ON LOWER AND UPPER SEMI-QUASICONTINUOUS FUNCTIONS

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ABSTRACT

In this paper, some interesting properties of lower and upper semi-quasicontinuous functions are investigated. The relationship of these functions with other well-known notions like lower (upper) semicontinuity are established.

Key words: Lower semi-quasicontinuity, Upper semi-quasicontinuity, Lower semicontinuity, Upper semicontinuity, Semi-continuity.

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INTRODUCTION

The notion of lower (upper) semi-quasicontinuity is similar to the well-known notion of lower (upper) semicontinuity. Some properties are similar but there are remarkable differences which are investigated and studied in this paper. Examples are provided wherever necessary. It is also proved that the set of all real valued lower (upper) semi-quasicontinuous functions defined on a topological space is closed under uniform limits.

In what follows \( X \) and \( \mathbb{R} \) stand for a topological space and the real line respectively. The symbols \( cl(\cdot) \) and \( Int(\cdot) \) denote the closure and the interior in \( X \) respectively. The symbol \( \chi_A \) represents the characteristic function of a subset \( A \) of \( X \).

1. PRELIMINARIES

The Definitions that are needed throughout this paper are presented in this section.

1.1 Definition [2]: A function \( f : X \rightarrow \mathbb{R} \) is said to be lower semi-quasicontinuous (\( lsqc \)) at a point \( x_0 \in X \) if for every \( \varepsilon > 0 \) and every neighborhood \( U \) of \( x_0 \) in \( X \) there exists a non-empty open set \( W \subset U \) such that

\[
-f(x) - f(x_0) < \varepsilon \quad \forall \ x \in W.
\]

If \( f \) is \( lsqc \) at every point of \( X \) then \( f \) is said to be \( lsqc \) on \( X \). The real set of all valued \( lsqc \) functions on \( X \) is denoted by \( \mathcal{R}^+(X) \).

1.2 Definitions [2]: A function \( f : X \rightarrow \mathbb{R} \) is said to be upper semi-quasicontinuous (\( usqc \)) at a point \( x_0 \in X \) if for every \( \varepsilon > 0 \) and every neighborhood \( U \) of \( x_0 \) in \( X \) there exists a non-empty open set \( W \subset U \) such that

\[
f(x) - f(x_0) < \varepsilon \quad \forall \ x \in W.
\]

If \( f \) is \( usqc \) at every point of \( X \) then \( f \) is \( usqc \) on \( X \). The set of all real valued \( usqc \) functions on \( X \) is denoted by \( \mathcal{R}^-(X) \).
1.3 Definition [5]: A function \( f : X \to \mathbb{R} \) is said to be lower semicontinuous (lsc) at a point \( x_0 \in X \) if for every \( \varepsilon > 0 \) there exists a neighborhood \( G \) of \( x_0 \) in \( X \) such that
\[
-\varepsilon < f(x) - f(x_0) \quad \forall \ x \in G.
\]
If \( f \) is lsc at every point of \( X \) then \( f \) is lsc on \( X \). The set of all real valued lsc functions on \( X \) is denoted by \( \mathcal{L}_s(X) \).

1.4 Definition [5]: A function \( f : X \to \mathbb{R} \) is said to be upper semicontinuous (usc) at a point \( x_0 \in X \) if for every \( \varepsilon > 0 \) there exists a neighborhood \( G \) of \( x_0 \) in \( X \) such that \( f(x) - f(x_0) < \varepsilon \quad \forall \ x \in G \).
If \( f \) is usc at every point of \( X \) then \( f \) is usc on \( X \). The set of all real valued usc functions on \( X \) is denoted by \( \mathcal{U}_s(X) \).

1.5 Definition [3]: A function \( f : X \to \mathbb{R} \) is said to be semi-continuous at a point \( x_0 \in X \) if for every \( \varepsilon > 0 \) there exists a non-empty open set \( U \) such that
\[
(f(x) - f(x_0)) < \varepsilon \quad \forall \ x \in U.
\]
If \( f \) is semi-continuous at every point of \( X \) then it is said to be semi-continuous on \( X \). Some Authors call these functions Quasicontinuous (see [2]).

