International Journal of Mathematical Archive-6(4), 2015, 49-59
Available online through www.ijma.info ISSN 2229 – 5046

SOLUTION OF SECOND KIND VOLTERRA INTEGRAL AND INTEGRO-DIFFERENTIAL EQUATION BY HOMOTOPY ANALYSIS METHOD

KUNJAN SHAH*
Applied Mathematics and Humanities Department
S. V. National Institute of Technology, Surat, Gujarat, India.

TWINKLE SINGH
Applied Mathematics and Humanities Department
S. V. National Institute of Technology, Surat, Gujarat, India.

(Received On: 12-02-15; Revised & Accepted On: 30-04-15)

ABSTRACT

In this paper, we introduce a solution of second kind Volterra integral and integro-differential equations by a numerical approach based on homotopy analysis method (HAM). From the computational point of view, we have compared the exact solution and homotopy analysis method solution and also shown the effectiveness of the method and conveniency for solving second kind Volterra integral and integro-differential equation.

Mathematics Subject Classification: *45DXX, 45JXX, 35C10, 35C05.*

Keywords: *Integral Equation, Integro-differential Equation, Homotopy Analysis Method, Closed form Solution.*

1. INTRODUCTION

Volterra integral equation arise in engineering, physics, chemistry and biological problems such as parabolic boundary value problems, the spatio-temporal development of the epidemic, population dynamics and semi-conductor device. Many initial and boundary value problems associated with the ordinary and partial differential equations can be cast into the Volterra integral equation types. The Volterra integral equation was first used by Vito Volterra [36] in 1884.

Many mathematical formulation of physical phenomena contain integro-differential equations, these equations arise in fluid dynamics, biological models and chemical kinetics; such as glass-forming process, nano-hydrodynamics, heat transfer, diffusion process in general, neutron diffusion and biological species coexisting together with increasing and decreasing rates of generating, and wind ripple in the desert. This type of equations was introduced by Volterra for the first time in the early 1900. Volterra investigated the population growth, focussing his study on the hereditary influences, where through his research work the topic of integro-differential equations was established.

A variety of analytic and numerical methods have been used to solve Volterra integral equations. For Example, Taylor series expansion method is used for second kind Volterra integral equation in [25]. Application of Collocation method on Volterra integral equations are investigated in [8, 9]. In [40] Variational iteration method is applied to solve integral equation. [26] used Chebyshev polynomials to find numerical solution of nonlinear Volterra integral equations of the second kind. Numerical solution of the second kind Volterra integral equation using an expansion method is found in [30], A new approach to solve Volterra integral equation by using Bernstein's approximation is employed in [27]. Application of Adomian decomposition method to solve integral equations are found in [37, 38].

Corresponding Author: **Kunjan Shah***
Applied Mathematics and Humanities Department
S. V. National Institute of Technology, Surat, Gujarat, India.

Integro-differential equations are usually difficult to solve analytically so it is required to obtain an efficient approximate solution. Therefore, they have been of great interest by several authors. In literature, there exist many numerical and semi-analytical-numerical techniques to solve Integro-differential equation. For Example, Wavelet-Galerkin method (WGM) to solve integro-differential equation can be found in [6]. Comparison between Wavelet-Galerkin method and Adomian decomposition method to solve integro differential equation is found in [12]. In [31] Lagrange interpolation method is applied to solve integro-differential equation. The Tau method is applied to the integro-differential equation in [16]. Application of Adomian's decomposition method on Integro-differential equation are investigated in [14, 38, 39]. [23] used Taylor polynomials to solve high-order Volterra integro-differential equation. In [24] rationalized Haar functions method is applied on system of linear integro-differential equations. In [5, 10] integro-differential equation is studied by using the differential transform method. Solution of forth-order integro-differential equation using variational iteration method can be found in [34]. In [32] Collocation method is used to solve fractional integro-differential equation. Application of He's homotopy perturbation method to solve Volterra integro-differential equation are found in [11, 13].

The Homotopy Analysis Method (HAM) is useful for obtaining both a closed form and the explicit solution and numerical approximations of linear or nonlinear differential equations and it is also quite straightforward to write computer codes. This method has been applied to obtain a formal solution to a wide class of stochastic and deterministic problems in science and engineering involving algebraic, differential, integro-differential, differential delay, integral and partial differential equations. This method has been firstly employed by Liao [18]-[22] in 1992. In recent years, many researchers have been successfully applying HAM to various nonlinear problems in science and engineering. For example, Homotopy analysis method is applied on Kawahara equation in [2]. Application of homotopy analysis method to solve nonlinear equation arising in heat transfer is found in [1]. [4] investigated the application of Homotopy analysis method for linear integral equations. This method is employed to find the convergent solution of nonlinear Klein-Gordon equation in [7]. The applications of this method on fractional diffusion-wave equation are found in [17]. A comparison between homotopy analysis method and homotopy perturbation method is presented in [15]. In [3] generalized Hirota-Satsuma coupled KdV equation is studied by using homotopy analysis method. This method is used to solve quadratic Riccati differential equation in [35]. The application of this method on Burgers-Huxley equation can be found in [28]. This method is modified to solve nonlinear boundary value problems arise in fingero-imbibition phenomenon in double phase flow through porous media in [33].

