring with center Z(R). In this paper we study the commutativity of prime rings under certain conditions with generalized (θ,θ) −derivations and give some results on semiprime rings with generalized $(\theta,\theta)-$ derivations.

***Key words**: prime ring, semiprime ring, left $\theta-$ centralizers, derivation, generalized derivation.*

INTRODUCTION

This paper consists of two sections. In section one, we recall some basic definitions and other concepts which will be used in our paper, we will explain these concepts by examples, remarks, and theorems. And we will give a result similar in [1], but in different certain conditions, to obtain the commutativity of prime ring with generalized (θ,θ) – derivation. Also, we will give some results on semiprime ring with generalized $(\theta,\theta)-$ derivation.

1. BASIC CONCEPTS

Definition 1.1: [2] A ring R is called a prime ring if for any a, b$\in R, aRb=\{0\}$,implies that either a=0 or b=0.

Examples 1.2: [2]
1. Any domain is a prime ring.
2. Any matrix ring over an integral domain is a prime ring.

Definition 1.3: [2] A ring R is called a semiprime ring if for any a$\in R, aRa=\{0\}$, implies that a=0.

Remark 1.4: [2] Every prime ring is a semiprime ring, but the converse in general is not true.

Definition 1.5: [3] A ring R is said to be n-torsion free, where n$\neq$ 0 is an integer if whenever na=0, with a$\in R$, then a=0.

Definition 1.6:[3] Let R be a ring. Define a lie product[. , .] on as follows.
$$[x, y] = xy - yx, \text{ for all } x, y\in R.$$

Properties 1.6: [3] Let R be a ring. Then for all x, y, z$\in R$, we have
1. [x, yz] = y[x, z] + [x, y]z
2. [xy, z] = x[y, z] + [x, z]y
3. [x+y, z] = [x, z] + [y, z]
4. [x, y+z] = [x, y] + [x, z]

Definition 1.7: [3] Let R be a ring, the center of R denoted by Z(R) and is defined by:
$$Z(R) =\{x\in R: xr=rx, for\ all\ r\in R\}$$

Definition 1.8: [4] Let R be a ring, then an additive map d: R→R is called derivation, if:
$$d(xy) = d(x)y + xd(y), \text{ for all } x, y \in R.$$

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Example 1.9: [4] Let R= $\{\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}, a, b \in N$, where N is the ring of integers} be a ring of 2×2 matrices with respect to usual addition and multiplication.

Let d: R→R, defined by d$\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}$, for all $\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \in R$

Then d is a derivation of R.

Definition 1.10: [5] Let R be a ring. An additive mapping F: R→ R is called a generalized derivation associated with d if there exists a derivation d: R → R, Such that

F(xy) = F(x)y + x d(y), for all x, y∈ R.

Example 1.11: [5] Let R= $\{\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}, a, b, c \in Z$, the set of integers}.

The additive maps F, d :R → R define the following :

F$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} 0 & a+c \\ 0 & 0 \end{pmatrix}$ $,or\ all\ a, b, c \in Z$

and d$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} 0 & a-c \\ 0 & 0 \end{pmatrix}, for\ all\ a, b, c \in Z$

Then F is a generalized derivation of R associated with d.

Definition 1.12: [1] A left (right) centralizer of a ring R is an additive mapping T: R→R, which satisfies:

T(xy)= T(x)y(resp.T(xy)=xT(y)), for all x, y∈ R.

A centralizer of a ring R is both left and right centralizer.

Example 1.13: [1] Let F be a field, and D_2 (F) be a ring of all diagonal matrices of order 2 over F. Let T: D_2 (F) → D_2 (F) be an additive mapping defined as.

T$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix}, for\ all\ a, b \in F.$

Then T is a centralizer.

Definition 1.14:[1] An additive mapping T:R→ R is called a left (right)θ-centralizer if for all x,y∈ R, T(xy)=T(x) $\theta(y)$(T(xy)=θ(x)T(y)).Aθ −centralizer of R is both left and right θ −centralizer, where θ is a homomorphism on R.