1.6 Definition [4]: A subset \( A \) of \( X \) is said to be semi-open if \( A \subseteq cl(\text{Int}(A)) \) or equivalently if there exists an open set \( O \) in \( X \) such that \( O \subseteq A \subseteq cl(O) \).

1.7 Definition [1]: A subset \( A \) of \( X \) is said to be semi-closed if the complement \( A^c \) of \( A \) is semi-open in \( X \).

2. RELATIONSHIP AMONG LSQC, USQC, LSC AND USC FUNCTIONS

2.1 Proposition: If \( f : X \to \mathbb{R} \) is semi-continuous at a point \( x_0 \in X \) then \( f \) is both lsqc and usqc at \( x_0 \).

Proof: Let \( f : X \to \mathbb{R} \) be semi-continuous at \( x_0 \in X \). Let \( \varepsilon > 0 \) be given and let \( U \) be a neighborhood of \( x_0 \) in \( X \).

Then there exists a non-empty open set \( W \subseteq U \) such that
\[
|f(x) - f(x_0)| < \varepsilon \quad \forall \ x \in W
\]
\[
\Rightarrow -\varepsilon < f(x) - f(x_0) < \varepsilon \quad \forall \ x \in W
\]
\[
\Rightarrow -\varepsilon < f(x) - f(x_0) \quad \text{and} \quad f(x) - f(x_0) < \varepsilon \quad \forall \ x \in W
\]
\[
\Rightarrow f \text{ is both lsqc and usqc at } x_0.
\]

2.2 Remark: The converse of the above Proposition 2.1 is not true as is evident from the following example.

2.3 Example: Define \( f : [0,1] \to \mathbb{R} \) by
\[
f(x) = \begin{cases} 
0 & \text{if } \quad 0 \leq x < \frac{1}{2} \\
\frac{1}{2} & \text{if } \quad x = \frac{1}{2} \\
1 & \text{if } \quad \frac{1}{2} < x \leq 1
\end{cases}
\]
Clearly \( f \) is both lsqc and usqc at \( x_0 = \frac{1}{2} \), but it is not semi-continuous at \( x_0 = \frac{1}{2} \).

**2.4 Proposition:**

(a) \( f \in C^+(X) \Rightarrow f \in C^+(X) \)

(b) \( f \in C^-(X) \Rightarrow f \in C^-(X) \)

**2.5 Remark:** The converses of (a) and (b) in the Proposition 2.4 are not true. The function in the example 2.3 is both lsqc and usqc at \( x = 0 \), but it is not semi-continuous at \( x = 0 \).

\( f \in C^+(X) \).

\( f \) is not usc at \( x = 0 \).

It can be easily verified that \( f \) is not lsc at \( x = 0 \).

**2.6 Proposition:** \( f \in C^-(X) \Leftrightarrow -f \in C^+(X) \)

**Proof:** Let \( x_0 \in X \). \( f \) is usqc at \( x_0 \) \iff for every \( \varepsilon > 0 \) and every neighborhood \( U \) of \( x_0 \) in \( X \) there exists a non-empty open set \( W \subset U \) such that

\[
(f(x) - f(x_0) < \varepsilon) \quad \forall \quad x \in W
\]

\[
(-f(x) - (-f(x_0)) = -[f(x) - f(x_0)] > -\varepsilon) \quad \forall \quad x \in W
\]

\( f \) is lsqc at \( x_0 \).

Hence \( f \in C^+(X) \Leftrightarrow -f \in C^-(X) \).

**3. CHARACTERIZATIONS OF LSQC AND USQC FUNCTIONS**

**3.1 Proposition:** Let \( A \subset X \). Then

(a) \( f \in C^+(X) \) if and only if for every \( s \in \mathbb{R} \) the set \( G_s = \{ x \in X \mid f(x) > s \} \) is semi-open \( X \).