The present work is aimed at producing analytic and approximate solutions which are obtained in rapidly convergent series with elegantly computable components by the homotopy analysis method. It is well covered in the literature that the homotopy analysis method provides the solution in a rapidly convergent series where the series may lead to the solution in a closed form if it exists. The rapid convergence of the solution is guaranteed by work done by [29]. The homotopy analysis method provides an analytical solution by using the initial condition only. It produces an efficient explicit solution with high accuracy and minimal calculation. The homotopy analysis method proved by many authors to be reliable and promising. It can be used for all types of differential equations, linear or nonlinear, homogeneous or inhomogeneous. The technique has many advantages over the classical techniques. It avoids perturbation in order to find solutions of given nonlinear equations. In some cases it is impossible to find the exact values of the components of the solution. Thus we have to approximate the components numerically.

In this paper, we propose homotopy analysis method to solve second kind Volterra integral and integro-differential equations. We have introduced that the HAM is very powerful and efficient technique in finding analytical solutions for wide classes of second kind Volterra integral and integro-differential equations.

The organization of this paper is as follows.

In Section 2, we review the HAM, which is used to solve integral and integro-differential equations. Four numerical experiments are introduced in Section 3 and we have shown the graphical representation between exact solution and 3rd-order approximate solution homotopy analysis method using MATLAB (R2009a). Remark and Conclusion are given in Section 4 and Section 5

2. DESCRIPTION OF HOMOTOPY ANALYSIS METHOD

Consider

$$N[u(x)] = 0; \tag{1}$$

where, N is nonlinear operator, $u(x)$ is an unknown function of independent variable x.

Let $u_0(x)$ denote an initial approximation of $u(x)$ and L denote an auxiliary linear operator with the property,

$$L[f(x)] = 0 \text{ when } f(x) = 0 \tag{2}$$

Liao [19] constructs the so-called zero-order deformation equation

$$(1-q)L[\varphi(x;q)-u_0(x)] = qhH(x)N[\varphi(x;q)]; \tag{3}$$

where, $q \in [0,1]$ is an embedding parameter, $\varphi(x,q)$ is a function of x and q, h is a non-zero auxiliary parameter and $h(x)$ is a non-zero auxiliary function.

When $q = 0$, we have,

$$L[\varphi(x;q)-u_0(x)] = 0.$$

Using Eq. (2), we get

$$\varphi[(x;q)]_{q=0} = u_0(x). \tag{4}$$

When $q = 1$, we have,

$$N[\varphi(x;1)] = 0,$$

$$\Rightarrow \varphi[(x;q)]_{q=1} = u(x). \tag{5}$$

As the embedding parameter q increases from 0 to 1, the solution $\varphi(x;q)$ of the Eq. (1) depends upon the embedding parameter q and varies from the initial approximation $u_0(x)$ to the solution $u(x)$ of Eq. (1). In topology, such a kind of continuous variation is called deformation.

Now, expand $\varphi(x;q)$ in Taylor series with respect to q, we have

$$\varphi(x;q) = \varphi(x;0) + \sum_{m=1}^{\infty} \left.\frac{1}{m!}\frac{\partial^m \varphi(x;q)}{\partial q^m}\right\}_{q=0} q^m. \tag{6}$$

Define

$$u_m(x) = \left.\frac{1}{m!}\frac{\partial^m \varphi(x;q)}{\partial q^m}\right\}_{q=0}$$

and using Eq. (4), we get

$$\varphi(x;q) = u_0(x) + \sum_{m=1}^{\infty} u_m(x) \tag{7}$$

Assume that the auxiliary parameter h, the auxiliary function $H(x)$, the initial approximation $u_0(x)$ and the auxiliary linear operator L are so properly chosen that the series (7) converges at $q = 1$.

$$\therefore u(x) = u_0(x) + \sum_{m=1}^{\infty} u_m(x). \tag{8}$$

Now, using Eq. (5), we get,

$$u(x) = u_0(x) + \sum_{m=1}^{\infty} u_m(x). \tag{9}$$

The above expression provides us with a relationship between the initial guess $u_0(x)$ and the exact solution $u(x)$ by means of the terms $u_m(x) (m = 1,2,3,\dots)$, which are unknown up to now.

Define the vector

$$\vec{u}_n = \{u_0(x), u_1(x), \dots, u_n(x)\} \tag{10}$$

Differentiating Eq. (3) m-times with respect to q and then setting $q = 0$ and finally dividing by $m!$, we get so-called mth order deformation equation [19],

$$L[u_m(x) - \chi_m u_{m-1}(x)] = hH(x)R_m(\vec{u}_{m-1}), \tag{11}$$

subject to the initial condition

$$u_m(0) = 0, \tag{12}$$

where

$$R_m(\vec{u}_{m-1}) = \left.\frac{1}{(m-1)!}\frac{\partial^{m-1} N[\varphi(x;q)]}{\partial q^{m-1}}\right\}_{q=0} \tag{13}$$

and

$$\chi_m = \begin{cases} 0, & m \leq 1 \\ 1, & otherwise \end{cases}. \tag{14}$$

Taking L^{-1} on the both sides of Eq. (11), we get

$$u_m(x) = \chi_m u_{m-1}(x) + hL^{-1}[H(x)R_m(\vec{u}_{m-1})] + C_1, \tag{15}$$

where the constant C_1 is determined by the initial condition (12). In this way, it is easily to obtain $u_m(x)$ for $m \geq 1$.

