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ON RAINBOW CONNECTIONS OF SOME CLASSES OF GRAPHS

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ABSTRACT

The rainbow connection number, rc(G), of a connected graph G is the minimum number of colors needed to color its edges, so that every pair of vertices is connected by at least one path in which no two edges are colored the same. And the strong rainbow connection number, src(G) of a connected graph G is the minimum number of colors needed to color its edges, so that every pair of vertices is connected by at least one geodesic in which no two edges are colored the same.

In this paper we determine the rainbow connection number and strong rainbow connection number of the barbell graph, lollipop graph and tadpole graph and the rainbow connection number of the power graph of a path. Also we have modified the rainbow connection number and strong rainbow connection number of a sun graph, given by Gema, Lyra and Sy in [3].

***Key Words**: rainbow coloring, strong rainbow coloring, rainbow connection number, strong rainbow connection number, sun graph, barbell graph, lollipop graph, tadpole graph, power graph of a path.*

1. INTRODUCTION

The concept of rainbow connection was introduced by Chartrand *et al.* in 2008. It actually came as a graph theoretical model of the following situation:

The Department of Homeland Security of USA was created in 2003 in response to the weaknesses discovered in the transfer of classified information after the September 11, 2001 terrorist attacks. Ericksen made the following observation: An unanticipated aftermath of those deadly attacks was the realization that law enforcement and intelligence agencies couldn't communicate with each other through their regular channels, from radio systems to databases. The technologies utilized were separate entities and prohibited shared access, meaning that there was no way for officers and agents to cross check information between various organizations.

While the information needs to be protected since it relates to national security, there must also be procedures that permit access between appropriate parties. This two-fold issue can be addressed by assigning information transfer paths between agencies which may have other agencies as intermediaries while requiring a large enough number of passwords and firewalls that is prohibitive to intruders, yet small enough to manage (that is, enough so that one or more paths between every pair of agencies have no password repeated). An immediate question arises: What is the minimum number of passwords or firewalls needed that allows one or more secure paths between every two agencies so that the passwords along each path are distinct?

The rainbow connection number is defined as follows:

For a nontrivial connected graph G and a positive integer k, let c: $E(G) \rightarrow \{1,2, \dots ,k\}$ be an edge coloring of G, where adjacent edges of G are permitted to be colored the same. A path in this edge colored graph G is called a rainbow path if no two of its edges are assigned the same color. The graph G is rainbow- connected (with respect to c) if G contains a rainbow u-v path for every pair u, v of vertices of G. And such a coloring c is called a rainbow coloring of G. The minimum k for which there exists a rainbow k-coloring of the edges of G is the rainbow connection number *rc(G*) of G [2]. A rainbow coloring of G using rc(G) colors is called a minimum rainbow coloring of G.

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The graph G is *strongly rainbow connected* if G contains a rainbow *u-v* geodesic for every two vertices *u* and *v* of G. In this case, the coloring *c* is called a *strong rainbow coloring* of G. The minimum *k* for which there exists a coloring $c: E(G) \rightarrow \{1,2, \dots ,k\}$ of the edges of G such that G is strongly rainbow connected is the *strong rainbow connection number src(G) of G[2]*.

Since every coloring that assigns distinct colors to the edges of a connected graph is both a rainbow coloring and a strong rainbow coloring, every connected graph is rainbow connected and strongly rainbow connected with respect to some coloring of the edges of G. Thus the rainbow connection numbers *rc(G)* and *src(G)* are defined for every connected graph G.

2. PRELIMINARIES

In this section we include the definitions of Power graph of a path, Sun graph, Barbell graph, Lollipop graph, and Tadpole graph [3], [4], [5].

Definition: 2.2 [4] (Power of a graph). The p^{th} Power of a graph G, denoted by G^p where p ≤ 1, is defined as follows: $V(G^p) = V(G)$. Two vertices u and v are adjacent in G^p if and only if the distance between vertices u and v in G, $dist\ G(u, v) \leq p$.

In this paper we denote the p^{th} power of the path with n vertices as $P_n^{\,p}$.

Definition: 2.3 [3] The Sun graph is the graph obtained from a cycle $C_n = v_1, v_2, \dots, v_n, v_1$ by adding *n* vertices u_i and *n* pendant edges $e_i = v_i u_i$ for $n = 1, 2, \dots, n$.

In this paper we denote the sun graph obtained from the cycle C_n as S_n.

Definition: 2.4 [5] An n-Barbell graph is a simple graph obtained by connecting two copies of a complete graph K_n by an edge.

Definition: 2.5 [5] A lollipop graph $L_{m,n}$ is a simple graph obtained by joining one copy of a complete graph K_n and a copy of a path P_n by an edge.

