SOME FIXED POINT THEOREMS IN TWO FUZZY METRIC SPACES

T. Veerapandi
Associate Professor of Mathematics, P. M. T College, Melanelithanallur-627953, India

T. Thiripura Sundari*
Department of Mathematics, Sri K. G. S Arts College, Srivaikuntam, India

J. Paulraj Joseph
Associate Professor of Mathematics, Manonmaniam Sundaranar University, Tirunelveli, India

(Received on: 19-07-12; Revised & Accepted on: 11-08-12)

ABSTRACT

In this paper we prove some fixed point theorems for generalized contraction mappings in two complete fuzzy metric spaces.

Key words and Phrases: fixed point, common fixed point and complete fuzzy metric space.

AMS Mathematics Subject Classification: 47H10, 54H25.

1. INTRODUCTION

Fuzzy set was defined by Zadeh [9] in 1965, has lead to a rich growth of fuzzy mathematics. Kramosil and Michalek [7] introduced fuzzy metric space, George and Veeramani [5] modified the notion of fuzzy metric spaces with the help of continuous t-norms. Many authors namely Deng [2], Erceg [3] used the concept of fuzzy mathematics in different ways. Recently there are several authors proved many kinds of fixed point theorems in fuzzy metric spaces viz. [4] In this paper we prove some fixed point theorems in two complete fuzzy metric spaces for contractive type mappings and non-expansive mappings by generalizing the results of Veerapandi et al [8] on fuzzy metric space.

Now we begin with some known definitions and preliminary concepts.

Definition 1.2: A fuzzy set A in X is a function with domain X and values in [0,1].

Definition 1.3 [10]: A binary operation \( \ast : [0, 1] \times [0, 1] \rightarrow [0,1] \) is a continuous t-norm if it satisfies the following conditions:
(i) \( \ast \) is commutative and associative,
(ii) \( \ast \) is continuous,
(iii) \( a \ast 1 = a \) for all \( a \in [0,1] \),
(iv) \( a \ast b \leq c \ast d \) whenever \( a \leq c \) and \( b \leq d \), for each \( a, b, c, d \in [0,1] \).

Examples of t-norm are \( a \ast b = ab \) and \( a \ast b = \min(a,b) \).

Definition 1.4 [7]: A 3-tuple \((X, M, \ast)\) is said to be a fuzzy metric space if \(X\) is an arbitrary set, \(\ast\) is a continuous t-norm and \(M\) is a fuzzy set on \(X^2 \times [0, \infty)\) satisfying the following conditions, for all \(x, y, z \in X\) and \(s, t > 0\),

\[
\begin{align*}
(f-1) & \quad M(x, y, t) > 0, \\
(f-2) & \quad M(x, y, t) = 1 \text{ if and only if } x = y, \\
(f-3) & \quad M(x, y, t) = M(y, x, t), \\
(f-4) & \quad M(x, y, t) \ast M(y, z, s) \leq M(x, z, t+s), \\
(f-5) & \quad M(x, y, .) : [0, \infty) \rightarrow [0,1] \text{ is left continuous } \forall x, y, z \in X \text{ and } t, s > 0.
\end{align*}
\]

Then \(M\) is called a fuzzy metric on \(X\) and \(M(x, y, t)\) denotes the degree of nearness between \(x\) and \(y\) with respect to \(t\).

Example 1.5 [5]: (Induced fuzzy metric): Let \((X, d)\) be a metric space, define \(a \ast b = \min\{a, b\}\) for all \(a, b \in [0,1]\) and \(M_d\) be fuzzy set on \(X^2 \times [0, \infty)\) defined as,
\[ M_d(x, y, t) = \frac{t}{t + d(x, y)} \]

for all \( x, y \in X \) and \( t > 0 \). Then \((X, M_d, \star)\) is a fuzzy metric space. We call this fuzzy metric \( M_d \) as the standard intuitionistic fuzzy metric.

**Definition 1.6[6]:** Let \((X, M, \star)\) be a fuzzy metric space. Then

1. A sequence \( \{x_n\} \) in \( X \) is said to be convergent to a point \( x \in X \), if
   \[ \lim_{n \to \infty} M(x_n, x, t) = 1 \]
   for all \( t > 0 \).
2. A sequence \( \{x_n\} \) in \( X \) is said to be a Cauchy sequence if
   \[ \lim_{n \to \infty} M(x_{n+p}, x_n, t) = 1 \]
   for all \( t > 0 \) and \( p > 0 \).
3. A fuzzy metric space is said to be complete if every Cauchy sequence is convergent to a point in it.

**Remark 1.7:** Since \( \star \) is continuous, it follows from (f-4), that the limit of the sequence in fuzzy metric space is uniquely determined.

Let \((X, M, \star)\) be a fuzzy metric space with the following conditions.

\[ \lim_{t \to \infty} M(x, y, t) = 1 \]

**Lemma 1.8[6]:** For all \( x, y \in X \), \( M(x, y, .) \) is non-decreasing.

**Lemma 1.9[1]:** Let \( \{x_n\} \) be a sequence in a fuzzy metric space \((X, M, \star)\) with the condition (f-6). If there exists a number \( q \in (0,1) \) such that
\[ M(x_{n+2}, x_{n+1}, qt) \geq M(x_{n+1}, x_n, t) \]
for all \( t > 0 \) and \( n = 1, 2, \ldots \), then \( \{x_n\} \) is a Cauchy sequence in \( X \).