Therefore, the mth-order approximation of $u(x)$ is given by,

$$u(x) = \sum_{m=0}^{k} u_m(x). \tag{16}$$

When $k \to \infty$, we get an accurate approximation of the original Eq. (1).

It is very important to note that the series solution of (1) is obtained by using the initial condition only. The solution (16) obtained by this method generally converge very rapidly in real physical problems. Furthermore, by the rapid convergence of the series solution, for sufficiently large values of k we can consider Eq. (16) as an approximation of the solution of (1). Therefore homotopy analysis method provides a numerical method for solving different classes of problems.

3. NUMERICAL EXPERIMENTS

In this section, we apply HAM to solve two nonlinear integral equations and two Integro-differential equations. These four examples are solved numerically in [38] by Adomian decomposition method.

3.1 Volterra integral equation of second kind:

Volterra integral equations of the second kind represented in the form:

$$u(x) = f(x) + \lambda \int_0^x K(x,t)\, u(t)dt, \tag{17}$$

where, $K(x,t)$ is the kernel of the integral equation and λ is a parameter.

As a first example, consider the following Volterra integral equation of the second kind [38]

$$u(x) = 1 - \int_0^x u(t)dt, \tag{18}$$

with the exact solution $u(x) = e^{-x}$.

Here we can noticed that $f(x) = 1, \lambda = -1$ and $K(x,t) = 1$.

Let $u_0(x) = 1$ and we define the auxiliary linear operator as $L[\varphi(x;q)] = \varphi[x;q]$.

We now define nonlinear operator as

$$N[\varphi(x;q)] = \varphi(x;q) - 1 + \int_0^x \varphi(t)dt. \tag{19}$$

Now, by assuming $H(x) = 1$, we construct so-called zero order deformation equation as

$$(1-q)L[\varphi(x;q) - u_0(t)] = qhN[\varphi(x;q)]. \tag{20}$$

Clearly, when $q = 0$ and $q = 1$, we have

$$\varphi(x;0) = u_0(x), \varphi(x;1) = u(x). \tag{21}$$

The m^{th}-order deformation equation is given by,

$$L[u_m(x) - \chi_m u_{m-1}(x)] = hR_m(\vec{u}_{m-1}(x)), \tag{22}$$

where

$$R_m(\vec{u}_{m-1}(x)) = u_{m-1}(x) - 1(1-\chi_m) + \int_0^x u_{m-1}(t)dt. \tag{23}$$

According to the homotopy analysis method we obtain the following components

$$u_1(x) = hx,$$
$$u_2(x) = hx + h^2x + \frac{h^2x^2}{2}, \qquad (24)$$
$$u_3(x) = hx + h^2x + \frac{h^2x^2}{2} + h^3x + h^2x + \frac{h^3x^2}{2} + \frac{h^2x^2}{2} + \frac{h^3x^2}{2} + \frac{h^3x^3}{6}$$

...

Therefore according to (16), we have

$$u(x) = 1 + hx + hx + h^2x + \frac{h^2x^2}{2} + hx + h^2x + \frac{h^2x^2}{2} + h^3x + h^2x + \frac{h^3x^2}{2} + \frac{h^2x^2}{2} + \frac{h^3x^2}{2} + \frac{h^3x^3}{6} + \cdots$$

By taking $h = -1$, we get

$$u(x) = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots$$

This has the closed form $u(x) = e^{-x}$, which is the exact solution of the problem.

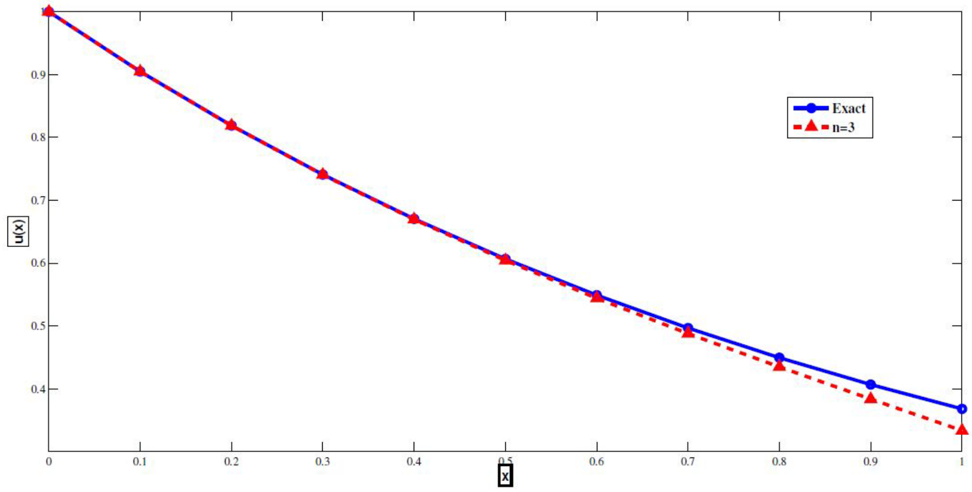

Figure-1: 3rd-order approximate solution by HAM and exact solution of (18)

As a second example, consider the following Volterra integral equation of second kind [38]

$$u(x) = 1 - x - \frac{x^2}{2} - \int_0^x (t - x)u(t)dt, \qquad (25)$$

with the exact solution $u(x) = 1 - \sinh x$.