Remark: 2.1 Note that a lollipop graph $L_{m,n}$ will have diameter $n + 1$:

Definition: 2.6 [5] A tadpole graph (Dragon graph), $T_{m,n}$ is a simple graph obtained by joining one copy of a cycle C_m and a copy of a path P_n by an edge.

Remark: 2.2 Note that the diameter of a tadpole graph $T_{m,n}$ is $\lfloor m/2 \rfloor + n$.

Also note that,

Remark: 2.3 [2] If $G(n, m)$ is a nontrivial connected graph of whose diameter (the largest distance between two vertices of G) denoted by *diam(G),* then

$$diam(G) \leq rc(G) \leq src(G) \leq m. (1)$$

3. MAIN RESULTS

This section includes the rainbow connection number and strong rainbow connection number of the Power graph of a path, Lollipop graph, Tadpole graph and n-Barbell graph. Also for n > 2, we discuss a modification for the rainbow connection number of a Sun graph S_n given by Gema, Lyra and Sy [3].

In the following proposition, given by Chartrand *et al.* [1], it is determined which connected graphs G attain the extreme values 1, 2 and m.

Proposition: 3.1 [2] Let G be a nontrivial connected graph of size m. Then
(a) *src(G)* = 1 if and only if G is a complete graph,
(b) *rc(G)* = 2 if and only if *src(G) = 2*,
(c) *rc(G)* = m if and only if G is a tree.

In the same paper they have determined the rainbow connection number and the strong rainbow connection number of the cycle with n vertices,

Proposition: 3.2 [2] For each integer $n \geq 4$, $rc(C_n) = src(C_n) = \lceil n/2 \rceil$.

Theorem: 3.1 *Let P_n, $n \geq 2$ be a path and let p be any positive integer. Then* $rc(P_n^p) = \left\lceil \frac{rc(P_n)}{p} \right\rceil$

Proof: *Let $P_n = v_1, v_2, \ldots, v_n$ be a path of length n and let $p \geq 2$* be any positive integer. Let $\left\lceil \frac{rc(P_n)}{p} \right\rceil = k$. Note that for a path P_n, $rc(P_n)=n-1=diam(P_n)$. And so $\left\lceil \frac{rc(P_n)}{p} \right\rceil = \left\lceil \frac{diam\ (P_n)}{p} \right\rceil = diam(P_n^p)$. Then by Remark 2.3 to prove the result it is enough to show that there exists a rainbow coloring of P_n^p using k colors.

The proof is by induction on n. If $n \leq p+1$ *then,* $P_n^p \cong K_n$ and hence by Proposition 3.1, we get

$$rc(P_n^p) = 1 = \left\lceil \frac{n-1}{p} \right\rceil = k$$

Now assume that the result is true for all graphs, P_n^p with $n < N$. Consider the graph P_N^p. P_N^p can be obtained by inserting a vertex v_N in P_{N-1}^p, which is adjacent to the vertices $v_{N-1}, v_{N-2}, \ldots, v_{N-p}$. Let $rc(P_{N-1}^p) = \left\lceil \frac{N-2}{p} \right\rceil = d$. Since the vertex v_N is adjacent to the vertices $v_{N-1}, v_{N-2}, \ldots, v_{N-p}$, there exists a rainbow path of length 1 between v_N - v_j; for $N-1 \leq j \leq N-p$. Let $\{c_i: 1 \leq i \leq n\}$ be the set of colors. Now, consider the two cases,

Case (i): *diam* $(P_N^p) = \left\lceil \frac{N-1}{p} \right\rceil = d.$

Hence by Remark 2.3, we have, $rc(P_N^p) \geq d$. Color all the adjacent vertices of v_N with a color c_d. So there exists a rainbow path between the vertices $v_N - v_j$, *for* $N-1 \leq j \leq N-p$. Now, if we remove the vertices v_l, $N-1 \leq l \leq N-p$ from P_N^p, the resulting graph, $G = P^p_{N-p}$, will have a diameter $d-1$since, $diam(P_N^p)$ and hence by induction $rc(G)=d-1$. So, there exists a rainbow path between the vertices of P^P_{N-p} with atmost $d-1$ colors. That is a rainbow path in P_N^p as well. And the rainbow path between any vertex $v \in V(G) \backslash \{v_{N-p}\}$ and v_{N-p} along with the edge $v_{N-p}v_N$ will be a rainbow path in P_N^p between the vertices in G and v_N. And if we remove the vertex v_N from P_N^p we get P_{N-1}^p and hence by induction there exists a rainbow coloring of P_{N-1}^p using d colors. So there exists a rainbow path between every pair of vertices $v_1, v_2, \ldots, v_{N-1}$ in P_N^p as well as in P_{N-1}^p.