(b) \( f \in C^-(X) \) if and only if for every \( s \in \mathbb{R} \) the set \( H_s = \{ x \in X \mid f(x) < s \} \) is semi-open \( X \).

(c) \( \xi \in C^+(X) \) if and only if \( A \) is semi-open \( X \).

(d) \( \xi \in C^-(X) \) if and only if \( A \) is semi-closed \( X \).

**Proof:**

(a) Suppose that \( f \in C^+(X) \). Let \( s \in \mathbb{R} \). Let \( x_0 \in G_s \)

\[
f(x_0) > s
\]

\[
f(x_0) - s > 0
\]

\( \varepsilon = f(x_0) - s \). Let \( U \) be a neighborhood of \( x_0 \) in \( X \).

Since \( f \) is lsqc at \( x_0 \), there exists a non-empty open set \( W \subset U \) such that \( x \in W \)

\[
-f(x) - f(x_0) < -\varepsilon
\]

\[
-f(x) + s < f(x) - f(x_0)
\]

\[
f(x) > s
\]

\( x \in G_s \)

\( \therefore W \subset G_s \) and hence \( W \subset U \bigcap G_s \).
Thus for each \( x_0 \in G_s \) there exists a non-empty open set \( W(x_0) \subseteq U \cap G_s \) such that \( \bigcup_{x_0 \in G_s} W(x_0) \). Then \( O \) is a non-empty open set in \( X \) such that \( O \subseteq G_s \).

Assume that \( G_s \cap (\text{cl}(O))^c \neq \emptyset \)

\[ \Rightarrow \quad \text{there exists a point } p \in X \text{ such that } p \in G_s \cap (\text{cl}(O))^c \]

\[ \Rightarrow \quad p \in G_s \quad \text{and} \quad p \notin \text{cl}(O) \]

Since \( p \notin \text{cl}(O) \), there exists a neighborhood \( U \) of \( p \) in \( X \) such that \( U \cap O = \emptyset \).

Since \( p \in G_s \), there exists a non-empty open set \( W(p) \) such that \( W(p) \subseteq U \cap G_s \).

This is a contradiction.

\[ \therefore \quad G_s \cap (\text{cl}(O))^c = \emptyset \Rightarrow G_s \subseteq \text{cl}(O) \]

Thus there is a non-empty open set \( O \) in \( X \) such that \( O \subseteq G_s \subseteq \text{cl}(O) \)

\[ \Rightarrow \quad G_s \text{ is semi-open in } X. \text{ Thus for every } s \in \mathbb{R}, \text{ the set } G_s = \{ x \in X \mid f(x) > s \} \text{ is semi-open in } X. \]

Conversely suppose that for every \( s \in \mathbb{R} \), the set \( G_s = \{ x \in X \mid f(x) > s \} \) is semi-open in \( X \).

Now we prove that \( f \in \mathcal{R}^+(X) \).

Let \( t \in X \) and \( U \) be a neighborhood of \( t \) in \( X \). Let \( \varepsilon > 0 \) be given.

Put \( s = f(t) - \varepsilon \). For this \( s \in \mathbb{R} \), the set \( G_s = \{ x \in X \mid f(x) > f(t) - \varepsilon \} \) is semi-open in \( X \).

Clearly \( t \in G_s \) and hence \( U \cap G_s \) is a semi-neighborhood of \( t \) in \( X \).

\[ \Rightarrow \quad \text{there exists a non-empty open set } W \text{ such that } W \subseteq U \cap G_s \subseteq \text{cl}(W) \]

\[ \therefore \quad x \in W \Rightarrow x \in U \cap G_s \]

\[ \Rightarrow \quad x \in U \quad \text{and} \quad x \in G_s \]

\[ \Rightarrow \quad f(x) > f(t) - \varepsilon. \]