In the given example, we have $f(x) = 1 - x - \frac{x^2}{2}$, $\lambda = -1, K(x,t) = t - x$.

We begin with

$$u_0(x) = 1 - x - \frac{x^2}{2}. \qquad (26)$$

For HAM solution, we choose the linear operator as

$$L[\varphi(x;q)] = \varphi(x;q). \qquad (27)$$

A nonlinear operator defined as

$$N[\varphi(x;q)] = \varphi(x;q) - 1 + x + \frac{x^2}{2} + \int_0^x (t - x)\varphi(t)\, dt. \qquad (28)$$

We construct the zero order deformation equation

$$(1 - q)L[\varphi(x;q) - u_0(x)] = qhH(x)N[\varphi(x;q)]. \qquad (29)$$

We can take $H(x) = 1$ and the m^{th}-order deformation equation is

$$L[u_m(x) - \chi_m u_{m-1}(x)] = hR_m(\vec{u}_{m-1}(x)), \qquad (30)$$

where

$$R_m\left(\vec{u}_{m-1}(x)\right) = u_{m-1}(x) - \left(1 - x - \frac{x^2}{2}\right)(1-\chi_m) + \int_0^x (t-x)u_{m-1}(t)dt. \tag{31}$$

Thus we have

$$u_1(x) = h\left[-\frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}\right],$$

$$u_2(x) = h\left[-\frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}\right] + h^2\left[\frac{x^2}{2!} + \frac{x^3}{3!} + \frac{2x^4}{4!} - \frac{x^5}{5!} - \frac{x^6}{6!}\right] \tag{32}$$

$$u_3(x) = h\left[-\frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}\right] + h^2\left[-x^2 + \frac{x^3}{3} + \frac{x^4}{3!} - \frac{x^5}{60} - \frac{x^6}{360}\right] + h^3\left[-\frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{8} - \frac{x^5}{60} - \frac{x^6}{240} + \frac{x^7}{5040} + \frac{x^8}{40320}\right]$$

...

Therefore the series solution expression by HAM can be written in the form

$$\begin{aligned} u(x) &= u_0(x) + u_1(x) + u_2(x) + u_3(x) + \cdots \\ &= 1 - x - \frac{x^2}{2!} + h\left[-\frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}\right] + h\left[-\frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}\right] \\ &\quad + h^2\left[\frac{x^2}{2!} + \frac{x^3}{3!} + \frac{2x^4}{4!} - \frac{x^5}{5!} - \frac{x^6}{6!}\right] + h\left[-\frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}\right] + h^2 \\ &\quad \left[-x^2 + \frac{x^3}{3} + \frac{x^4}{3!} - \frac{x^5}{60} - \frac{x^6}{360}\right] + h^3\left[-\frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{8} - \frac{x^5}{60} - \frac{x^6}{240} + \frac{x^7}{5040} + \frac{x^8}{40320}\right] + \cdots \end{aligned} \tag{33}$$

If we take $h = 1$, we have

$$\begin{aligned} u(x) &= 1 - x - \frac{x^3}{3!} - \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \\ &= 1 - \left[x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots\right] \end{aligned}$$

with the following closed form $u(x) = 1 - \sinh x$.

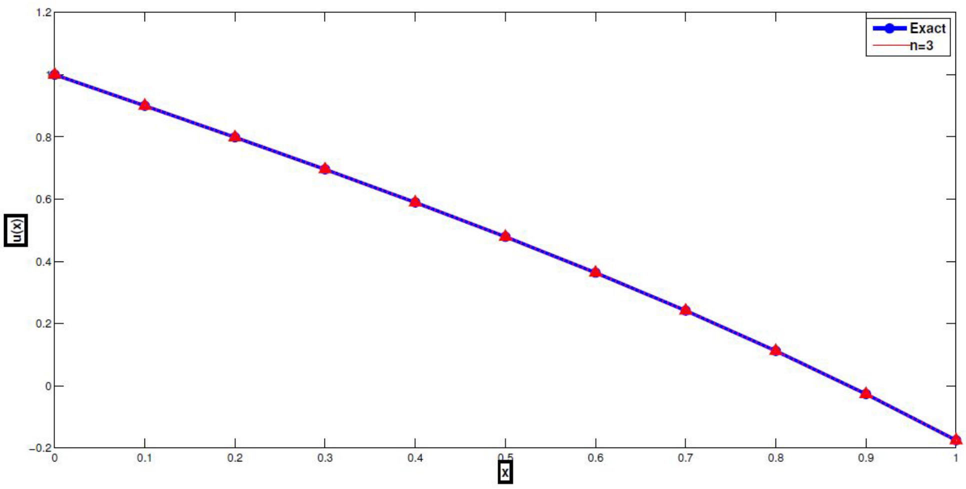

Figure-2: 3rd-order approximate solution by HAM and exact solution of (25)

