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# $R_\alpha$— OPEN SETS IN TOPOLOGICAL SPACES

S. P. NANDHINI*[1], M. GEETHA[2]

[1]Assistant Professor of Mathematics,
The Standard Fireworks Rajaratnam College for Women, Sivakasi-626 123, Tamilnadu, India.

[2]Master of Degree in Mathematics,
The Standard Fireworks Rajaratnam College for Women, Sivakasi-626 123, Tamilnadu, India.

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***ABSTRACT***

*In this paper, we investigate a new class of regular open sets called $R_\alpha$-open sets in topological spaces and its properties are studied.*

***Keywords:*** *Regular open sets, α-closed sets, $R_\alpha$-open sets.*

## 1. INTRODUCTION AND PRELIMINARIES

Throughout this paper, a space means a topological space on which no separation axioms are assumed unless otherwise explicitly stated. In 1963 Levine [10] initiated semi open sets and gave their properties. Mathematicians gave in several papers interesting and different new types of sets. In 2007, A.H.Shareef [14] initiated regular open sets and gave their properties. In 1965, O. Njastad [12] introduced α- closed sets. We recall the following definitions and characterizations. The closure (resp., interior ) of a subset A of X is denoted by cl A (resp., int A),A subset A of X is said to be regular open [14] (resp, semi open [10],pre open [11], α- open [12]) set if A= int cl A (resp., A⊂ cl int A, A⊂ int clA, A⊂ int cl int A) The complement of regular open (resp., semi open, pre open, α- open) set is said to be regular closed (resp., semi closed, pre closed, α- closed) The intersection of all regular closed (resp., semi closed, pre closed, α-closed) sets of X containing A is called regular closure (resp., semi closure, pre closure, α-closure) and denoted by rcl A (resp., scl A, pcl A, αcl A). The union of all semi open (resp., pre open, α- open) sets of X contained in A is called the semi interior (resp., pre interior, α-interior) and denoted by s int A (resp., p int A, α int A). The family of all regular open (resp., semi open, pre open, α- open, semi closed, pre closed, α- closed, regular closed) subsets of a topological space X is denoted by RO(X) (resp., SO(X), PO(X), αO(X), SC(X), PC(X), αC(X),RC(X) ).

**Definition: 1.1** A topological space (X, τ) is said to be

1. Extremally disconnected of cl V∈τ, for every V∈τ.
2. Locally indiscrete if every open subset of X is closed.

**Lemma: 1.2**

1. If X is a locally is indiscrete space, then each semi open subset of X is closed and hence each semi closed subset of X is open [3].
2. A topological space X is hyperconnected if and only if RO(X) = {∅, X} [7]

**Theorem: 1.3** Let (X, τ) be a topological space. Then SO(X, τ) =SO(X, αO(X)) [4].

**Theorem: 1.4** Let (X, τ) be a topological space.

1. Let A⊂X. Then A∈RO (X, τ) if and only if int A = int cl A.
2. If {Aγ: γ∈Γ} is a collection of regular open sets in a topological space (X, τ), then ∪{Aγ: γ∈Γ} is regular open.

**Proof:** Obvious.

*Corresponding Author: S. P. Nandhini*[1], [1]Assistant Professor of Mathematics,*
*The Standard Fireworks Rajaratnam College for Women, Sivakasi-626 123, Tamilnadu, India.*

**Theorem: 1.5** Let (X, τ) be a topological space. If A∈τ, and B∈RO(X), then A∩B∈RO(X).

**Proof:** Given (X, τ) is a topological space and A∈τ, and B∈RO(X)**.** Let x ∈A∩B then x∈A and x ∈B.
Since B∈RO(X)⇒x∈A∩B⊂B∈RO(X).Hence A∩B∈RO(X).

**Result: 1.6** Every closed set is α-closed.

**Theorem: 1.7[8]** A space X is extremally disconnected if and only if RO(X) =RC(X).

**2. $R_\alpha$-OPEN SETS**

In this section, we introduce and study the concept of $R_\alpha$- open sets in topological spaces and study some of its properties.

**Definition: 2.1** A regular open set A of a topological space X is said to be $R_\alpha$-open if for each x∈A, there exists a α-closed set F such that x∈F⊆A. A subset B of a topological space X is $R_\alpha$-closed, if X-B is $R_\alpha$-open.
The family of $R_\alpha$-open subsets of X is denoted by $R_\alpha O(X)$.