**Lemma 1.10:** If for all \( x, y \in X \), \( t > 0 \) and for a number \( q \in (0, 1) \),
\[ M(x, y, qt) \geq M(x, y, t) \]
then \( x = y \).

### 2. MAIN RESULTS

**Theorem 2.1.** Let \((X, M_1, \star)\) and \((Y, M_2, \star)\) be two complete fuzzy metric spaces. If \( T \) is a mapping from \( X \) into \( Y \) and \( S \) is a mapping from \( Y \) into \( X \), satisfying the following conditions:

\[ M_2(Tx, TSy, qt) \geq \min\{M_1(x, Sy, t), M_2(y, Tx, t), M_2(y, TSy, t)\} \]
\[ M_1(Sy, STx, qt) \geq \min\{M_1(x, Sy, t), M_1(x, STx, t), M_2(y, Tx, t)\} \]

for all \( x \in X \) and \( y \in Y \) where \( q < 1 \), then \( ST \) has a unique fixed point \( z \) in \( X \) and \( TS \) has a unique fixed point \( w \) in \( Y \). Further \( Tz = w \) and \( Sw = z \).

**Proof:** Let \( x_0 \) be an arbitrary point in \( X \). Define a sequence \( \{x_n\} \) in \( X \) and \( \{y_n\} \) in \( Y \), as follows:
\[ x_n = (ST)^n x_0, \quad y_n = T(x_{n-1}) \]
for \( n = 1, 2, \ldots \). We have

\[ M_1(x_n, x_{n+1}, qt) \]
\[ = M_1((ST)^n x_0, (ST)^{n+1} x_0, qt) \]
\[ = M_1(S(T(ST)^n x_0), ST(x_{n+1}), qt) \]
\[ = M_1(ST(x_{n+1}), STx_0, qt) \]
\[ = M_1(STx_0, STx_0, qt) \]
\[ \geq \min\{M_1(x_n, Sy_n, t), M_2(y_n, Tx_n, t)\} \]

(since by (2))

\[ M_2(y_n, y_{n+1}, qt) \]
\[ = \min\{M_1(x_n, y_{n+1}, t), M_2(y_n, y_{n+1}, t)\} \]
\[ \geq M_2(y_n, y_{n+1}, t) \]

\[ \geq M_2(y_{n+1}, y_{n+2}, t) \]
Now
\[ M_2(y_n, y_{n+1}, t) = M_2(Tx_n, Tx_{n-1}, t) \]
\[ = M_2(Tx_n, TSy_n, t) \]
\[ \geq \min \{ M_1(x_{n-1}, Sy_n, t/q), M_2(y_n, Tx_n, t/q) \ast M_2(y_n, TSy_n, t/q) \} \quad \text{(since by (1))} \]
\[ = \min \{ M_1(x_{n-1}, x_n, t/q), M_2(y_n, y_{n+1}, t/q) \ast M_2(y_n, y_{n+1}, t/q) \} \]
\[ \geq M_1(x_{n-1}, x_n, t/q) \]
Hence
\[ M_1(x_n, x_{n+1}, t/q) \geq M_2(y_n, y_{n+1}, t) \geq M_1(x_{n-1}, x_n, t/q) \]
\[ \vdots \]
\[ \geq M_1(x_0, x_1, t/q^{n-1}) \rightarrow 1 \text{ as } n \rightarrow \infty \quad \text{(since } q < 1) \]
Thus \( \{x_n\} \) is a Cauchy sequence in \( X \). Since \( (X, M_1, \ast) \) is complete, it converges to a point \( z \) in \( X \). Similarly, we can prove that the sequence \( \{y_n\} \) is also a Cauchy sequence in \( Y \) and it converges to a point \( w \) in \( Y \).

Now we prove \( Tz = w \).

Suppose \( Tz \neq w \).

We have
\[ M_2(Tz, w, qt) = \lim_{n \rightarrow \infty} M_2(Tz, y_{n+1}, qt) \]
\[ = \lim_{n \rightarrow \infty} M_2(Tz, TSy_n, qt) \]
\[ \geq \lim_{n \rightarrow \infty} \min \{ M_1(z, Sy_n, t), M_2(y_n, Tz, t) \ast M_2(y_n, TSy_n, t) \} \]
\[ = \lim_{n \rightarrow \infty} \min \{ M_1(z, x_n, t), M_2(y_n, Tz, t) \ast M_2(y_n, y_{n+1}, t) \} \]
\[ = \min \{ M_1(z, z, t), M_2(w, Tz, t) \ast M_2(w, w, t) \} \]
\[ = \min \{ 1, M_2(w, Tz, t) \ast 1 \} \]
\[ \geq M_2(Tz, w, t) \quad \text{(since } q < 1), \text{ which is a contradiction.} \]
Thus \( Tz = w \).

Now we prove \( Sw = z \).

Suppose \( Sw \neq z \).

We have
\[ M_1(Sw, z, qt) = \lim_{n \rightarrow \infty} M_1(Sw, x_{n+1}, qt) \]
\[ = \lim_{n \rightarrow \infty} M_1(Sw, STx_n, qt) \]
\[ \geq \lim_{n \rightarrow \infty} \min \{ M_1(x_n, Sw, t) \ast M_1(x_n, STx_n, t), M_2(w, Tz, t) \} \]
\[ = \lim_{n \rightarrow \infty} \min \{ M_1(x_n, Sw, t) \ast M_1(x_n, x_{n+1}, t), M_2(w, y_{n+1}, t) \} \]
\[ = \min \{ M_1(z, Sw, t) \ast 1, 1 \} \]
\[ \geq M_1(z, Sw, t) \quad \text{(since } q < 1), \text{ which is a contradiction.} \]
Thus \( Sw = z \).