2. GENERALIZED (θ, θ) −DERIVATIONS

Definition 2.1: Let R be a ring. An additive mapping F: R→ R is called a generalized (θ, θ)-derivation associated with d, where $\theta: R$ → R is a mapping of R, if there exists a (θ, θ) − derivation d:R → R such that

F(xy) = F(x)$\theta(y) + \theta(x)d(y)$, for all x,y∈ R

Example 2.2: Let R=$\{\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$, a, b, c∈ Z, the set of integers}.

The additive mapsF,d :R → R define the following:

F$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} 0 & a+c \\ 0 & 0 \end{pmatrix}$, for all a, b, c∈ Z, and

d$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} 0 & a-c \\ 0 & 0 \end{pmatrix}$, for all a,b,c∈ Z, and

Suppose that $\theta: R$ → R is a mapping such that

$\theta\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$, for all a, b, c∈ Z.

It is clear that d is a (θ, θ)-derivation of R.

Then F is a generalized (θ, θ)-derivation of R associated with d.

Theorem 2.3: Let R be a prime ring. If R admits a generalized (θ, θ)-derivation F associated with a nonzero (θ, θ)-derivationd, whereθ is an automorphism on R, such that: d(x)F(y) - $\theta(x)\theta(y) \in Z$(R), for all x, y∈ R. Then R is commutative.

Proof: we have

d(x)F(y) - $\theta(x)\theta(y) \in Z$(R), for all x, y∈ R (1)

Replacing y by yz in (1) we get:

$$d(x)F(y)\,\theta(z) + d(x)\,\theta(y)d(z) - \theta(x)\theta(y)\theta(z) \in Z(R), \textit{for all } x,y,z \in R$$

for all x, y, z$\in R$ (2)

That is:

$$[(d(x)F(y) - \theta(x)\theta(y))\theta(z) + d(x)\theta(y)d(z), \theta(z)] = 0, \text{ for all } x, y, z \in R \tag{3}$$

The above expression implies that:

$$[d(x)\theta(y)d(z), \theta(z)] = 0, \text{ for all } x, y, z \in R \tag{4}$$

Then we have:

$$d(x)[\theta(y)d(z), \theta(z)] + [d(x), \theta(z)]\theta(y)d(z) = 0, \textit{for all } x,y,z \in R \tag{5}$$

Replacing $\theta(y)$by d(x)$\theta(y)$ in (5), and using (4), to get:

$$[d(x), \theta(z)]d(x)\theta(y)d(z) = 0, \textit{for all } x,y,z \in R \tag{6}$$

By primeness of R and d$\neq$ 0, $we\ get$:

$$[d(x), \theta(z)]d(x) = 0, \textit{for all } x,z \in R \tag{7}$$

Replace z by zy in (7), and using (7), we obtain:

$$[d(x), \theta(z)]\theta(y)d(x) = 0, \textit{for all } x,y,z \in R \tag{8}$$

Again, since R is prime and d$\neq$ 0, we get:

$$[d(x)\theta(z)] = 0\ , \textit{for all } x,y,z \in R \tag{9}$$

Especially, we get:

$$[d(x), x] = 0,\ \textit{for all } x \in R \tag{10}$$

By using [1, Lemma 1.1.30], we get R is commutative.

We will obtain the left $\theta - centralizer$from a generalized $(\theta,\theta) - derivation$ insemiprime ring under certain conditions, as following.

Where θ is an automorphism of R.

Theorem 2.4: Let R be a semiprim ring .If R admits a generalized $(\theta,\theta) - derivation$ F associated with $(\theta,\theta) - derivation\ d$,where θ is an automorphism of R, such that d(x)$F(y) = 0$, for all $x,y \in R$, then F is a left $\theta - centralizer$.

Proof: we have

$$d(x)F(y) = 0, \text{for all } x,y \in R \tag{1}$$

Replacing yx for y in (1) and using (1), we have

$$d(x)\theta(y)d(x) = 0, \textit{for all } x,y \in R \tag{2}$$

By semiprimeness of R, (2) gives:

$$d(x)=0, \text{ for all } x \in R \tag{3}$$

That is:

$$F(xy)= F(x)\,\theta(y), \text{ for all } x,y \in R \tag{4}$$

Hence, F is a left $\theta - centralizer$.