Thus for every \( \varepsilon > 0 \) and every neighborhood \( U \) of \( t \) in \( X \) there exists a non-empty open set \( W \subseteq U \) such that \( -\varepsilon < f(x) - f(t) \forall x \in W \)

\[ \Rightarrow \quad f \text{ is lsqc at } t. \text{ Thus } f \text{ is lsqc on } X. \text{ Hence } f \in \mathcal{R}^+(X). \]

(b) Suppose that \( f \in \mathcal{R}^-(X) \)

\[ \Leftrightarrow \quad -f \in \mathcal{R}^-(X) \]

\[ \Leftrightarrow \quad \text{the set } \{ x \in X \mid -f(x) > -s \} \text{ is semi-open for } s \in \mathbb{R} \]

\[ \Leftrightarrow \quad \text{the set } H_s = \{ x \in X \mid f(x) < s \} \text{ is semi-open for every } s \in \mathbb{R}. \]

(c) Suppose that \( \xi_s \in \mathcal{R}^+(X) \)

\[ \text{for } s = \frac{1}{2}, \text{ the set } G_s = \{ x \in X \mid \xi_s(x) > s \} = A \text{ is semi-open.} \]
Conversely suppose that \( A \) is semi-open in \( X \).

\[
G_s = \begin{cases} 
\emptyset & \text{if } s \geq 1 \\
A & \text{if } 0 \leq s < 1 \\
A \cup A^c & \text{if } s < 0
\end{cases}
\]

\( \Rightarrow \) \( G_s = \{ x \in X / \xi_A(x) > s \} \) is semi-open for every \( s \in \mathbb{R} \).

\( \Rightarrow \) \( \xi_A \in \mathcal{H}(X) \).

(d) Suppose that \( \xi_A \in \mathcal{H}(X) \)

\( \Rightarrow \) for \( s = \frac{1}{2} \), the set \( H_s = \{ x \in X / \xi_A(x) < s \} = A^c \) is semi-open.

\( \Rightarrow \) \( A \) is semi-closed in \( X \).

Conversely suppose that \( A \) is semi-closed in \( X \).

\[
H_s = \begin{cases} 
\emptyset & \text{if } s \leq 1 \\
A & \text{if } 0 < s \leq 1 \\
A \cup A^c & \text{if } s > 1
\end{cases}
\]

\( \Rightarrow \) \( H_s = \{ x \in X / \xi_A(x) < s \} \) is semi-open for every \( s \in \mathbb{R} \).

\( \Rightarrow \) \( \xi_A \in \mathcal{H}(X) \).

4. ALGEBRAIC PROPERTIES

4.1 Proposition: If \( \lambda \geq 0 \) and \( f \in \mathcal{H}(X) \) then \( \lambda f \in \mathcal{H}(X) \)

Proof: If \( \lambda = 0 \) then \( \lambda f \in \mathcal{H}(X) \).

Now suppose that \( \lambda > 0 \). Let \( \varepsilon > 0 \) be given.

Let \( x \in X \) and \( U \) be a neighborhood of \( x_0 \) in \( X \).

Since \( f \) is \( \text{usqc} \) at \( x_0 \), there exists a non-empty open set \( W \subset U \) such that

\[
f(x) - f(x_0) < \frac{\varepsilon}{\lambda} \quad \forall \quad x \in W
\]

\( \Rightarrow \) \( (\lambda f)(x) - (\lambda f)(x_0) < \varepsilon \quad \forall \quad x \in W
\)

\( \Rightarrow \) \( \lambda f \) is \( \text{usqc} \) at \( x_0 \). Hence \( \lambda f \in \mathcal{H}(X) \).

4.2 Proposition: If \( \lambda \geq 0 \) and \( f \in \mathcal{H}(X) \) then \( \lambda f \in \mathcal{H}(X) \)

4.3 Proposition: If \( f \in \mathcal{H}(X) \) and \( g \in \mathcal{H}(X) \) then \( f + g \in \mathcal{H}(X) \)

Proof: Let \( x \in X \) and let \( \varepsilon > 0 \) be given.