Therefore we have \( STz = Sw = z \) and \( TSw = Tz = w \). Thus the point \( z \) is a fixed point of \( ST \) and the point \( w \) is a fixed point of \( TS \).

**Uniqueness:** let \( z' \) be another fixed point of \( ST \) such that \( z = z' \).

We have
\[ M_1(z, z', qt) = M_1(STz, STz', qt) \]
\[ \geq \min \{ M_1(z', STz, t) \ast M_1(z', STz', t), M_2(Tz, Tz', t) \} \]
\[ = \min \{ M_1(z', z, t), M_2(Tz, Tz', t) \} \]
\[ \geq M_2(Tz, Tz', t) \]
Also we have
\[ M_2(Tz, Tz', t) = M_2(Tz', TSTz', t) \]
\[ \geq \min \{ M_1(z, STz', t/q), M_2(Tz', Tz, t/q) \ast M_2(Tz', TSTz', t/q) \} \]
\[
M_1(z, z', t/q) \geq M_2(Tz, Tz', t) \geq M_1(z, z', t/q) \quad (\text{since } q<1),
\]

which is a contradiction.

Thus \( z = z' \).

So the point \( z \) is the unique fixed point of \( ST \). Similarly, we prove the point \( w \) is also a unique fixed point of \( TS \).

This completes the proof.

**Remark 2.2:** If \((X, M_1, *)\) and \((Y, M_2, *)\) are the same fuzzy metric spaces, then by the above theorem 2.1, we get the following theorem, as corollary.

**Corollary 2.3:** Let \((X, M, *)\) be a complete fuzzy metric space. If \( S \) and \( T \) are mappings from \( X \) into itself satisfying the following conditions:

\[
M_1(Tx, TSy, qt) \geq \min \{M(x, Sy, t), M(y, Tx, t), M(y, TSy, t)\}
\]

\[
M_2(Sy, STx, qt) \geq \min \{M(x, Sy, t), M(x, STx, t), M(y, Tx, t)\}
\]

for all \( x, y \) in \( X \) where \( q < 1 \), then \( ST \) has a unique fixed point \( z \) in \( X \) and \( TS \) has a unique fixed point \( w \) in \( X \). Further \( Tz = w \) and \( Sw = z \).

**Theorem 2.4:** Let \((X, M_1, *)\) and \((Y, M_2, *)\) be two complete fuzzy metric spaces. If \( T \) is a mapping from \( X \) into \( Y \) and \( S \) is a mapping from \( Y \) into \( X \), satisfying following conditions:

\[
M_2(Tx, TSy, qt) \geq \min \{M(x, Sy, t), M(y, Tx, t), M(y, TSy, t)\} \quad (1)
\]

\[
M_1(Sy, STx, qt) \geq \min \{M(x, Sy, t), M(x, STx, t), M(y, Tx, t)\} \quad (2)
\]

for all \( x \) in \( X \) and \( y \) in \( Y \) where \( q < 1 \), then \( ST \) has a unique fixed point \( z \) in \( X \) and \( TS \) has a unique fixed point \( w \) in \( Y \). Further \( Tz = w \) and \( Sw = z \).

**Proof:** Let \( x_0 \) be an arbitrary point in \( X \). Define a sequence \( \{x_n\} \) in \( X \) and \( \{y_n\} \) in \( Y \), as follows:

\[
x_n = (ST)^n x_0 , \quad y_n = T(x_{n-1}) \quad \text{for } n = 1, 2, \ldots.
\]

we have

\[
M_1(x_n, x_{n+1}, qt) = M_1((ST)^n x_0, (ST)^{n+1} x_0, qt) = M_1(S(T(ST)^{n-1} x_0), ST(x_{n+1}, qt) \geq \min \{M_2(y_n, Tx_n, t), M_1(x_n, STx_n, t)\}
\]

\[
M_2(y_n, y_{n+1}, qt) = M_2(Tx_{n-1}, TSy_n, qt) \geq \min \{M_1(x_{n-1}, Sy_n, t), M_2(y_n, Tx_{n-1}, t) \}
\]

Now

\[
M_2(y_n, y_{n+1}, qt) = \min \{M_1(x_{n-1}, Sy_n, t), M_2(y_n, Tx_{n-1}, t) \}
\]

\[
= \min \{M_1(x_{n-1}, x_n, t), M_2(y_n, y_{n+1}, t) \}
\]

\[
= \min \{M_1(x_{n-1}, x_n, t), 1 \}
\]

\[
= \min \{M_1(x_{n-1}, x_n, t), M_2(y_n, y_{n+1}, t) \}.
\]

\[
M_2(y_n, y_{n+1}, qt) = \min \{M_1(x_{n-1}, x_n, t), M_2(y_n, y_{n+1}, t) \}
\]

\[
= \min \{M_1(x_{n-1}, x_n, t), M_2(y_n, y_{n+1}, t) \}.
\]

\[
M_1(x_{n-1}, x_n, t) = \min \{M_1(x_{n-1}, x_n, t), M_2(y_n, y_{n+1}, t) \}.
\]
Thus \( \{x_n\} \) is a Cauchy sequence in \( X \). Since \( (X, M_1, \delta) \) is complete, it converges to a point \( z \) in \( X \). Similarly, we can prove that the sequence \( \{y_n\} \) is also a Cauchy sequence in \( Y \) and it converges to a point \( w \) in \( Y \).

Now we prove \( Tz = w \)

Suppose \( Tz \neq w \).