Hence $rc(P_N^p) \leq d$:

Case (ii): *diam* $(P_N^p) = \left\lceil \frac{N-1}{p} \right\rceil = d+1$

Hence by Remark 2.3, we have, $rc(P_N^p) \geq d+1$. Color all the adjacent vertices of v_N with a color c_{d+1}. So there exists a rainbow path between the vertices $v_N - v_j$, *for* $N-1 \leq j \leq N-p$. If we remove the vertex v_N from P_N^p we get the graph $G' = P_{N-1}^p$, which has rainbow connection number d (by induction). Hence there exists a rainbow path between every pair of vertices $v_1, v_2, \ldots, v_{N-1}$ in P_N^p as well as in P_{N-1}^p. Then the path v_i-v_{N-1}, $1 \leq i \leq N-2$ in P_{N-1}^p along with the edge v_{N-1}-v_N of color c_{d+1} will form a rainbow path v_i-v_N, $1 \leq i \leq N-2$ in ${}^PN_p$.

Hence $rc(P_N^p) \leq d+1$.

Hence,

$$rc(P_n^p) = \left\lceil \frac{rc(P_n)}{p} \right\rceil \tag{2}$$

Remark: 3.1 Since such a coloring gives a strong rainbow coloring as well using Remark 2.3 and Theorem 3.1 we have,

$src(P_n^p) = \left\lceil \frac{rc(P_n)}{p} \right\rceil$.

Theorem: 3.2 *Let G be an n- barbell graph wih $n \geq 2$. Then, $rc(G) = src(G) = 3$.*

Proof: Since the diameter of the n-barbell graph, $n \geq 2$ *is* 3, using Remark 2.3 and the coloring.

$$c(e)=\begin{cases} 1 if\ e \in E(K_n^1) \\ 2\ if\ e \in E(K_n^2) \\ 3\ \ if\ e\ is\ a\ bridge \end{cases}$$

where (K_n^1) $i \in \{1,2\}$ denote the i^{th} copy K_n, we have the result.

Theorem: 3.3 *For a lollipop graph with $n \geq 1$, $m \geq 2$, $rc(L_{m,n}) = src\,(L_{m,n}) = n+1$.*

Proof: From *Remark 2.1* we have, $diam\,(L_{m,n}) = n+1$. So, it is enough to show that there exists a rainbow coloring of $L_{m,n}$ with $n+1$colors. Let $v_1,v_2,...,v_m$ be the vertices of K_m and $u_1,u_2,...,u_n$ be the vertices of P_n. Let v_m be made adjacent to u_1 form $L_{m,n}$. So there exists exactly one path connecting the vertices u_n and v_m and it will be of length (n-1) +1 = n. Now color all these edges with different colors. Since the vertex v_m is adjacent to the vertices $v_1,v_2,...,v_{m-1}$ there will be a rainbow path between the vertices of P_n and there vertices $v_1,v_2,...,v_{m-1}$ if we color all these edges adjacent to v_m using an another $(n+1)^{th}$ color.

Hence the result follows.

Theorem: 3.4 *For a tadpole graph $T_{m,n}$ with $n \geq 1$, $m \geq 3$*

$$src(T_{m,n}) = rc\,(T_{m,n}) = \lfloor m/2 \rfloor + n$$

Proof: Let $u_1, u_2,...,u_n$ be the vertices of the path and let $v_1, v_2,..., v_m$ be the vertices of the cycle and the vertices v_m and u_1 are made adjacent to form $T_{m,n}$. Then u_1-u_2- ... -u_n –v_m with form a path of length n in $T_{m,n}$ and the there exists exactly one path between v_m and u_1. So we should color all these n edges with n different colors to get a rainbow v_m-u_1 path in $T_{m,n}$. Color the path v_m - v_1- v_2-...-$v_{\lfloor n/2 \rfloor}$ with colors say $n+1, n+2, ..., n+\lfloor m/2 \rfloor$ respectively and color the edges in the path v_m, - v_{m-1},-...-, -$v_{\lceil m/2 \rceil}$ with colors $n+\lfloor m/2 \rfloor$, $n+\lfloor m/2 \rfloor -1, ..., n+2, n+1$ respectively. Now color the remaining edge, if there is any, with any color $k \in \{1,2,...,n\}$. This will form a rainbow coloring as well as a strong rainbow coloring of $T_{m,n}$. Then the result follows using *Remark 2.2 and Remark 2.3*.

Remark: 3.2 [3] The rainbow connection number and strong rainbow connection number of a sun graph S_n, $n \geq 2$ are, $\lfloor n/2 \rfloor + n$.