Let \( U \) be a neighborhood of \( x_0 \) in \( X \).
Since \( f \) is usc at \( x_0 \), there exists a neighborhood \( G \) of \( x_0 \) in \( X \) such that
\[
f(x) - f(x_0) < \frac{\varepsilon}{2} \quad \forall \quad x \in G.
\]
Since \( g \) is usqc at \( x_0 \), there exists a non-empty open set \( W \subset U \cap G \) such that
\[
g(x) - g(x_0) < \frac{\varepsilon}{2} \quad \forall \quad x \in W.
\]
∴ \( x \in W \) \( \implies \) \((f + g)(x) - (f + g)(x_0) = f(x) - f(x_0) + g(x) - g(x_0) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\]
Thus for every \( \varepsilon > 0 \) and every neighborhood \( U \) of \( x_0 \), there exists a Non-empty open \( W \subset U \) such that
\((f + g)(x) - (f + g)(x_0) < \varepsilon \quad \forall \quad x \in W \)
\( \implies \) \( f + g \) is usqc at \( x_0 \). Hence \( f + g \in \mathcal{R}(X). \)

4.4 Proposition: If \( f \in \mathcal{R}(X) \) and \( g \in \mathcal{R}(X) \) then \( f + g \in \mathcal{R}(X) \).

4.5 Remark: If \( f \in \mathcal{R}(X) \) and \( g \in \mathcal{R}(X) \) then it is not necessary that \( f + g \in \mathcal{R}(X) \). Similarly \( f \in \mathcal{R}(X) \) and \( g \in \mathcal{R}(X) \) need not imply \( f + g \in \mathcal{R}(X) \). These facts are clear from the following example.

4.6 Example: Define \( f : [0, 1] \rightarrow \mathbb{R} \) and \( g : [0, 1] \rightarrow \mathbb{R} \) by
\[
f(x) = \begin{cases} 
3 & \text{if } 0 \leq x < \frac{1}{2} \\
1 & \text{if } x = \frac{1}{2} \\
\frac{1}{2} & \text{if } \frac{1}{2} < x \leq 1 
\end{cases}
\]
\[
g(x) = \begin{cases} 
0 & \text{if } 0 \leq x < \frac{1}{2} \\
\frac{1}{2} & \text{if } x = \frac{1}{2} \\
2 & \text{if } \frac{1}{2} < x \leq 1 
\end{cases}
\]

then \((f + g)(x) = \begin{cases} 
3 & \text{if } 0 \leq x < \frac{1}{2} \\
1.5 & \text{if } x = \frac{1}{2} \\
2.5 & \text{if } \frac{1}{2} < x \leq 1 
\end{cases} \)

Clearly \( f \) and \( g \) are usqc at \( x = \frac{1}{2} \) but \( f + g \) is not.
Since \( f \) and \( g \) are usqc on \([0,1]\), \(-f\) and \(-g\) are lsqc on \([0,1]\). But \(-(f + g)\) is not lsqc at \( x = \frac{1}{2} \).

**4.7 Proposition:** If \( f \in \mathcal{R}^+(X) \), \( g \in \mathcal{R}^+(X) \) and \( f + g \) is continuous on \( X \) then \( f \) is semi-continuous on \( X \).

**Proof:** Let \( p \in X \) and \( \varepsilon > 0 \) be given. Let \( U \) be a neighborhood of \( p \) in \( X \).

Since \( g \) is lsc at \( p \), there exists a neighborhood \( G \) of \( p \) such that
\[
-\frac{\varepsilon}{2} < g(x) - g(p) \quad \forall x \in G.
\]

Since \( f + g \) is continuous at \( p \), there exists a neighborhood \( H \) of \( p \) such that
\[
\left| (f + g)(x) - (f + g)(p) \right| < \frac{\varepsilon}{2} \quad \forall x \in H.
\]