**Theorem: 2.2** A subset A of a topological space X is $R_\alpha$-open if and only if A is regular open and it is a union of α-closed sets.

**Proof:** Let A be $R_\alpha$-open. Then A is regular open x∈A implies, there exists α-closed set $F_x$ Such that $x\in F_x\subset A$

Hence $\bigcup_{x\in A} F_x\subset A$. But x∈A, $x\in F_x$ implies $A\subset \bigcup_{x\in A} F_x$. This completes one half of the proof.

Let A be regular open and $A=\bigcup_{i\in I} F_i$, where each $F_i$ is α-closed. Let x∈A. Then x belongs to some $F_i \subset A$. Hence A is $R_\alpha$-open.

The following result shows that any union of $R_\alpha$-open sets is $R_\alpha$-open.

**Theorem: 2.3** Let $\{A_\alpha : \alpha\in\Delta\}$ be a family of $R_\alpha$-open sets in a topological space X. Then $\bigcup_{\alpha\in\Delta} A_\alpha$ is an $R_\alpha$-open set.

**Proof:** The union of an arbitrary regular open sets is regular open by theorem 1.4.

Suppose that $x\in \bigcup_{\alpha\in\Delta} A_\alpha$ . This implies that there exists $\alpha_0\in\Delta$ such that $x\in A_{\alpha_0}$ and as $A_{\alpha_0}$ is an $R_\alpha$-open set, there exists a α-closed set F in X such that $x\in F\subset A_{\alpha_0}\subset \bigcup_{\alpha\in\Delta} A_\alpha$ . Therefore $\bigcup_{\alpha\in\Delta} A_\alpha$ is a $R_\alpha$-open set.

From theorem 2.3, it is clear that any intersection of $R_\alpha$-closed sets of a topological space X is $R_\alpha$-closed. The following example shows that the intersection of two $R_\alpha$-open sets need not be $R_\alpha$-open.

**Example: 2.4** Let X= {a, b, c} and τ = {∅, {a}, {b}, {a, b}, X}
$R_\alpha$-open sets = {Φ, {a, c}, {b, c}, X},{a, c} ∩{b, c}={c} is not an $R_\alpha$-open set

**Theorem: 2.5** A subset G of the topological space X is $R_\alpha$-open if and only if for each x∈G, there exists an $R_\alpha$-open set H such that x∈H⊂G.

**Proof:** Let G be a $R_\alpha$-open set in X. Then for each x∈G, we have G is an $R_\alpha$-open set such that x∈G⊂G. Conversely, let for each x∈G, there exists an $R_\alpha$-open set H such that x∈H⊂G. Then G is a union of $R_\alpha$-open sets, hence by theorem 2.3, G is $R_\alpha$-open.

**Theorem: 2.6** If the subset A of X is pre open and semi closed then A is $R_\alpha$ –open set.

**Proof:** Let A be semi closed and pre open set in a topological space X. Then Int(Cl(A))⊆A and A⊆Int(Cl(A)) respectively. This implies that A=Int(Cl(A)).Therefore A is regular open. Then there exists α-closed set F such that x∈F⊆A. Hence A is $R_\alpha$ –open.

**Theorem: 2.7** If a space X is a T1 –space, then $R_\alpha O(X)$ = RO(X).

**Proof:** $R_\alpha O(X)\subset$ RO(X). Let A∈RO(X). Let x∈A. As X is a T1 –space, {x} is closed. Every closed set in X is α-closed. Hence x∈{x} ⊂ A∈ $R_\alpha O(X)$.This completes the proof.

 

**Theorem: 2.8** If the family of all regular open subsets of a topological space is a topology on X, then the family of $R_\alpha O(X)$ is also a topology on X.

**Proof:** Obvious.

**Theorem: 2.9** If a topological space X is locally indiscrete, then every regular open set is $R_\alpha$ –open.

**Proof:** Let A be a regular open set in X. Then A=Int(Cl(A)). A is also open in X, Int A=A. We get Int(A)= Int(Cl(A)). As X is locally indiscrete, Int(A) is closed. Hence Int(A)=Cl(Int(A)). So, Int(Cl(A))= Cl(Int(A)). Then there exists α-closed set F such that $x\in F\subseteq A$. Hence A is $R_\alpha$ –open.