We have

\[
M_2(Tz, w, qt) = \lim_{n \to \infty} M_2(Tz, y_{n+1}, qt)
\]

\[
= \lim_{n \to \infty} M_2(Tz, TSy_n, qt)
\]

\[
\geq \lim_{n \to \infty} \min \{M_1(z, Sy_n, t), M_2(y_n, Tz, t), M_2(y_{n+1}, Tz, t)\}
\]

\[
= \lim_{n \to \infty} \min \{M_1(z, x_n, t), M_2(y_n, Tz, t), M_2(y_{n+1}, Tz, t)\}
\]

\[
= \min \{M_2(z, z, t), M_2(w, Tz, t), M_2(w, Tz, t)\} \geq 0
\]

\[
(\text{since } q < 1)
\]

Thus \( Tz = w \).

Now we prove \( Sw = z \).

Suppose \( Sw \neq z \).

We have

\[
M_1(Sw, z, qt) = \lim_{n \to \infty} M_1(Sw, x_{n+1}, qt)
\]

\[
= \lim_{n \to \infty} M_1(Sw, STx_n, qt)
\]

\[
\geq \lim_{n \to \infty} \min \{M_2(w, Tz, t), M_1(x_n, Sw, t), M_1(x_n, Sw, t)\}
\]

\[
= \lim_{n \to \infty} \min \{M_2(w, y_{n+1}, t), M_1(x_n, Sw, t), M_1(x_{n+1}, Sw, t)\}
\]

\[
= \min \{M_2(w, w, t), M_2(z, w, t), M_2(z, Sw, t)\} \geq 0
\]

\[
(\text{since } q < 1)
\]

Thus \( Sw = z \).

Therefore we have \( STz = Sw = z \) and \( TSw = Tz = w \). Thus the point \( z \) is a fixed point of \( ST \) and the point \( w \) is a fixed point of \( TS \).

**Uniqueness:** Let \( z' \) be another fixed point of \( ST \) such that \( z = z' \).

We have

\[
M_1(z', z, qt) = M_1(STz', STz, qt)
\]

\[
\geq \min \{M_2(Tz', Tz, t), M_1(z, STz', t), M_2(z, STz', t)\}
\]

\[
\geq \min \{M_2(Tz', Tz, t), M_1(z, z', t), M_1(z, z', t)\}
\]

\[
\geq M_2(Tz', Tz, t)
\]

Also we have

\[
M_2(Tz', Tz, qt) = M_2(Tz', TSTz, qt)
\]

\[
\geq \min \{M_1(z', STz, t), M_2(Tz, Tz', t), M_2(Tz, Tz', t)\}
\]

\[
= \min \{M_1(z', z, t), M_2(Tz, Tz', t), M_2(Tz, Tz', t)\}
\]

\[
\geq M_1(z', t)
\]

(i.e.) \( M_2(Tz', Tz, t) \geq M_1(z', t/q) \)
Hence
\[ M_1(z',z, qt) \geq M_2(Tz', Tz, t) \geq M_1(z, z', t/q) \text{ (since } q < 1 \text{)} \]
which is a contradiction.

Thus \( z = z' \).

So the point \( z \) is the unique fixed point of \( ST \). Similarly, we prove the point \( w \) is also a unique fixed point of \( TS \). This completes the proof.

**Remark 2.5:** If \((X, M_1, *)\) and \((Y, M_2, *)\) are the same fuzzy metric spaces, then by the above theorem 2.4, we get the following theorem, as corollary.

**Corollary 2.6:** Let \((X, M, *)\) be a complete fuzzy metric space. If \( S \) and \( T \) are mappings from \( X \) into itself satisfying the following conditions:

\[
M(Tx, TSy, qt) \geq \min\{M(x, Sy, t), M(y, Tx, t), M(y, TSy, t) \}
\]

for all \( x, y \) in \( X \) where \( q < 1 \), then \( ST \) has a unique fixed point \( z \) in \( X \) and \( TS \) has a unique fixed point \( w \) in \( X \). Further \( Tz = w \) and \( Sw = z \).

**Theorem 2.7:** Let \((X, M_1, *)\) and \((Y, M_2, *)\) be two complete fuzzy metric spaces. If \( T \) is a mapping from \( X \) into \( Y \) and \( S \) is a mapping from \( Y \) into \( X \), satisfying the following conditions:

\[
M_2(Tx, TSy, qt) \geq \min\{M_1(x, Sy, t), M_2(y, Tx, t), M_2(y, TSy, t) \}
\]

for all \( x, y \) in \( X \) where \( q < 1 \), then \( ST \) has a unique fixed point \( z \) in \( X \) and \( TS \) has a unique fixed point \( w \) in \( Y \). Further \( Tz = w \) and \( Sw = z \).

**Proof:** Let \( x_0 \) be an arbitrary point in \( X \). Define a sequence \( \{x_n\} \) in \( X \) and \( \{y_n\} \) in \( Y \), as follows:

\[
x_n = (ST)^n x_0, \quad y_n = T(x_{n-1}) \quad \text{for } n = 1, 2, \ldots .
\]

we have

\[
M_1(x_n, x_{n+1}, qt) = M_1((ST)^n x_0, (ST)^{n+1} x_0, qt) = M_1(ST(x_{n-1}), STx_{n+1}, qt) = M_1(Sy_n, SY_{n+1}, qt) \geq \min\{M_2(y_n, x_{n+1}, t), M_1(x_{n+1}, Sy_n, t), M_2(x_{n+1}, STx_{n+1}, qt) \}
\]

for all \( x, y \) in \( X \) where \( q < 1 \), then \( ST \) has a unique fixed point \( z \) in \( X \) and \( TS \) has a unique fixed point \( w \) in \( Y \). Further \( Tz = w \) and \( Sw = z \).