Since \( f \) is lsqc at \( p \), there exists a non-empty open set \( W \subset U \cap G \cap H \) such that
\[
-\varepsilon < f(x) - f(p) \quad \forall x \in W.
\]

\[
\Rightarrow -\varepsilon < f(x) - f(p) < g(x) + g(p) - \frac{\varepsilon}{2} < \varepsilon.
\]

Thus for every \( \varepsilon > 0 \) and every neighborhood \( U \) of \( p \) in \( X \), there exists a non-empty open set \( W \subset U \) such that
\[
\left| (f(x) - f(p)) \right| < \varepsilon \quad \forall x \in W.
\]

\[
\Rightarrow f \text{ is semi-continuous at } p.
\]

**4.8 Proposition:** If \( f \in \mathcal{R}^-(X) \), \( g \in \mathcal{R}^-(X) \) and \( f + g \) is continuous on \( X \) then \( f \) is semi-continuous on \( X \).

**4.9 Remark:** From the following example it is clear that \( f \wedge g \) and \( fg \) are not necessarily lsqc (usqc) eventhough \( f \) and \( g \) are lsqc (usqc).

**4.10 Example:** Define \( f : [0,1] \rightarrow \mathbb{R} \) and \( g : [0,1] \rightarrow \mathbb{R} \) by
\[
\begin{align*}
f(x) &= \begin{cases} 1 & \text{if } 0 \leq x < \frac{1}{2} \\
\frac{1}{2} & \text{if } x = \frac{1}{2} \\
0 & \text{if } \frac{1}{2} \leq x \leq 1
\end{cases} \\
g(x) &= \begin{cases} 0 & \text{if } 0 \leq x < \frac{1}{2} \\
\frac{1}{2} & \text{if } x = \frac{1}{2} \\
1 & \text{if } \frac{1}{2} < x \leq 1
\end{cases}
\end{align*}
\]
Then \((f \land g)(x) = \begin{cases} 0 & \text{if } x \in [0,1] - \left\{ \frac{1}{2} \right\} \\ 1/2 & \text{if } x = 1/2 \end{cases}\)

It can be easily verified that \(f\) and \(g\) are \(lsqc\) at \(x = 1/2\). But \(f \land g\) and \(fg\) are not \(lsqc\) at \(x = 1/2\).

4.11 Remark: Similar examples can be constructed to show \(f \lor g\) and \(fg\) are not necessarily \(usqc\) even though \(f\) and \(g\) are.

4.12 Proposition:
(a) If \(S \subset C^+(X)\) and \(S \subset C^+(X)\) then \(S \subset C^+(X)\)
(b) If \(S \subset C^-(X)\) and \(S \subset C^-(X)\) then \(S \subset C^-(X)\)

Proof: (a) Suppose that \(S \subset C^+(X)\) and \(S \subset C^+(X)\).

Then for every \(s \in \mathbb{R}\) the sets \(\{x \in X / f(x) > s\}\) and \(\{x \in X / g(x) > s\}\) are semi-open. Since \(\{x \in X \mid (f \lor g)(x) > s\} = \{x \in X / f(x) > s\} \cup \{x \in X / g(x) > s\}\), the set \(\{x \in X / (f \lor g)(x) > s\}\) is semi-open for every \(s \in \mathbb{R}\).

\[ \Rightarrow f \lor g \in C^{+}(X). \]

The proof of (b) follows from (a) and by Proposition 2.6.

4.13 Proposition: Let \(S \subset C^+(X)\). If \(h(x) = \sup \{f(x) / f \in S\}\) exists for all \(x \in X\) then \(h \in C^{+}(X)\).