**Theorem: 2.10** If a topological space X is locally indiscrete, then every regular closed set is $R_\alpha$ –open.

**Proof:** Let A be a regular closed set in X. Then A=Cl(Int(A)). As X is locally indiscrete, Int(A) is closed. Hence Int(A)=Cl(Int(A)). So, Cl(Int(A))=Int(A)⊂A. Then there exists α- closed set F such that $x\in F\subseteq A$. Hence A is $R_\alpha$ –open.

**Theorem: 2.11** If a topological space (X, τ) is locally indiscrete, then $\tau\subset R_\alpha O(X)$.

**Proof:** Let (X, τ) be locally indiscrete then $\tau\subset RO(X) \subset R_\alpha O(X)$.

**Theorem: 2.12** If B is clopen subset of a space X and A is $R_\alpha$–open in X, then $A\cap B\in R_\alpha O(X)$.

**Proof:** Let A be $R_\alpha$ –open. So A is regular open. B is open and closed in x. Then by theorem 1.5, A∩B is regular open in X. Let $x \in A\cap B$. Then $x\in A$ and $x\in B$. Since A is $R_\alpha$ –open, there exists a α –closed set F such that $x\in F\subseteq A$. B is closed and hence α –closed. F∩B is α –closed. $x\in F\cap B\subseteq A\cap B$. So A∩B is $R_\alpha$ –open.

**Theorem: 2.13** Let X be a locally indiscrete and A⊂X, B⊂X. If $A\in R_\alpha O(X)$ and B is open, and then A∩B is $R_\alpha$ –open in X.

**Proof:** Follows from theorem 2.12.

**Theorem: 2.14** Let X be externally disconnected and A⊂X, B⊂X. If $A\in R_\alpha O(X)$ and $B\in RO(X)$ then A∩B is $R_\alpha$ –open in X.

**Proof:** Let $A\in R_\alpha O(X)$ and $B\in RO(X)$. Hence A is regular open. By Theorem 1.5, $A\cap B\in RO(X)$. Let $x\in A\cap B$. This implies $x\in A$ and $x\in B$. As A is $R_\alpha$ –open, there exists a α –closed set F such that $x\in F\subseteq A$. X is extremally disconnected. By Theorem 1.7, B is a regular closed set. This implies F∩B is α-closed. $x\in F\cap B\subseteq A\cap B$. So A∩B is $R_\alpha$ – open.

## 3. $R_\alpha$-OPERATIONS

**Definition: 3.1** A subset N of a topological space X is called $R_\alpha$ - neighbourhood of a subset A of X, if there exists an $R_\alpha$-open set U such that A⊂U⊂N. When A={x}, we say N is $R_\alpha$- neighbourhood of x.

**Definition: 3.2** A point x∈X is said to be an $R_\alpha$- interior point of A, if there exists an $R_\alpha$-open set U containing x such that $x\in U\subset A$. The set of all $R_\alpha$-interior points of A is said to be $R_\alpha$- interior of A and it is denoted by $R_\alpha$- int A.

**Theorem: 3.3** Let A be any subset of a topological space X. If x is a $R_\alpha$-interior point of A, then there exists a α-closed set F of X containing x such that F⊂A.

**Proof:** Let $x\in R_\alpha$- int A. Then there exists a $R_\alpha$-open set U containing x such that U⊄A. Since U is in $R_\alpha$-open set, there exists a α-closed set F such that $x\in F\subset U\subset A$.

**Theorem: 3.4** For any subset A of a topological space X, the following statements are true

1. The $R_\alpha$-interior of A is the union of all $R_\alpha$-open sets contained in A.
2. $R_\alpha$-int A is the largest $R_\alpha$-open set contained in A.
3. A is $R_\alpha$-open set if and only of A=$R_\alpha$-int A.