Thus \( \{x_n\} \) is a Cauchy sequence in \( X \). Since \((X, M_1, *)\) is complete, it converges to a point \( z \) in \( X \). Similarly, we can prove that the sequence \( \{y_n\} \) is also a Cauchy sequence in \( Y \) and it converges to a point \( w \) in \( Y \).

Now we prove \( Tz = w \)
Suppose $Tz \neq w$.

We have

$$M_2(Tz, w, qt) = \lim_{n \to \infty} M_2(Tz, y_{n+1}, qt)$$

$$\geq \lim_{n \to \infty} \min\{M_1(z, y_n, t), M_2(y_n, Tz, t), M_2(y_n, TSy_n, t), M_2(z, STz, t)\}$$

Also

$$M_1(z, STz, qt) = \lim_{n \to \infty} M_1(x_n, STz, qt)$$

$$\geq \lim_{n \to \infty} \min\{M_2(y_n, Tz, t), M_1(x_n, z, t), M_1(x_n, STx_n, t), M_2(Tz, TSx_n, t)\}$$

(hence) $M_1(z, STz, t) \geq M_2(Tz, w, t/q)$

Thus $Tz = w$.

Now we prove $Sw = z$.

Suppose $Sw \neq z$.

Then we have

$$M_1(Sw, z, qt) = \lim_{n \to \infty} M_1(Sw, x_{n+1}, qt)$$

$$\geq \lim_{n \to \infty} \min\{M_2(w, Tz, t), M_1(x_n, Sw, t), M_1(x_n, STz, t), M_2(Tz, TSw, t)\}$$

(i.e), $M_2(Tz, w, qt) \geq M_2(w, Tz, t/q)$, which is a contradiction. (Since $q < 1$)

Hence $M_1(Sw, z, qt) \geq M_2(w, Tz, t/q)$.
Thus $Sw = z$.

Therefore we have $STz = Sw = z$ and $TSw = Tz = w$. Thus the point $z$ is a fixed point of $ST$ and the point $w$ is a fixed point of $TS$.

**Uniqueness:** Let $z'$ be the another fixed point of $ST$ such that $z \neq z'$.

Now

$$M_1(z, z', \eta t) = \lim_{n \to \infty} M_1(Sw, STz', \eta t)$$

$$\geq \lim_{n \to \infty} \min \{M_2(w, Tz', \eta t), M_1(z', Sw, \eta t), M_1(z', STz', \eta t), M_2(Tz', w, \eta t)\}$$

$$= \min \{ M_2(w, Tz', \eta t), M_1(z', z, \eta t), 1, M_2(Tz', w, \eta t)\}$$

$$\geq M_2(Tz', w, \eta t)$$

Also

$$M_2(Tz', w, \eta t) = \lim_{n \to \infty} M_2(Tz', y_{n+1}, \eta t)$$

$$\geq \lim_{n \to \infty} \min \{M_1(z', y_{n+1}, \eta t), M_2(y_n, Tz', \eta t), M_2(y_n, STy_n, \eta t), M_1(z', STz', \eta t)\}$$

$$= \min \{ M_1(z', z, \eta t), M_2(w, Tz', \eta t), 1, 1\}$$

$$\geq M_1(z', z, \eta t)$$

(i.e), $M_2(Tz', w, \eta t) \geq M_1(z', z, \eta t)$

Therefore we have

$$M_1(z, z', \eta t) \geq M_2(Tz', w, \eta t) \geq M_1(z, z', \eta t)$$

Which is a contradiction. (since $q < 1$)

Thus $z = z'$.

So the point $z$ is the unique fixed point of $ST$. Similarly, we prove the point $w$ is also a unique fixed point of $TS$.

This completes the proof.

**Remark 2.8:** If $(X, M_1, \ast)$ and $(Y, M_2, \ast)$ are the same fuzzy metric spaces, then by the above theorem 2.7, we get the following theorem, as corollary.

**Corollary 2.9:** Let $(X, M, \ast)$ be a complete fuzzy metric space. If $S$ and $T$ are mappings from $X$ into itself satisfying the following conditions:

$$M(Tx, TSy, \eta t) \geq \min \{M(x, Sy, t), M(y, Tx, t), M(y, TSy, t), M(x, STx, t)\}$$

$$M(Sy, STx, \eta t) \geq \min \{M(y, Tx, t), M(x, Sy, t), M(x, STx, t), M(Tx, TSy, t), M(y, TSy, t)\}$$

for all $x, y$ in $X$ where $q < 1$, then $ST$ has a unique fixed point $z$ in $X$ and $TS$ has a unique fixed point $w$ in $Y$. Further $Tz = w$ and $Sw = z$.

**Theorem 2.10:** Let $(X, M_1, \ast)$ and $(Y, M_2, \ast)$ be two complete fuzzy metric spaces. If $T$ is a mapping from $X$ into $Y$ and $S$ is a mapping from $Y$ into $X$, satisfying following conditions:

$$M_1(Tx, TSy, \eta t) \geq \min \{M_2(x, Sy, t), M_2(y, Tx, t), M_2(y, TSy, t), M_2(x, STx, t), M_1(Sy, STx, t)\}$$

(1)

$$M_2(Sy, STx, \eta t) \geq \min \{M_2(x, Tx, t), M_2(x, Sy, t), M_2(x, STx, t), M_2(Tx, TSy, t), M_2(y, TSy, t)\}$$

(2)

for all $x$ in $X$ and $y$ in $Y$ where $q < 1$, then $ST$ has a unique fixed point $z$ in $X$ and $TS$ has a unique fixed point $w$ in $Y$. Further $Tz = w$ and $Sw = z$.