3.2 Volterra Integro-Differential Equations of second kind

Volterra Integro-Differential Equations of second kind is represented in the form

$$u^{(n)}(x) = f(x) + \lambda \int_0^x K(x,t)u(t)dt, \tag{34}$$

where $u^{(n)}(x) = \frac{d^n u}{dx^n}$. The Volterra integro-differential equations may be observed when we convert an initial value problem to an integral equation by using Leibnitz rule. To determine a solution for the integro-differential equation, the initial conditions should be given, and this may be clearly seen as a result of involving $u(x)$ and its derivatives. The initial conditions are needed to determine the exact solution.

As a third example, consider the following Volterra integro-differential equation [38]

$$u'(x) = 1 - \int_0^x u(t)dt, u(0) = 0 \tag{35}$$

with the exact solution $u(x) = \sin x$.

To solve Eq. (34) by means of homotopy analysis method, according to the initial conditions denoted in Eq. (34), we choose

$$u_0(x) = x. \tag{36}$$

We choose the linear operator as

$$L[\varphi(x;q)] = \frac{\partial\varphi(x;q)}{\partial x} - 1 + \int_0^x \varphi(t)dt. \tag{37}$$

We construct zero order deformation equation as

$$(1-q)L[\varphi(x;q) - u_0(x)] = qhH(x)N[\varphi(x;q)]. \tag{38}$$

We can take $H(x) = 1$, and the mth-order deformation equation is,

$$L[u_m(x) - \chi_m u_{m-1}(x)] = hR_m(\vec{u}_{m-1}(x)). \tag{39}$$

subject to initial condition $u_m(0) = 0$, where

$$R_m(\vec{u}_{m-1}(x)) = \frac{d}{dx}u_{m-1}(x) - 1(1-\chi_m) + \int_0^x u_{m-1}(t)\,dt \tag{40}$$

According to the homotopy analysis method we obtain the following components

$$\begin{aligned} u_1(x) &= \frac{hx^3}{3!}, \\ u_2(x) &= \frac{hx^3}{3!} + \frac{h^2x^3}{3!} + \frac{h^2x^5}{5!}, \\ u_3(x) &= \frac{hx^3}{3!} + \frac{h^2x^3}{3} + \frac{h^3x^3}{6} + \frac{h^2x^5}{60} + \frac{h^3x^5}{60} + \frac{h^3x^7}{7!}, \end{aligned} \tag{41}$$

...

Thus we obtain

$$u(x) = x + \frac{hx^3}{3!} + \frac{hx^3}{3!} + \frac{h^2x^3}{3!} + \frac{h^2x^5}{5!} + \frac{hx^3}{3!} + \frac{h^2x^3}{3} + \frac{h^3x^3}{6} + \frac{h^2x^5}{60} + \frac{h^3x^5}{60} + \frac{h^3x^7}{7!} + \cdots$$

If we take $h = -1$, we get

$$u(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

So the solution in a closed form is $u(x) = \sin x$.

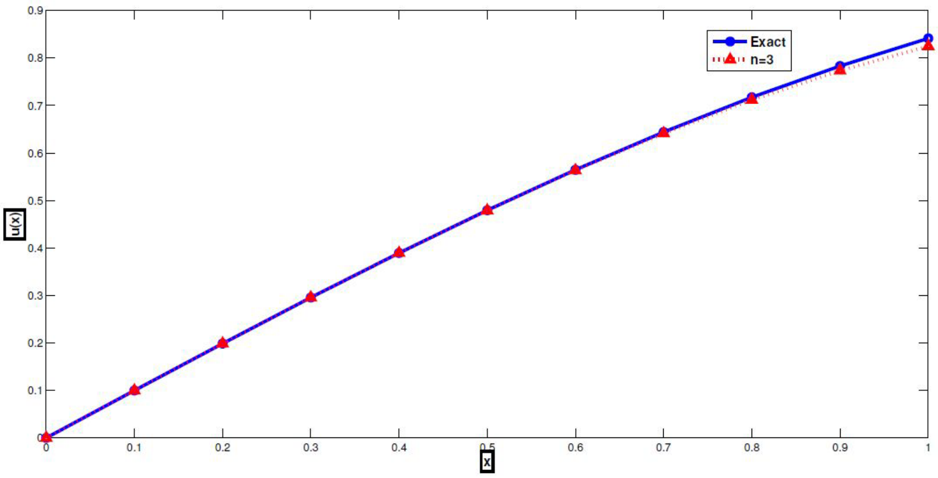

Figure-3: 3rd-order approximate solution by HAM and exact solution of (34)

As a fourth example, consider the following Volterra integro-differential equation of second kind [38]

$$u''(x) = 1 + x + \int_0^x (x-t)u(t)dt, u(0) = 1, u'(0) = 1. \tag{42}$$

Beginning with

$$u_0(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}. \tag{43}$$

We choose linear operator as

$$L[\varphi(x;q)] = \frac{\partial^2 \varphi(x;q)}{\partial x^2}, \tag{44}$$

with the property $L[C_1 x + C_2]$, where C_1 and C_2 are constants.