**Proof:** obvious.

**Theorem: 3.5** If A and B are any subsets of a topological space X. Then,

1. $R_\alpha$ int∅ =∅ and $R_\alpha$ int X=X
2. $R_\alpha$ int A⊂A

3. If $A \subset B$, then $R_\alpha$ int $A \subset R_\alpha$ int B
4. $R_\alpha$ int $A \cup R_\alpha$ int $B \subset R_\alpha$int $(A \cup B)$
5. $R_\alpha$ int $(A \cap B) \subset R_\alpha$ int $A \cap R_\alpha$ int B
6. $R_\alpha$ int (A-B)$\subset R_\alpha$ int A-$R_\alpha$ int B

**Proof:** 1-5, obvious.

6. Let $x \in R_\alpha$ int (A-B). There exists an $R_\alpha$-open set U such that $x \in U \subset A-B$. That is $U \subset A$. $U \cap B = \emptyset$ and $x \notin B$. Hence $x \in R_\alpha$ int A, $x \notin R_\alpha$ int B. Hence $x \in R_\alpha$ int A-$R_\alpha$ int B. This completes the proof.

**Definition: 3.6** Intersection of all $R_\alpha$-closed sets containing F is called the $R_\alpha$-closure of F and is denoted by $R_\alpha$ cl F.

**Theorem: 3.7** Let A be a subset of the space X. $x \in X$ is in $R_\alpha$-closure of A if and only if $A \cap U \neq \emptyset$, for every $R_\alpha$-open set U containing x.

**Proof:** To prove the theorem, let us prove the contra positive. $x \notin$rcl A $\Leftrightarrow$There exists an $R_\alpha$-open set U containing x that does not intersect A. Let $x \notin R_\alpha$ cl A. X-$R_\alpha$ cl A is an $R_\alpha$-open set containing x that does not intersect A. Let U be an $R_\alpha$-open set containing x that does not intersect A. X-U is a $R_\alpha$ –closed set containing A. $R_\alpha$ cl $A \subset$ (X-U) , $x \notin X-U \Rightarrow x \notin R_\alpha$cl A.

**Theorem: 3.8** Let A be any subset of a space X .$A \cap F \neq \emptyset$ for every α-closed set F of X containing x, then the point x is in the $R_\alpha$- closure of A.

**Proof:** Let U be any $R_\alpha$- open set containing x. So, there exists a α-closed set F such that $x \in F \subset U$. $A \cap F \neq \emptyset$ implies $A \cap U \neq \emptyset$ for every $R_\alpha$ -open set U containing x. Hence $x \in R_\alpha$ cl A, by theorem 3.7.

**Theorem: 3.9** For any subset F of a topological space X, the following statements are true.

1. $R_\alpha$ cl F is the intersection of all $R_\alpha$- closed sets in X containing F.
2. $R_\alpha$ cl F is the smallest $R_\alpha$ -closed set containing F.
3. F is $R_\alpha$ closed if and only if F= $R_\alpha$ cl F.

**Proof:** Obvious.

**Theorem: 3.10** If F and E are any subsets of a topological space X, then

1. $R_\alpha$ cl $\emptyset$ =$\emptyset$ and $R_\alpha$ cl X=X.
2. For any subset F of X, $F \subset R_\alpha$ cl F.
3. If $F \subset E$, then $R_\alpha$ cl $F \subset R_\alpha$ cl E.
4. $R_\alpha$ cl $F \cup R_\alpha$ cl $E \subset R_\alpha$ cl $(F \cup E)$.
5. $R_\alpha$ cl $(F \cap E) \subset R_\alpha$ cl $F \cap R_\alpha$ cl E.

**Proof:** Obvious.

**Theorem: 3.11** For any subset A of a topological space X, the following statements are true.

1. X- $R_\alpha$ cl A= $R_\alpha$ int(X-A).
2. X- $R_\alpha$ int A= $R_\alpha$ cl A.
3. $R_\alpha$ int A= X- $R_\alpha$ cl A.

**Proof:** 1. X- $R_\alpha$ cl A is a $R_\alpha$-open set contained in (X-A). Hence X- $R_\alpha$ cl $A \subset R_\alpha$ int (X-A).

If X- $R_\alpha$ cl $A \neq R_\alpha$ int(X-A), then X- $R_\alpha$ int (X-A) is a $R_\alpha$ closed set properly contained in $R_\alpha$ cl A, a contradiction. Hence X- $R_\alpha$ cl A= $R_\alpha$ int(X-A). 2&3 follow from 1.

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