**Proof:** Let $x_0$ be an arbitrary point in $X$. Define a sequence $\{x_n\}$ in $X$ and $\{y_n\}$ in $Y$, as follows:

$$x_n = (ST)^n x_0, \quad y_n = T(x_{n-1}) \text{ for } n = 1, 2, \ldots$$
we have
\[ M_1(x_n, x_{n+1}, qt) = M_1((ST)^n x_0, (ST)^{n+1} x_0, qt) = M_1(ST x_{n-1}, ST x_n, qt) \]
\[ \geq \min \{ M_2(y_n, T x_{n+1}, t), M_1(x_n, S x_n, t), M_1(x_n, S T x_n, t), M_2(T x_{n+1}, T S x_n, t), M_2(y_n, T S x_n, t) \} \]
\[ = \min \{ M_2(y_n, y_{n+1}, t), 1, M_1(x_n, x_{n+1}, t), 1, M_2(y_n, y_{n+1}, t) \} \]
\[ \geq M_2(y_n, y_{n+1}, t) \]

Also we have
\[ M_2(y_n, y_{n+1}, qt) = M_2(T x_{n+1}, T x_n, qt) \]
\[ = M_2(T x_n, T S y_n, qt) \]
\[ \geq \min \{ M_1(x_n, S y_n, t), M_2(y_n, T x_n, t), M_2(y_n, T S y_n, t), M_1(x_n, S T x_n, t), M_1(x_n, S T x_n, t) \} \]
\[ = \min \{ M_1(x_n, x_{n+1}, t), M_2(y_n, y_{n+1}, t), M_1(x_n, x_{n+1}, t), M_1(x_n, x_{n+1}, t) \} \]
\[ \geq M_1(x_n, x_{n+1}, t) \]

Now
\[ M_1(x_n, x_{n+1}, qt) \geq M_2(y_n, y_{n+1}, t) \]
\[ \geq M(x_n, x_{n+1}, t/q) \]
\[ \geq M_1(x_n, x_{n+1}, t/q^{n-1}) \rightarrow 1 \text{ as } n \rightarrow \infty \] (since \( q < 1 \))

Thus \( \{ x_n \} \) is a Cauchy sequence in \( X \). Since \((X, M_1, \leq)\) is complete, it converges to a point \( z \) in \( X \). Similarly, we can prove that the sequence \( \{ y_n \} \) is also a Cauchy sequence in \( Y \) and it converges to a point \( w \) in \( Y \).

Now we prove \( Tz = w \)

We have
\[ M_2(Tz, w, qt) = \lim_{n \to \infty} M_2(Tz, y_{n+1}, qt) \]
\[ = \lim_{n \to \infty} M_2(Tz, T S y_n, qt) \]
\[ \geq \lim_{n \to \infty} \min \{ M_1(z, S y_n, t), M_2(y_n, T z, t), M_2(y_n, T S y_n, t), M_1(z, S T z, t), M_1(z, S T z, t) \} \]
\[ = \lim_{n \to \infty} \min \{ M_1(z, x_n, t), M_2(y_n, T z, t), M_2(y_n, y_{n+1}, t), M_1(z, S T z, t), M_1(z, S T z, t) \} \]
\[ = \min \{ 1, 1, M_2(w, T z, t), 1, M_1(z, S T z, t), M_1(z, S T z, t) \} \]
\[ \geq M_1(z, S T z, t) \]

Hence
\[ M_2(Tz, w, qt) \geq M_2(Tz, w, t/q) \]

Thus \( Tz = w \).

Now we prove \( S w = z \).

Suppose \( S w \neq z \).
\[ M_1(S w, z, qt) = \lim_{n \to \infty} M_1(S w, x_{n+1}, qt) \]
\[ = \lim_{n \to \infty} M_1(S w, S T x_n, qt) \]
Now
\[ M_2(w,TSw,qt) = \lim_{n \to \infty} M_2(y_{n+1},TSw,qt) \]
\[ = \lim_{n \to \infty} M_2(Tx_n,TSw,qt) \]
\[ \geq \lim_{n \to \infty} \min \{ M_1(x_n,Sw,t), M_2(w,STx_n,t), M_2(Sw,STx_n,t) \} \]
\[ = \lim_{n \to \infty} \min \{ M_1(x_n,x_{n+1},t), M_1(x_{n+1},TSw,t), M_2(y_{n+1},TSw,t) \} \]
\[ = \min \{ M_1(z,Sw,t), 1 , M_2(w,TSw,t), 1 \} \]
\[ \geq M_2(w,TSw,t) \]

Hence
\[ M_1(Sw,z,qt) \geq M_2(w,TSw,t) \]
\[ \geq M_1(Sw,z,t/q) \]

Which is a contradiction. (since $q < 1$)

Thus $Sw = z$.

Therefore we have $STz = Sw = z$ and $TSw = Tz = w$. Thus the point $z$ is a fixed point of $ST$ and the point $w$ is a fixed point of $TS$.