Proof: Since \(f \in C^+(X)\), the set \(\{x \in X / f(x) > s\}\) is semi-open for every \(s \in \mathbb{R}\). Since \(\{x \in X / h(x) > s\} = \bigcup_{s \in S} \{x \in X / f(x) > s\}\), the set \(\{x \in X / h(x) > s\}\) is semi-open for every \(s \in \mathbb{R}\).

\[ \Rightarrow h \in C^{+}(X). \]

4.14 Proposition: Let \(S \subset C(X)\). If \(k(x) = \inf \{f(x) / f \in S\}\) exists for all \(x \in X\) then \(k \in C(X)\).

5. SOME OTHER CHARACTERIZATIONS

5.1 Definition: Let \(f : X \to \mathbb{R}\) be a bounded function.

We define \(f^\uparrow (x) = \sup \{g(x) / g \in C(X), g \leq f\} \quad (x \in X)\)

\[ f^\downarrow (x) = \inf \{h(x) / h \in C(X), h \geq f\} \]

5.2 Proposition: Let \(f : X \to \mathbb{R}\) and \(g : X \to \mathbb{R}\) be bounded functions. Then

(i) \(f^\uparrow \leq f \leq f^\downarrow\)
(ii) \(f^\uparrow \in C^+(X)\) and \(f^\downarrow \in C^+(X)\)
(iii) If \(h \leq f, h \in C^+(X)\), then \(h \leq f^\uparrow\)
(iv) If \(k \geq f, k \in C(X)\), then \(k \geq f^\downarrow\)
(v) If \(f \leq g\) then \(f^\uparrow \leq g^{\uparrow}\) and \(f^\downarrow \leq g^{\downarrow}\)
(vi) \(\inf \{f^\uparrow (x) / x \in X\} \geq \inf \{f(x) / x \in X\} = \inf \{f^\uparrow (x) / x \in X\}\)
(vii) \(\sup \{f^\downarrow (x) / x \in X\} = \sup \{f(x) / x \in X\} \geq \sup \{f^\downarrow (x) / x \in X\}\)
5.3 Proposition: Let \( f : X \to \mathbb{R} \) be a bounded function. Then

(i) \( f \in \mathcal{R}^+(X) \iff f = f^\uparrow \)

(ii) \( f \in \mathcal{R}^-(X) \iff f = f^\downarrow \)

(iii) \( f \) is semi-continuous on \( X \) \( \implies f^\uparrow = f^\downarrow \)

6. UNIFORM LIMITS

6.1 Proposition: If \( f_n \in \mathcal{R}^+(X) \), \( n = 1, 2, 3, \ldots \), \( f_n \to f \) uniformly on \( X \) then \( f \in \mathcal{R}^+(X) \).

Proof: Let \( \varepsilon > 0 \) be given. Let \( x_0 \in X \) and \( U \) be a neighborhood of \( x_0 \) in \( X \).

Since \( f_n \to f \) uniformly on \( X \), there exists an integer \( N \) such that \( n \geq N \)

\[
\Rightarrow \left| f_n(x) - f(x) \right| < \frac{\varepsilon}{3} \quad \forall \ x \in W
\]

Since \( f_N \) is \( usqc \) at \( x_0 \), there exists a non-empty open set \( W \subseteq U \) such that

\[
f_N(x) - f_N(x_0) < \frac{\varepsilon}{3} \quad \forall \ x \in W.
\]

\[
\therefore x \in W \Rightarrow f_N(x) - f_N(x_0) = f(x) - f_N(x) - f_N(x_0) + f_N(x_0) - f(x_0)
\]

\[
< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon
\]

Thus for every \( \varepsilon > 0 \) and every neighborhood \( U \) of \( x_0 \) there exists a non-empty open set \( W \subseteq U \) such that

\( f(x) - f(x_0) < \varepsilon \quad \forall \ x \in W \)

\( \Rightarrow f \) is \( usqc \) at \( x_0 \).

Hence \( f \in \mathcal{R}^+(X) \).

6.2 Proposition: If \( f_n \in \mathcal{R}^+(X) \), \( n = 1, 2, 3, \ldots \), \( f_n \to f \) uniformly on \( X \) then \( f \in \mathcal{R}^+(X) \).

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REFERENCES


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