Theorem 2.5: Let R be a semiprime ring. If R admits a generalized $(\theta,\theta) - derivation$ F associated with $(\theta,\theta) - derivation\ d$,where θ is an automorphism of R, such that$(x) = \theta(x)$, for all $x \in R$, then F is a left $\theta - centralizer$.

Proof: we have:

$$F(x) = \theta(x), \text{ for all } x \in R \tag{1}$$

Replacing x by xy in (1) and using (1), we get:

$$\theta(x)d(y) = 0, \text{for all } x,y \in R \tag{2}$$

Left multiplication of (2) by d(y), leads to:

$$d(y)\theta(x)d(y)=0, \text{for all } x,y\in R \tag{3}$$

By semiprimeness of R, we have

$$\text{d(y)=0, for all } y\in R \tag{4}$$

That is:

$$\text{F(xy)= F(x) } \theta(y), \text{ for all } x,y\in R \tag{5}$$

And thus, F is a left $\theta - centralizer$.

In the following theorems, we will give an important relation to a generalized $(\theta,\theta)-derivation$ F associated with $(\theta,\theta)-derivation$ d:

Theorem 2.6: Let R be a 2- torsion free semiprime ring. If R admits a generalized $(\theta,\theta)-derivation$ F associated with $(\theta,\theta)-derivation\ d$, such that $d^2(x)=F(x)$, for all $x\in R$, then d =F=0, where θ is an automorphism of R.

Proof: We have

$$d^2(x)=F(x), \text{ for all } x\in R \tag{1}$$

Substituting xy for x in (1), we obtain:

$$d^2(x)\theta(y)+2d(x)d(y)+\theta(x)d^2(y)=F(x)\theta(y)+\theta(x)d(y),$$

for all $x,y\in R$ (2)

From (1), (2) we get

$$2d(x)d(y)+\theta(x)d^2(y)=\theta(x)d(y), \text{for all } x,y\in R \tag{3}$$

The above relation can be written as:

$$2d(x)d(y)+\theta(x)d^2(y)=\theta(x)d(y)+\theta(x)F(y)-\theta(x)F(y), \text{for all } x,y\in R \tag{4}$$

Again, from (1) and (4), one obtains:

$$2d(x)d(y)=\theta(x)(d(y)-F(y)), \text{ for all } x,y\in R \tag{5}$$

Replacing yx for x in (5), we have:

$$2d(y)\,\theta(x)d(y)+2\,\theta(y)d(x)d(y)=\theta(y)\theta(x)\big(d(y)-F(y)\big), \text{for all } x,y\in R \tag{6}$$

Left multiplication of (5) by $\theta(y)$, to get:

$$2\,\theta(y)d(x)d(y)=\theta(y)\theta(x)(d(y)-F(y)), \text{ for all } x,y\in R \tag{7}$$

Comparing (6) and (7), we obtain:

$$2\,d(y)\theta(x)d(y)=0, \text{for all } x,y\in R \tag{8}$$

Since R is 2 – torsion free, we get:

$$d(y)\theta(x)d(y)=0, \text{for all } x,y\in R \tag{9}$$

Since R is semiprime, we obtain:

$$d(y)=0\ , \text{ for all } y\in R \tag{10}$$

Hence: F(y) = 0, for all $y\in R$

Theorem 2.7: Let R be a 2- torsion free semiprime ring. If R admits a generalized $(\theta,\theta)-derivation$ Fassociated with $(\theta,\theta)-derivation\ d$,where θ is an automorphism of R, such that F(x) = - d(x), for all $x\in R, then\ d=F=0$.

Proof: we have

$$F(x)+d(x)=0, \text{ for all } x,y\in R \tag{1}$$

Replacing x by xy in (1) and using (1), we get:

$$F(x)\,\theta(y)+2\theta(x)d(y)+d(x)\theta(y)=0, \text{ for all } x,y\in R \tag{2}$$

From (1), (2) we get:

$$2\theta(x)d(y)=0, \text{for all } x,y\in R \tag{3}$$

Since R is 2-torsion free, we get:

$\theta(x)d(y) = 0$, for all $x, y \in R$ (4)

Left multiplication of (4) by d(y), and since R is semiprime, we get:

d(y)=0, for all $y \in R$ (5)

Thus, from (1) and (5), we obtain:

F(y) = 0, for all $y \in R$.

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