Now, we define a non-linear operator as

$$N[\varphi(x;q)] = \frac{\partial^2 \varphi(x;q)}{\partial x^2} - 1 - x - \int_0^x (x-t)\varphi(t)dt. \tag{45}$$

We construct zero order deformation equation as

$$(1-q)L[\varphi(x;q) - u_0(x)] = qhH(x)N[\varphi(x;q)]. \tag{46}$$

By assuming $H(x) = 1$, we obtain the mth-order deformation equation,

$$L[u_m(x) - \chi_m u_{m-1}(x)] = hR_m(\vec{u}_{m-1}(x)). \tag{47}$$

subject to initial conditions

$$u_m(0) = 0 \;\; u'_m(0) = 0, \tag{48}$$

where

$$R_m(\vec{u}_{m-1}(x)) = \frac{\partial^2 u_{m-1}}{\partial x^2} - (1+x)(1-\chi_m) + \int_0^x (x-t)\, u_{m-1}(t)dt. \tag{49}$$

Accordingly, the solution to the Eq. (42) for m > 1 becomes

$$u_m(x) = \chi_m u_{m-1}(x) + hL^{-1}[R_m\left(\left(\vec{u}_{m-1}(x)\right)\right] + C_1 x + C_2, \tag{50}$$

where C_1 and C_2 are determined by Eq. (48).

Therefore, using HAM, we obtain components of the solution successively as follows

$$u_1(x) = -\frac{hx^4}{4!} - \frac{hx^5}{5!} - \frac{hx^6}{6!} - \frac{hx^7}{7!},$$

$$u_2(x) = -\frac{hx^4}{4!} - \frac{hx^5}{5!} - \frac{hx^6}{6!} - \frac{hx^7}{7!} - \frac{h^2x^4}{4!} - \frac{h^2x^5}{5!} - \frac{h^2x^6}{6!} - \frac{h^2x^7}{7!} + \frac{h^2x^8}{8!} + \frac{h^2x^9}{9!} + \frac{h^2x^{10}}{10!} + \frac{h^2x^{11}}{11!},$$

$$\begin{aligned} u_3(x) = {} & \frac{hx^4}{4!} + \frac{h^2x^4}{12} + \frac{h^3x^4}{24} - \frac{hx^5}{5!} - \frac{h^2x^5}{60} - \frac{h^3x^5}{5!} - \frac{hx^6}{6!} - \frac{h^2x^6}{360} - \frac{h^3x^6}{720} + \frac{hx^7}{5040} \\ & - \frac{h^2x^7}{2520} - \frac{h^3x^7}{5040} - \frac{h^2x^8}{20160} - \frac{h^3x^8}{20160} + \frac{h^2x^9}{181440} + \frac{h^3x^9}{181440} + \frac{h^2x^{10}}{1814400} \\ & + \frac{h^3x^{10}}{1814400} + \frac{h^2x^{11}}{19958400} + \frac{h^3x^{11}}{19958400} + \frac{h^3x^{12}}{479001600} - \frac{h^3x^{13}}{6227020800} \\ & - \frac{h^3x^{14}}{87178291200} - \frac{h^3x^{15}}{1307674368000}, \\ & \ldots \end{aligned}$$

Then the series solution expression by HAM can be written in the form

$$\begin{aligned} u(x) &= u_0(x) + u_1(x) + u_2(x) + u_3(x) + \cdots \\ &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} - \frac{hx^4}{4!} - \frac{hx^5}{5!} - \frac{hx^6}{6!} - \frac{hx^7}{7!} - \frac{hx^4}{4!} - \frac{hx^5}{5!} - \frac{hx^6}{6!} - \frac{hx^7}{7!} - \frac{h^2x^4}{4!} \\ &\quad - \frac{h^2x^5}{5!} - \frac{h^2x^6}{6!} - \frac{h^2x^7}{7!} + \frac{h^2x^8}{8!} + \frac{h^2x^9}{9!} + \frac{h^2x^{10}}{10!} + \frac{h^2x^{11}}{11!}\frac{hx^4}{4!} + \frac{h^2x^4}{12} + \frac{h^3x^4}{24} \end{aligned}$$

$$-\frac{hx^5}{5!}-\frac{h^2x^5}{60}-\frac{h^3x^5}{5!}-\frac{hx^6}{6!}-\frac{h^2x^6}{360}-\frac{h^3x^6}{720}+\frac{hx^7}{5040}-\frac{h^2x^7}{2520}-\frac{h^3x^7}{5040}-\frac{h^2x^8}{20160}$$
$$-\frac{h^3x^8}{20160}+\frac{h^2x^9}{181440}+\frac{h^3x^9}{181440}+\frac{h^2x^{10}}{1814400}+\frac{h^3x^{10}}{1814400}+\frac{h^2x^{11}}{19958400}+\frac{h^3x^{11}}{19958400}$$
$$+\frac{h^3x^{12}}{479001600}-\frac{h^3x^{13}}{6227020800}-\frac{h^3x^{14}}{87178291200}-\frac{h^3x^{15}}{1307674368000}+\cdots$$

If we take $h=-1$, we get

$$u(x)=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\frac{x^5}{5!}+\frac{x^6}{6!}+\frac{x^7}{7!}+\frac{x^8}{8!}+\frac{x^9}{9!}+\frac{x^{10}}{10!}+\cdots,$$

which implies the following closed form of the solution

$u(x)=e^x$.