Note that this result is true only for $n = 2$. In the case of $n > 2$ *is* discussed as follows:

Theorem: 3.5 *The rainbow connection number and the strong rainbow connection number for a sun graph Sn, $n \geq 2$ are,*

$$rc(S_n)=src\,(S_n)= \begin{cases} n \text{ if } n \text{ is odd,} \\ n+1 \text{ if } n \text{ is even} \end{cases}$$

Proof: Let S_n consists of the cycle C_n: $u_1 - u_2 - ... -u_n - u_{n+1} = u_1$,$n$ pendant vertices $v_1, v_2,...,v_n$ and edges $e_i = v_iu_i$, $1 \leq i \leq n$.

Claim: $rc(S_n) \geq n$.

Suppose on the contrary that $rc(S_n) \leq n-1$. Let c be such a rainbow coloring of S_n. Now since S_n contains n pendant vertices there must exists two pendant edges say $e = v_iu_i$ and $f = v_ju_j$, $1 \leq i, j \leq n$, $i \neq j$ with $c(e)= c\,(f)$.Now since u_i and u_j are pendant vertices any u_i - u_j path must contain the edges *e and f*. And so we can not have a rainbow u_i - u_j path in S_n. Hence the claim.

Now, consider the following cases;

Case (i): n is odd.

Consider the coloring $c\square: E(S_n) \rightarrow \{1,2,3,..., n\}$ defined by,

$$c\square(e)= \begin{cases} i & if\ e = v_i\,u_{i+1},\ 1 \leq i \leq n, \\ i + \lceil n/2 \rceil & if\ e = u_i\,u_{i+1},\ 1 \leq i \leq \frac{n-1}{2} \\ i - \lfloor n/2 \rfloor & if\ e = u_i\,u_{i+1},\ \frac{n-1}{2} \leq i \leq n. \end{cases}$$

Then clearly $c\square$ will give a rainbow coloring for S_n. Hence $rc(S_n) \leq n$, when n is odd. So we have when n is odd, $rc(S_n) = n$.

Case (ii): n is even

Consider the coloring $c^*: E(S_n) \rightarrow \{1,2,3,..., n, n+1\}$ defined by,

$$c^*(e)= \begin{cases} i & if\ e = v_i\,u_{i+1},\ 1 \leq i \leq n, \\ i + \frac{n}{2} + 1 & if\ e = u_i\,u_{i+1},\ 1 \leq i \leq \frac{n}{2} \\ i - \frac{n}{2} & if\ e = u_i\,u_{i+1},\ \frac{n}{2} + 1 \leq i \leq n. \end{cases}$$

Hence c^* is a n +1 – rainbow coloring of S_n, we have, $n \leq rc(S_n) \leq n+1$.

Claim: $rc(S_n) = n+1$.

Suppose on the contrary that $rc(S_n) = n$. Let c^{**} be a n – rainbow coloring of S_n. Then since no two pendant edge can have the same color, without loss of generality let $c^{**}(u_i v_i) = i,\ 1 \leq i \leq n$.

Each edge in the cycle should colored differently. *i.e.*, $c^{**}(g) \neq c^{**}(h)$, *for* $g \neq h$ *and* $g, h \in E(C_n)$. Suppose not. Let there exists two edges say g and h in $E(C_n)$ such that $c^{**}(g) = c^{**}(h) = k,\ 1 \leq k \leq n$. Let $g = u_i u_{i+1}$ *and* $h = u_j u_{j+1}$ *for some* $i \neq j,\ 1 \leq i, j \leq n$. Without loss of generality let i < j. Then any path connecting v_{i+1} and v_k will contain either g or h. So there does not exist a rainbow $v_k u_{i+1}$ path in S_n.

Now consider any vertex $v_a,\ 1 \leq a \leq n/2$. Let $v_1,\ 1 \leq l \leq n$ be the eccentric vertex of v_a. Since any $v_a - v_l$ path in S_n has length, the edges in the path $v_a - v_l$ (without loss of generality, let it be $u_a - u_{a+1}$ - .... – u_l) must be colored with $n/2$ different colors form the set $\{1, 2, \ldots, n\} \setminus \{a, l\}$. So at least one of the edges in this path should be colored with a color p such that $a \leq p \leq l$... So we can not have either a rainbow $v_a - v_p$ path or a rainbow $v_p - v_l$ path through any of the vertices u_{a+1}, u_{a+2} -... – u_{l-1}. And also since the path u_{l+1}- u_{l+1} -...-u_a must contain the edges colored with the colors a and l, we can never have a rainbow $v_k v_p$ path as well as a rainbow $v_p v_l$ path which has common edges with the path u_{l+1}- u_{l+1} -...-u_a. Thus we cannot have either a rainbow $v_a - v_p$ path or a rainbow v_p-v_i path. So, $rc(S_n) \neq n$. And so it must be $n+1$.

Now since c' and c* are strong rainbow coloring as well, using *Remark 2.3* the result followes:

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