**Uniqueness:** Let $z'$ be the another fixed point of $ST$ such that $z \neq z'$.

Now
\[ M_1(z,z',qt) = M_1(Sw,STz',qt) \]
\[ \geq \min \{ M_1(w,Tz',t), M_1(z',Sw,t), M_1(z',STz',t), M_2(Tz',TSw,t), M_2(w,TSw,t) \} \]
\[ = \min \{ M_1(w,Tz',t), M_1(z',z,t), 1 , M_2(Tz',w,t), 1 \} \]
\[ \geq M_2(Tz',w,t) \]
\[ M_2(Tz',w,qt) = M_2(Tz',TSw,t) \]
\[ \geq \min \{ M_1(z',Sw,t), M_2(w,Tz',t), M_1(z',TSz',t), M_1(z',z,t), M_1(z,STz',t) \} \]
\[ = \min \{ M_1(z',z,t), M_2(w,Tz',t), 1 , M_1(z',z,t), M_1(z, z',t) \} \]
\[ \geq M_2(z, z',t) \]

Hence
\[ M_1(z,z',qt) \geq M_2(Tz',w,t) \]
\[ \geq M_1(z,z',t/q) \]

Which is a contradiction. (since $q < 1$)

Thus $z = z'$.

So the point $z$ is the unique fixed point of $ST$. Similarly, we prove the point $w$ is also a unique fixed point of $TS$.

This completes the proof.

**Remark 2.11:** If $(X,M_1, \ast)$ and $(Y,M_2, \ast)$ are the same fuzzy metric spaces, then by the above theorem 2.10, we get the following theorem, as corollary.

**Corollary 2.12:** Let $(X, M, \ast)$ be a complete fuzzy metric space. If $S$ and $T$ are mappings from $X$ into itself satisfying the following conditions:

\[ M(Tx, TSy, qt) \geq \min \{ M(x, Sy, t), M(y, Tx, t), M(y, TSy, t), M(x, STx, t), M(Sy, STx, t) \} \]
\[ M(Sy, STx, qt) \geq \min \{ M(y, Tx, t), M(x, Sy, t), M(x, STx, t), M(Tx, TSy, t), M(y, TSy, t) \} \]
for all \( x, y \) in \( X \) where \( q < 1 \), then \( ST \) has a unique fixed point \( z \) in \( X \) and \( TS \) has a unique fixed point \( w \) in \( Y \). Further \( Tz = w \) and \( Sw = z \).

**Theorem 2.13:** Let \( (X,M_1,\ast) \) and \( (Y,M_2,\ast) \) be two complete fuzzy metric spaces. If \( T \) is a mapping from \( X \) into \( Y \) and \( S \) is a mapping from \( Y \) into \( X \), satisfying the following conditions:

\[
M_2(Tx, TSy, t) \geq \min\{M_1(x, Sy, t), M_1(Tx, STx, t), M_2(y, TSy, t), M_2(y, Ty, t)\} \\
M_1(Sy, STx, t) \geq \min\{M_1(x, Sy, t), M_1(Sy, STx, t), M_2(y, TSy, t), M_2(y, Tx, t)\}
\]

for all \( x \) in \( X \) and \( y \) in \( Y \) where \( q < 1 \), then \( ST \) has a unique fixed point \( z \) in \( X \) and \( TS \) has a unique fixed point \( w \) in \( Y \).

Further \( Tz = w \) and \( Sw = z \).

**Proof.** Let \( x_0 \) be an arbitrary point in \( X \). Define a sequence \( \{x_n\} \) in \( X \) and \( \{y_n\} \) in \( Y \), as follows:

\[
x_n = (ST)^n x_0, \quad y_n = T(x_{n-1}) \quad \text{for } n = 1, 2, \ldots.
\]

We have

\[
M_1(x_n, x_{n+1}, qt) = M_1((ST)^n x_0, (ST)^{n+1} x_0, qt) = M_1(S(T(ST)^n x_0), ST(x_n), qt) \geq \min\{M_1(x_n, SY, t), M_2(y_n, TY, t), M_1(x_n, x_{n+1}, t)\}
\]

Hence

\[
M_1(x_n, x_{n+1}, qt) \geq M_1(x_0, x_1, qt/q) \geq M_1(x_0, x_1, qt/q^{2n-1}) \to 1 \quad \text{as } n \to \infty \quad \text{(since } q < 1)\]

Thus \( \{x_n\} \) is a Cauchy sequence in \( (X,M_1,\ast) \). Since \( (X,M_1,\ast) \) is complete, it converges to a point \( z \) in \( X \). Similarly, we can prove that the sequence \( \{y_n\} \) is also a Cauchy sequence in \( (Y,M_2,\ast) \). Since \( (Y,M_2,\ast) \) is complete, it converges to a point \( w \) in \( Y \).

Now we prove \( Tz = w \).

Suppose \( Tz \neq w \).

We have

\[
M_2(Tz, w, qt) = \lim_{n \to \infty} M_2(Tz, y_{n+1}, qt) = \lim_{n \to \infty} M_2(Tz, Sy_n, qt) \geq \lim_{n \to \infty} \min\{M_1(z, Sy_n, t), M_1(x_n, STz, t), M_2(y_n, TSy_n, t), M_2(z, STz, t)\}
\]

Hence

\[
M_1(z, STz, t) \geq M_1(z, z, qt) \to 1 \quad \text{as } n \to \infty \quad \text{since } q < 1.
\]