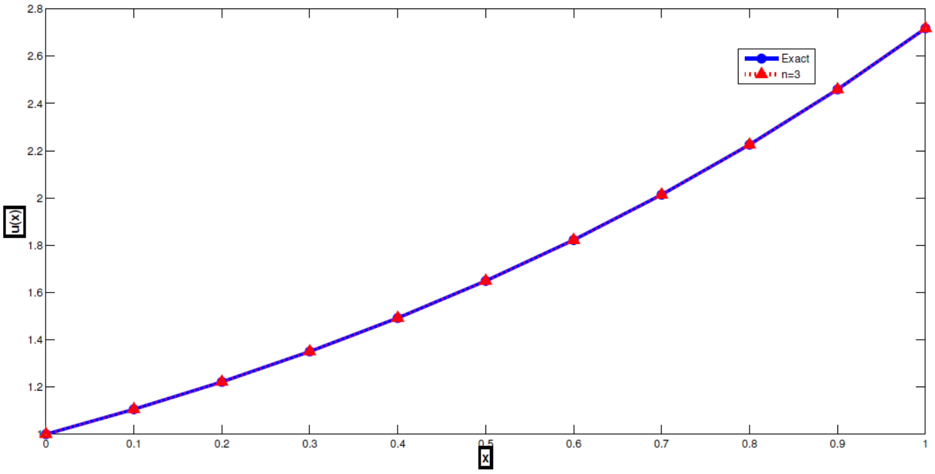

Figure-4: 3rd-order approximate solution by HAM and exact solution of (42)

4. REMARK

In Figure (1)-(4), we show the comparisons between the 3-term HAM solutions and the exact solutions. We observe that the results of the 3-term HAM are very close to the exact solutions which confirms the validity of the HAM.

All the numerical results obtained by the 3-term HAM are exactly same as the ADM solutions [38] for special case $h=-1, H(x)=1$. So, its means that the ADM is a special case of HAM. But HAM is more general and contains the auxiliary parameter h, which provides us to adjust and control the convergence region of solution series. As pointed out by Abbasbandy in [1], one had to choose a proper value of h to ensure the convergence of series solution for strongly integral equations.

5. CONCLUSION

In this paper, we have successfully used HAM for solving Volterra integral and integro-differential equations of second kind. It is apparently seen that HAM is a powerful and easy-to-use analytic tool for finding the solutions for wide classes of integral and integro-differential equations. It is worth pointing out that this method provides us a simple way to ensure the rapid convergence of series solutions, and therefore, HAM is valid even for strongly integral and integro-differential equations. Numerical experiments in comparison with other method such as ADM. They also do not require large computer memory and discrimination of the variable x. The results show the efficiency of the homotopy analysis method for solving integral and integro-differential equations.

REFERENCES

1. S. Abbasbandy, The application of Homotopy analysis method to nonlinear equations arising in heat transfer, Phys. Lett. A, 360 (2006), 10913.
2. S. Abbasbandy, Homotopy analysis method for the Kawahara equation, Nonlinear Anal.: Real World Appl., 11 (2010), 307-312.
3. S. Abbasbandy, The application of homotopy analysis method to solve a generalized Hirota-Satsuma coupled KdV equation, Phys. Latter A, 361 (2007), 478-483.
4. A. Adawi, F. Awawdeh, A numerical method for solving linear integral equations, Int. Jour. Contemp. Math. Sci., 10 (2009), 485-496.

5. A. Arikoglu, I. Ozkol, Solution of boundary value problems for integro-differential equations by using differential transform method, Appl. Math. Compu., 168 (2005), 1145-1158.
6. A. Avudainayagam, C. Vani, Wavelet-Galerkin method for integro-differential equations, Appl. Numer. Math., 32 (2000), 247-254.
7. Sh. Behzadi, The convergence of homotopy methods for nonlinear Klein Gordon equation, Jour. Appl. Math. Informatics, 28 (2010), 1227-1237.
8. H. Brunner, On the Numerical Solution of Nonlinear Volterra-Fredholm Integral Equations by Collocation Methods, SIAM Jour. Numer. Anal., 27 (1990), 987-1000.
9. H. Brunner, The numerical solution of weakly singular Volterra integral equations by collocation on graded meshes,Appl. Math. Compu., 45 (1985), 417-437.
10. P. Darnaia, A. Ebadian, A method for the numerical solution of the integro-differential equations, Appl. Math. Compu., 188 (2007), 657-668.
11. M. Dehghan, F. Shakeri, Solution of an integro-differential equation arising in oscillating magnetic field using He's homotopy Perturbation method, PIER, 78 (2008), 361-376.
12. S. El-Sayed, M. Abdel-Aziz, A comparison of Adomian's decomposition method and Wavelet-Galerkin method for integro-differential equations, Appl. Math. Compu., 136 (2003), 151-159.
13. M. El-Shahed, Application of He's Homotopy Perturbation Method to Volterra's Integro-differential equation, Int. Jour. Nonlinear Sci. Num. Simulat., 6 (2005), 163-168. 16
14. I. Hashim, Adomian decomposition method for solving BVPs for fourth-order integro- differential equations, Jour. Compu. Appl. Math., 193 (2006), 658-664.
15. J. He, Comparison of homotopy perturbation method and homotopy analysis method, Appl. Math. Compu., 156 (2004), 527-539.
16. S. Hosseini, S. Shahmorad, Numerical solution of a class of Integro-Differential equations by the Tau Method with an error estimation, Appl. Math. Compu., 136 (2003), 559-570.
17. H. Jafari, S. Seifi, Homotopy Analysis method for solving linear and non-linear diffusion-wave equation, Commun. Nonlinear Sci. Num. Simulat., 14 (2009), 2006-2012.
18. S. Liao, A kind of approximate solution technique which does not depend upon small parameters, a special example, Int. Jour. Nonlinear Mech., 30 (1995), 371-380.
19. S. Liao, Beyond Perturbation: Introduction to the Homotopy Analysis Method, Chapman and Hall, CRC Press, Boca Raton, (2003).
20. S. Liao, On the homotopy analysis method for nonlinear problems, Appl. Math. Compu., 147 (2004), 499-513.
21. S. Liao, Y. Tan, A general approach to obtain series solutions of nonlinear differential equations, Stud. Appl. Math., 119 (2007), 297-354.
22. S. Liao, Notes on the homotopy analysis method: some definitions and theorems, Commu. Nonlinear Sci. Number Simulat., 14 (2009), 983-997.
23. K. Maleknejad, Y. Mahmoudi, Taylor polynomial solution of high-order nonlinear Volterra-Fredholm integro-differential equations, Appl. Math. Compu., 145 (2003), 641-653.
24. K. Maleknejad, F. Mirzaee, S. Abbasbandy, Solving linear integro-differential equations system by using rationalized Haar function method, Appl. Math. Compu., 155 (2005), 317- 328.
25. K. Maleknejad, N. Agazadeh, Numerical solution of Volterra integral equations of the second kind with convolution kernel by using Taylor-series expansion method, Appl. Math. Compu., 161 (2005), 915-922.
26. K. Maleknejad, S. Sohrabi, Y. Rostami, Numerical solution of nonlinear Volterra integral equations of the second kind by using Chebyshev polyomials, Appl. Math. Compu., 188 (2007), 123-128. 17
27. K. Maleknejad, E. Hashemizadeh, R. Ezzati, A new approach to the numerical solution of Volterra integral equations by using Bernstein's approximation, Commu. Nonlinear Sci. Num. Simul., 16 (2011), 647-655.
28. A. Molabahrami, F. Khani, The homotopy analysis method to solve the Burgers-Huxley equation, Nonlinear Anal.: Real World Appl., 10 (2009), 589-600.
29. Z. Odibat, A study on the convergence of homotopy analysis method, Appl. Math. Compu., 217 (2010), 782-789.
30. M. Rabbani, K. Maleknejad, N. Aghazadeh, Numerical computational solution of the Volterra integral equations system of the second kind by using an expansion method, Appl.Math. Compu., 187 (2007), 1143-1146.
31. M. Rashed, Lagrange interpolation to compute the numerical solutions of differential, integral and integro-differential equations, Appl. Math. Compu., 151 (2004), 869-878.
32. E. Rawashdeh, Numerical solution of fractional integro-differential equation by collocation method, Appl. Math. Compu., 176 (2006), 1-6.
33. O. Samuel, Q-homotopy analysis method and application to fingero-imbibition phenomena in double phase flow through porous media, Asian Jour. Curr. Eng. Math. 2, 4 (2013), 283-286.
34. N. Sweilam, Fourth order integro-differential equations using variational iteration method, Compu. Math. Appl., 54 (2007), 1086-1091.
35. Y. Tan, S. Abbasbandy, Homotopy analysis method for quadratic Riccati differential equation, Comm. Nonlinear Sci. Num. Simul., 13 (2008). 539-546.

36. V. Volterra, Theory of functionals of integral and integro-differntial equations, Dover, New York, (1959).
37. A. Wazwaz, Two methods for solving integral equations, Appl. Math. Compu., 77 (1996), 79-89.
38. A. Wazwaz, Linear and Nonlinear Integral Equations: Methods and Applications, Higher education press, Springer, (2011).
39. A. Wazwaz, A reliable algorithm for solving boundary value problems for higher-order integro-differential equations, Appl. Math. Compu., 118 (2001), 327-342. 18.
40. L. Xu, Variational iteration method for solving integral equations, I. Jour. comp. math. appl., 54 (2007), 1071-1078.

Source of support: Nil, Conflict of interest: None Declared