Thus \( \{x_n\} \) is a Cauchy sequence in \( (X,M_1,\ast) \). Since \( (X,M_1,\ast) \) is complete, it converges to a point \( z \) in \( X \). Similarly, we can prove that the sequence \( \{y_n\} \) is also a Cauchy sequence in \( (Y,M_2,\ast) \). Since \( (Y,M_2,\ast) \) is complete, it converges to a point \( w \) in \( Y \).
Now
\[M_1(z, STz, qt) = \lim_{n \to \infty} M_1(x_n, STz, qt)\]
\[= \lim_{n \to \infty} M_1(Sy_n, STz, qt)\]
\[\geq \lim_{n \to \infty} \min\{M_1(z, Sy_n, t), M_1(z, STz, t), M_2(y_n, Tz, t), M_2(Tz, Sy_n, t)\}\]
\[= \lim_{n \to \infty} \min\{M_1(z, x_n, t) * M_1(z, STz, t), M_2(y_n, Tz, t), M_2(Tz, y_{n+1}, t)\}\]
\[= \min\{M_1(z, STz, t), 1, M_2(w, Tz, t), M_2(Tz, w, t)\}\]
\[\geq M_2(Tz, w, t)\]

Hence
\[M_2(Tz, w, qt) \geq M_1(z, STz, t) \geq M_2(Tz, w, t/q)\] (since q<1) which is a contradiction.

Thus \(Tz = w\).

Now we prove \(Sw = z\).

Suppose \(Sw \neq z\).

We have
\[M_1(Sw, z, qt) = \lim_{n \to \infty} M_1(Sw, x_n, qt)\]
\[= \lim_{n \to \infty} M_1(Sw, STx_n, qt)\]
\[\geq \lim_{n \to \infty} \min\{M_1(x_n, Sw, t), M_1(x_n, STx_n, t), M_2(w, TSw, t), M_2(Tx_n, TSw, t)\}\]
\[= \lim_{n \to \infty} \min\{M_1(x_n, Sw, t) * M_1(x_n, x_{n+1}, t), M_2(w, TSw, t), M_2(w, y_{n+1}, t), M_2(y_{n+1}, w, t)\}\]
\[\geq M_2(w, TSw, t)\]

Now
\[M_2(w, TSw, qt) = \lim_{n \to \infty} M_2(y_{n+1}, TSw, qt)\]
\[= \lim_{n \to \infty} M_2(Tx_n, TSw, qt)\]
\[\geq \lim_{n \to \infty} \min\{M_1(x_n, Sw, t), M_1(Sw, STx_n, t), M_2(w, Tx_n, t) * M_2(w, TSw), M_1(x_n, Tx_n, t)\}\]
\[= \lim_{n \to \infty} \max\{M_1(x_n, Sw, t), M_1(Sw, x_{n+1}, t), M_2(w, y_{n+1}, t) * M_2(w, TSw, t), M_1(x_n, y_{n+1}, t)\}\]
\[\geq M_1(z, Sw, t)\]

Hence
\[M_1(Sw, z, qt) \geq M_2(w, TSw, t) \geq M_1(z, Sw, t/q)\] (since q<1) which is a contradiction.

Thus \(Sw = z\).

We have \(STz = Sw = z\) and \(TSw = Tz = w\). Thus the point \(z\) is a fixed point of \(ST\) in \(X\) and the point \(w\) is a fixed point of \(TS\) in \(Y\).

**Uniqueness:** Let \(z' \neq z\) be the another fixed point of \(ST\) in \(X\).

We have
\[M_1(z, z', qt) = M_1(Sw, STz', qt)\]
\[\geq \min\{M_1(z, Sw, t), M_1(z', STz', t), M_2(w, TSw, t), M_2(w, Tz', t), M_2(Tz', TSw, t)\}\]
\[= \min\{M_1(z, Sw, t) * 1, M_2(w, Tz', t), M_2(Tz', w, t)\}\]
\[\geq M_2(Tz', w, t)\]
Now
\[ M_2(Tz',w,qt) = M_2(Tz',TSw,qt) \]
\[ \geq \min \{ M(z',Sw,t), M_1(Sw,STz',t), M_2(w,Tz',t)*M_2(w,TSw,t), M_1(z',STz',t) \} \]
\[ = \min \{ M_1(z',z,t), M_1(z,z',t), M_2(w,Tz',t)*1, 1 \} \]
\[ \geq M_1(z,z',t) \]

Hence
\[ M_1(z, z',qt) \geq M_2(Tz',w,t) \geq M_1(z,z',t/q) \]

which is a contradiction.

Thus \( z = z' \).

So the point \( z \) is a unique fixed point of \( ST \). Similarly, we prove the point \( w \) is also a unique point of \( TS \).

This completes the proof.

**Remark 2.14:** If \((X, M_1, \ast)\) and \((Y, M_2, \ast)\) are the same fuzzy metric spaces, then by the above theorem 2.13, we get the following theorem, as corollary.

**Corollary 2.15:** Let \((X, M, \ast)\) be a complete fuzzy metric space. If \( S \) and \( T \) are mappings from \( X \) into itself satisfying the following conditions:

\[ M(Tx,TSy,t) \geq \min \{ M(x,Sy,t), M(Sy,STx,t), M(y,Tx,t) \} \]
\[ M(Sy,STx,t) \geq \min \{ M(x,Sy,t)*M(x,STx,t), M(y,TSy,t), M(y,Tx,t), M(Tx,TSy,t) \} \]

for all \( x, y \) in \( X \) where \( q < 1 \), then \( ST \) has a unique fixed point \( z \) in \( X \) and \( TS \) has a unique fixed point \( w \) in \( Y \). Further \( Tz = w \) and \( Sw = z \).

**REFERENCES**


**Source of support: Nil, Conflict of interest: None Declared**